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On A System of Rational Difference Equations

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In this paper, we investigate behaviors of well-defined solutions of the following system

$$x_{n+1} = \frac{A_1 y_{n-(3k-1)}}{B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$

$$y_{n+1} = \frac{A_2 x_{n-(3k-1)}}{B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}},$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{Z}^+$ the coefficients $A_1, A_2, B_1, B_2, C_1, C_2$ and the initial conditions are arbitrary real numbers.

Keywords: System of difference equations, Asymptotic behavior, Periodicity, Closed form solution.
AMS Classification: 39A10

1 Introduction

There has been a great effort in studying periodic and asymptotic behaviors of solutions of difference equations (see e.g. [3,6,12,15,18,20-23,27,35,45,46]). Also, studying in system of difference equations has increased considerably (see, e.g. [5,7,8,16,17,19,28-30,32-34,37,38,40,43,47]).

Ozkan et al. [31] gave the solutions of the systems of the difference equations

$$\begin{aligned} x_{n+1} &= \frac{y_{n-2}}{-1 \mp y_{n-2} x_{n-1} y_n}, \\ y_{n+1} &= \frac{x_{n-2}}{-1 \mp x_{n-2} y_{n-1} x_n}, \\ z_{n+1} &= \frac{x_{n-2} + y_{n-2}}{-1 \mp x_{n-2} y_{n-1} x_n}, n \in \mathbb{N}_0. \end{aligned} \tag{1}$$

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In [39] it was showed that the system of difference equations, which is an extension of first and second equations of system (1) with respect to coefficients,

$$x_n = \frac{c_n y_{n-3}}{a_n + b_n y_{n-1} x_{n-2} y_{n-3}},$$
$$y_n = \frac{\gamma_n x_{n-3}}{\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3}}, n \in \mathbb{N}_0, \tag{2}$$

where the sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$, and the initial values $x_i, y_i, i \in \{1, 2, 3\}$ are real numbers, such that $c_n \neq 0, \gamma_n \neq 0, n \in \mathbb{N}_0$, can be solved in closed form, and for the case when all sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$ are constant it was described the asymptotic behavior of well-defined solutions of the system.

In [41] it was showed that an extension of system (2) with respect to indices

$$x_n = \frac{c_n y_{n-(2k-1)}}{a_n + b_n y_{n-(2k-1)} \prod_{i=1}^{k-1} y_{n-(2i-1)} x_{n-2i}},$$
$$y_n = \frac{\gamma_n x_{n-(2k-1)}}{\alpha_n + \beta_n x_{n-(2k-1)} \prod_{i=1}^{k-1} x_{n-(2i-1)} y_{n-2i}}, \tag{3}$$

where $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$, and the initial conditions $x_i, y_i, i \in \{1, 2, ...2k-1\}$ are real numbers, is solved in closed form, and the behavior of its well-defined solutions when all the sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n$ are constant was described. Related rational difference equations are studied, e.g. in [1,2,4,9-11,13,14,24-26,31,36,42,44,48].

In this paper we consider an other extension of system (2)

$$x_{n+1} = \frac{A_1 y_{n-(3k-1)}}{B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$
$$y_{n+1} = \frac{A_2 x_{n-(3k-1)}}{B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}}, \tag{4}$$

where $n \in \mathbb{N}_0$, k is a positive integer, the initial conditions and the coefficients A_1, A_2, B_1 , B_2, C_1, C_2 are arbitrary real numbers. We will consider only well-defined solutions, that is, $B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)} \neq 0$ and $B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)} \neq 0, n = 0, 1, 2, ...$

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2 Special Cases

2.1 The case $A_1 = 0$ or $A_2 = 0$

If $A_1 = 0$, we obtain directly $x_n = 0$ for $n > 0$. By using this, we get $y_n = 0$ for $n > 3k$. If $A_2 = 0$, we obtain directly $y_n = 0$ for $n > 0$. By using this, we get $x_n = 0$ for $n > 3k$. From now on both of A_1 and A_2 will be considered a non-zero real numbers.

System (4) is equivalent to the following system

$$x_{n+1} = \frac{y_{n-(3k-1)}}{b_1 + c_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$
$$y_{n+1} = \frac{x_{n-(3k-1)}}{b_2 + c_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}}, \tag{5}$$

where $n \in \mathbb{N}_0$, $b_i = \frac{B_i}{A_i}$ and $c_i = \frac{C_i}{A_i}, i = 1, 2$. So, we will consider system (5) instead of system (4).

2.2 The case $b_1 = 0$ or $b_2 = 0$

If $b_1 = 0$, from the first equation of system (5), we have $x_{n-2k} y_{n-k} x_n = \frac{1}{c_1}$, $n > 0$. Using this, we obtain directly $y_n = \alpha x_{n-3k}$, $n \geq k$, where $\alpha = \frac{c_1}{b_2 c_1 + c_2}$. From this and by the change of variables

$$z_n = \frac{y_n}{x_{n-3k}}, w_n = \frac{y_{n-3k}}{x_n}, n \geq k, \tag{6}$$

system (5) can be transformed into the system

$$w_{n+1} = c - c b_2 z_{n-(k-1)}, z_{n+1} = \alpha, n \geq k-1, \tag{7}$$

where $c = \frac{c_1}{c_2}$. The solutions are obtained easily as $z_n = w_n = \alpha$, $n \geq k$. This means every solution of system (5) is periodic with $6k$ periods, not necessarily prime period, such that $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n \geq 4k$.

If $b_2 = 0$, we get immediately $y_{n-2k} x_{n-k} y_n = \frac{1}{c_2}$, $n > 0$. From the first equation in system (5) and using this, we obtain $x_n = \beta y_{n-3k}$, $n \geq k$, where $\beta = \frac{c_2}{b_1 c_2 + c_1}$. The change of variables

$$u_n = \frac{x_n}{y_{n-3k}}, t_n = \frac{x_{n-3k}}{y_n}, n \geq k, \tag{8}$$

reduces system (5) to the system

$$t_{n+1} = \bar{c} - \bar{c} b_1 u_{n-(k-1)}, u_{n+1} = \beta, n \geq 2k-1, \tag{9}$$

where $\bar{c} = \frac{c_2}{c_1}$. The solutions of this system $t_n = u_n = \beta$, $n \geq 2k-1$, are obtained easily. So, every solution of system (5) is periodic with $6k$ periods, not necessarily prime period, such that $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n \geq 4k$.

Assume that $b_1 = 0$ and $b_2 = 0$. We have $x_{n-2k} y_{n-k} x_n = \frac{1}{c_1}$, $y_{n-2k} x_{n-k} y_n = \frac{1}{c_2}$, $n > 0$. Then, we get immediately $x_n = \frac{c_2}{c_1} y_{n-3k}$, $y_n = \frac{c_1}{c_2} y_{n-3k}$, $n > k$. Thus, we can write $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n > 4k$.

2.3 The case $c_1 = 0$ or $c_2 = 0$

If $c_1 = 0$, we have $x_n = \frac{1}{b_1} y_{n-3k}$, $n > 0$. From this and using the change of variables

$$v_n = \frac{1}{x_{n+3k} y_{n-k} x_n}, n > 0, \tag{10}$$

the second equation of system (5) implies the linear equation

$$v_{n+1} = b_1 b_2 v_{n-(2k-1)} + {b_1}^2 c_2, n = 0, 1, 2, ... \tag{11}$$

We can rewrite the equation (11) in the form of

$$v_{2kn+m} = b_1 b_2 v_{2k(n-1)+m} + b_1^2 c_2, \tag{12}$$

where $n \in \mathbb{N}_0$, $m = 1, 2, ..., k$. Considering the solution of a nonhomogeneous first order difference equation, we can give the solution of the equation (12) such that

$$v_{2kn+m} = (b_1 b_2)^n v_{m-2k} + b_1^2 c_2 \frac{1 - (b_1 b_2)^{n+1}}{1 - b_1 b_2}, n \geq 0. \tag{13}$$

when $b_1 b_2 \neq 1$. If $b_1 b_2 = 1$, the solution of the equation (12) can be written as

$$v_{2kn+m} = v_{m-2k} + (n+1) b_1^2 c_2, n \geq 0. \tag{14}$$

From (10), we have

$$x_{2kn+3k+m} = \tfrac{v_{2k(n-1)+m}}{v_{2kn+m}} x_{2kn-3k+m}.$$

Considering $x_n = \frac{1}{b_1} y_{n-3k}$, we obtain the solutions of system (5) as

$$x_{6kn+3k+m} = x_{-3k+m} \prod_{r=0}^{n} \frac{v_{6kr-2k+m}}{v_{6kr+m}}, y_{6kn+m} = b_1 x_{-3k+m} \prod_{r=0}^{n} \frac{v_{6kr-2k+m}}{v_{6kr+m}}, \tag{15}$$

$$x_{6kn+5k+m} = x_{-k+m} \prod_{r=0}^{n} \frac{v_{6kr+m}}{v_{6kr+2k+m}}, y_{6kn++2k+m} = b_1 x_{-k+m} \prod_{r=0}^{n} \frac{v_{6kr+m}}{v_{6kr+2k+m}}, \tag{16}$$

$$x_{6kn+7k+m} = x_{k+m} \prod_{r=0}^{n} \frac{v_{6kr+2k+m}}{v_{6kr+4k+m}}, y_{6kn+4k+m} = b_1 x_{k+m} \prod_{r=0}^{n} \frac{v_{6kr+2k+m}}{v_{6kr+4k+m}}, \tag{17}$$

$n \geq 0$ and $m = 1, 2, ..., 2k$.

Suppose that $c_2 = 0$. Then, we have $y_n = \frac{1}{b_2} x_{n-3k}$, $n > 0$. From this and using the change of variable

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$$u_n = \frac{1}{y_{n+3k}y_{n+k}y_{n-k}}, n > 0, \tag{18}$$

the first equation of system (5) implies the linear equation

$$u_{n+1} = b_1b_2u_{n-(2k-1)} + b_2^2c_1, n \geq 0. \tag{19}$$

By similar processes just as we did, we can rewrite the equation (19) as

$$u_{2kn+m} = b_1b_2u_{2k(n-1)+m} + b_2^2c_1, n \geq 0, \tag{20}$$

where $m = 1, 2, ..., k$. We obtain the solution of the equation (20)

$$u_{2kn+m} = (b_1b_2)^n u_{m-2k} + b_2^2c_1 \frac{1-(b_1b_2)^{n+1}}{1-b_1b_2}, n \geq 0, \tag{21}$$

when $b_1b_2 \neq 1$. When $b_1b_2 = 1$, the solution of the equation (20)

$$u_{2kn+m} = u_{m-2k} + (n+1)\, b_2^2c_1, n \geq 0. \tag{22}$$

From (18), we have

$$y_{n+3k} = \frac{1}{u_n y_{n+k} y_{n-k}}, n > 0,$$

and

$$y_{2kn+3k+m} = \frac{u_{2k(n-1)+m}}{u_{2kn+m}} y_{2kn-3k+m}.$$

Considering $y_n = \frac{1}{b_2}x_{n-3k}$, $n > 0$, we obtain the solutions of system (5) as

$$x_{6kn+m} = b_2y_{-3k+m}\prod_{r=0}^{n}\frac{u_{6kr-2k+m}}{u_{6kr+m}}, y_{6kn+3k+m} = y_{-3k+m}\prod_{r=0}^{n}\frac{u_{6kr-2k+m}}{u_{6kr+m}}, \tag{23}$$

$$x_{6kn++2k+m} = b_2y_{-k+m}\prod_{r=0}^{n}\frac{u_{6kr+m}}{u_{6kr+2k+m}}, y_{6kn+5k+m} = y_{-k+m}\prod_{r=0}^{n}\frac{u_{6kr+m}}{u_{6kr+2k+m}}, \tag{24}$$

$$x_{6kn+4k+m} = b_2y_{k+m}\prod_{r=0}^{n}\frac{u_{6kr+2k+m}}{u_{6kr+4k+m}}, y_{6kn+7k+m} = y_{k+m}\prod_{r=0}^{n}\frac{u_{6kr+2k+m}}{u_{6kr+4k+m}}, \tag{25}$$

$n \geq 0$ and $m = 1, 2, ..., 2k$.

Suppose that both c_1 and c_2 are equal to zero. We get immediately $x_{n+1} = \frac{1}{b_1}y_{n-(3k-1)}$, $y_{n+1} = \frac{1}{b_2}x_{n-(3k-1)}$, $n \geq 0$. From this result, we obtain $x_{n+1} = \frac{1}{b_1b_2}x_{n-(6k-1)}, y_{n+1} = \frac{1}{b_1b_2}y_{n-(6k-1)}$, $n \geq 3k$. So, we have $x_{6kn+3k+m} = \frac{1}{b_1b_2}$ $x_{6kn-3k+m}$, $y_{6kn+3k+m} = \frac{1}{b_1b_2}y_{6kn-3k+m}$ and from this $x_{6kn+3k+m} = \left(\frac{1}{b_1b_2}\right)^{n+1}$ x_{-3k+m}, $y_{6kn+3k+m} = \left(\frac{1}{b_1b_2}\right)^{n+1} y_{-3k+m}$, $n \geq 0$, $m = 1, 2, ..., 6k$.

5

3 Main Case

In this section, we will need the following results, given in the reference [16], in the proofs of our results.

Consider the first order Riccati difference equation

$$x_{n+1} = \frac{a + bx_n}{c + dx_n}, n = 0, 1, ..., \tag{26}$$

where the parameters and the initial condition x_0 are arbitrary real numbers.

Theorem 1 *The followings are true:*

1) *Eq.(26) has a prime period-2 solution if and only if* $b + c = 0$.

2) *Suppose* $b + c = 0$. *Then every solution* $\{x_n\}$ *of Eq. (26) with* $x_0 \neq 0$ *is periodic with period 2.*

Theorem 2 *Assume that* $d \neq 0, bc - ad \neq 0,\ b + c \neq 0$ *and* $R = \frac{bc-ad}{(b+c)^2} < \frac{1}{4}$. *Then the forbidden set* F *of Eq.(26) is given as follows:*

$$F = \left\{ \tfrac{b+c}{d} \left(\tfrac{\lambda_1 \lambda_2^n - \lambda_2 \lambda_1^n}{\lambda_2^n - \lambda_1^n} \right) - \tfrac{c}{d} : n \geq 1 \right\} .$$

For any well-defined solution $\{x_n\}$ *of Eq. (26), we have*

$$x_n = \tfrac{b+c}{d} \left(\tfrac{c_1 \lambda_1^{n+1} - c_2 \lambda_2^{n+1}}{c_1 \lambda_1^n - c_2 \lambda_2^n} \right) - \tfrac{c}{d},$$

for $n = 0, 1, ...,$ *where* $\lambda_1 = \frac{1-\sqrt{1-4R}}{2}$, $\lambda_2 = \frac{1+\sqrt{1-4R}}{2}, c_1 = \frac{\lambda_2(b+c)-(dx_0+c)}{(\lambda_2-\lambda_1)(b+c)}$ *and* $c_2 = \frac{(dx_0+c)-\lambda_1(b+c)}{(\lambda_2-\lambda_1)(b+c)}$.

Corollary 1 *Assume that the conditions in Theorem2 hold. Let* $\{x_n\}$ *be a well-defined solution of Eq. (26). Then*

$$lim_{n\to\infty} x_n = \tfrac{\lambda_2(b+c)-c}{d}.$$

Theorem 3 *Assume that* $d \neq 0, bc - ad \neq 0,\ b + c \neq 0$ *and* $R = \frac{bc-ad}{(b+c)^2} = \frac{1}{4}$. *Then the forbidden set* F *of Eq.(26) is given as follows:*

$$F = \left\{ \tfrac{n(b-c)-(b+c)}{2dn} : n \geq 1 \right\}.$$

For any well-defined solution $\{x_n\}$ *of Eq. (26), we have*

$$x_n = \tfrac{b+c}{d} \left(\tfrac{(b+c)+(n+1)(2dx_0+(c-b))}{2(b+c)+2n(2dx_0+(c-b))} \right) - \tfrac{c}{d},$$

for $n = 0, 1, ...$

Corollary 2 *Assume that the conditions in Theorem3 hold. Let* $\{x_n\}$ *be a well-defined solution of Eq.(26). Then*

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$$lim_{n\to\infty} x_n = \frac{b-c}{2d}.$$

Now we consider the system (5) with b_1, b_2, c_1, c_2 parameters and the initial conditions are non-zero real numbers. By the change of variables (6), the system (5) reduces to

$$z_{n+1} = \frac{w_{n-(k-1)}}{\frac{1}{\alpha}w_{n-(k-1)} - \gamma_2}, w_{n+1} = \frac{1}{\beta} - \gamma_1 z_{n-(k-1)}, n \geq k, \tag{27}$$

where $\gamma_1 = \frac{c_1 b_2}{c_2}, \gamma_2 = \frac{c_2 b_1}{c_1}, \alpha = \frac{c_1}{b_2 c_1 + c_2}, \beta = \frac{c_2}{b_1 c_2 + c_1}$. We can rewrite the system (27) such that

$$z_{n+1} = \frac{\frac{1}{\beta} - \gamma_1 z_{n-(2k-1)}}{\left(\frac{1}{\alpha\beta} - \gamma_2\right) - \frac{\gamma_1}{\alpha} z_{n-(2k-1)}}, w_{n+1} = \frac{\left(\frac{1}{\alpha\beta} - \gamma_1\right) w_{n-(2k-1)} - \frac{\gamma_2}{\beta}}{\frac{1}{\alpha}w_{n-(2k-1)} - \gamma_2}, \tag{28}$$

$n \geq 2k$. Each of the equation in (28)is a $2k$th order Riccati difference equation. Furthermore, the equations in (28) can be rewritten such that

$$\begin{aligned} z_{2kn+1+i} &= \frac{\frac{1}{\beta} - \gamma_1 z_{2k(n-1)+1+i}}{\left(\frac{1}{\alpha\beta} - \gamma_2\right) - \frac{\gamma_1}{\alpha} z_{2k(n-1)+1+i}}, \\ w_{2kn+1+i} &= \frac{\left(\frac{1}{\alpha\beta} - \gamma_1\right) w_{2k(n-1)+1+i} - \frac{\gamma_2}{\beta}}{\frac{1}{\alpha}w_{2k(n-1)+1+i} - \gamma_2}, \end{aligned} \tag{29}$$

$n > 0, i = 0, 1, ..., (2k-1)$. Note that the equations in (29) are first order Riccati difference equation in variables z_{2kn+i}, w_{2kn+i}, for $i = 1, 2, ..., 2k$.

Theorem 4 *Assume that $b_1 b_2 = -1$ and $\{x_n, y_n\}$ is a well-defined solution of system (5). Then,*

$$\begin{aligned} x_{2kn+1+i} &= \frac{x_{2k(n-2)+1+i}x_{2k(n-3)+1+i}}{x_{2k(n-5)+1+i}}, \\ y_{2kn+1+i} &= \frac{y_{2k(n-2)+1+i}y_{2k(n-3)+1+i}}{y_{2k(n-5)+1+i}}, \end{aligned}$$

for $n \geq 4$, $i = 0, 1, ..., (2k-1)$.

***Proof* 1** *Consider system (29) and suppose that $b_1 b_2 = -1$. Then, we have*

$$\begin{aligned} \frac{1}{\alpha\beta} - \gamma_1 - \gamma_2 &= \frac{1}{\frac{c_1}{b_2 c_1 + c_2}\frac{c_2}{b_1 c_2 + c_1}} - \frac{c_1 b_2}{c_2} - \frac{c_2 b_1}{c_1} \\ &= \frac{b_1 b_2 c_1 c_2 + c_1^2 b_2 + c_2^2 b_1 + c_1 c_2 - c_1^2 b_2 - c_2^2 b_1}{c_1 c_2} \\ &= \frac{c_1 c_2 (b_1 b_2 + 1)}{c_1 c_2} \\ &= 0. \end{aligned}$$

So, from Theorem1(2) we conclude that every solution of each equation in system (29) *is periodic with period* $4k$*, that is,*

$$z_{2kn+1+i} = z_{2k(n-2)+1+i}, w_{2kn+1+i} = w_{2k(n-2)+1+i}, \tag{30}$$

for $n \geq 2,\ i = 0, 1, ..., (2k-1)$.

From (6), we have

$x_n = \frac{z_{n-3k}}{w_n} x_{n-6k}, y_n = \frac{z_n}{w_{n-3k}} y_{n-6k},$ (31)

for $n \geq 4k$. *System* (31) *can be written such that*

$$x_{2kn+1+i} = \frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}} x_{2kn+1+i-6k}, y_{2kn+1+i} = \frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}} y_{2kn+1+i-6k} \tag{32}$$

for $n \geq 2,\ i = 0, 1, ..., (2k-1)$. *From* (6), (30) *and* (32), *we get*

$$\begin{aligned} x_{2kn+1+i} &= \frac{z_{2k(n-2)+1+i-3k}}{w_{2k(n-2)+1+i}} x_{2kn+1+i-6k} \\ &= \frac{\frac{y_{2k(n-2)+1+i-3k}}{x_{2k(n-2)+1+i-6k}}}{\frac{y_{2k(n-2)+1+i-3k}}{x_{2k(n-2)+1+i}}} x_{2kn+1+i-6k} \\ x_{2kn+1+i} &= \frac{x_{2k(n-2)+1+i} x_{2k(n-3)+1+i}}{x_{2k(n-5)+1+i}} \end{aligned} \tag{33}$$

and similarly

$y_{2kn+1+i} = \frac{y_{2k(n-2)+1+i} y_{2k(n-3)+1+i}}{y_{2k(n-5)+1+i}}$ (34)

for $n \geq 4,\ i = 0, 1, ..., (2k-1)$.

Theorem 5 *Assume that* $\{x_n, y_n\}$ *is a well-defined solution of system* (5). *Then the followings are true:*

i) *Assume that* $b_1 b_2 = 1$. *Then every solution converges to a periodic solution with period* $6k$.

ii) *Assume that* $b_1 b_2 \neq 1$. *Then,*

a) *If* $b_1 b_2 < -1$ *or* $b_1 b_2 > 1$, *then*

$$lim_{n\to\infty} \frac{x_n}{x_{n-6k}} = lim_{n\to\infty} \frac{y_n}{y_{n-6k}} = \frac{b_2 c_1 + c_2}{b_2 c_2 (b_1 b_2 c_1 + c_2 b_1 + c_1)}.$$

b) *If $-1 < b_1b_2 < 1$, then every solution converges to a periodic solution with period $6k$.*

Proof 2

i) Consider system (29) with $\gamma_1 = \frac{c_1b_2}{c_2}, \gamma_2 = \frac{c_2b_1}{c_1}, \alpha = \frac{c_1}{b_2c_1+c_2}, \beta = \frac{c_2}{b_1c_2+c_1}$. Suppose that $b_1b_2 = 1$. Then, we have

$$\begin{aligned} \frac{-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha})}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} &= \frac{\gamma_1\gamma_2}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} \\ &= \frac{\frac{c_1b_2}{c_2}\frac{c_2b_1}{c_1}}{\left(-\frac{c_1b_2}{c_2}+\frac{1}{\frac{c_1}{b_2c_1+c_2}\frac{c_2}{b_1c_2+c_1}}-\frac{c_2b_1}{c_1}\right)^2} \\ &= \frac{b_1b_2}{(b_1b_2+1)^2} \\ &= \frac{1}{4}. \end{aligned} \tag{35}$$

Similarly, it can be seen that

$$\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2} = \frac{\gamma_1\gamma_2}{(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2)^2} = \frac{1}{4}. \tag{36}$$

So, from (31), (32) and Theorem3, we obtain

$$\begin{aligned} \lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}} &= \lim_{n\to\infty}\frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}} \\ &= \frac{\frac{-\gamma_1-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{2\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)-(-\gamma_2)}{2\left(\frac{1}{\alpha}\right)}} = 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}} &= \lim_{n\to\infty}\frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}} \\ &= \frac{\frac{-\gamma_1-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{2\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)-(-\gamma_2)}{2\left(\frac{1}{\alpha}\right)}} = 1, \end{aligned}$$

$i = 0, 1, ..., (2k-1)$. *Thus, we have* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-6k}$ *and* $\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n-6k}$. *So, theproof of (i) is finished.*

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ii)a) Assume that $b_1b_2 < -1$ *or* $b_1b_2 > 1$. *From* (35) *and* (36), *we get that*

$-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}\left(-\frac{\gamma_1}{\alpha}\right))\frac{1}{\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)^2}<\frac{1}{4}$

(36)

and

$\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha})\frac{1}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}<\frac{1}{4}.$

So, from (31), (32) and Theorem2, we obtain that

$$\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}}=\lim_{n\to\infty}\frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}}$$

$$=\frac{\dfrac{\frac{1+\sqrt{1-4\frac{-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}\left(-\frac{\gamma_1}{\alpha}\right)}{\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)^2}}}{2}\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{\left(-\frac{\gamma_1}{\alpha}\right)}}{\dfrac{\frac{1+\sqrt{1-4\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}}}{2}\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)-(-\gamma_2)}{\frac{1}{\alpha}}}$$

$$=\frac{\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)-\left(b_1b_2+1+\frac{c_1b_2}{c_2}\right)}{-\frac{c_1b_2}{c_2}\left(\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)+\frac{c_2b_1}{c_1}\right)} \tag{37}$$

$$=\frac{b_2c_1+c_2}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$$

and

$\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}}=\lim_{n\to\infty}\frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}}$

$=(b_2c_1+c_2)\frac{1}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)},$

$i=0,1,...,(2k-1)$. *Thus, we have* $\lim_{n\to\infty}\frac{x_n}{x_{n-6k}}=\lim_{n\to\infty}\frac{y_n}{y_{n-6k}}=\frac{b_2c_1+c_2}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$.

b) Assume that $-1 < b_1b_2 < 1$. *From (35) and (36), we get that*

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-$\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha}))\frac{1}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2}<\frac{1}{4}$

and

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$\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}\right)\frac{1}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}<\frac{1}{4}$. *So, from (31), (32), (37) and Theorem2, we obtain*

$$\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}}=\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}}$$

$$=\frac{\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)-\left(b_1b_2+1+\frac{c_1b_2}{c_2}\right)}{-\frac{c_1b_2}{c_2}\left(\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)+\frac{c_2b_1}{c_1}\right)}$$

$$=\frac{b_1b_2+\frac{c_1b_2}{c_2}}{\frac{c_1b_2}{c_2}\left(1+\frac{c_2b_1}{c_1}\right)}$$

$$=\frac{b_1b_2c_2+c_1b_2}{b_1b_2c_2+c_1b_2}=1,$$

$i=0,1,...,(2k-1)$. *So, we get immediately* $\lim_{n\to\infty}x_n=\lim_{n\to\infty}x_{n-6k}$ *and* $\lim_{n\to\infty}y_n=\lim_{n\to\infty}y_{n-6k}$ *and then the proof of is finished.*

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A Numerical Approach Based on Subdivision Schemes for Solving Non-Linear Fourth Order Boundary Value Problems

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Abstract

This paper presents an iterative collocation numerical approach based on interpolating subdivision schemes for the solution of non-linear fourth order boundary value problems involving ordinary differential equations. Numerical evidence suggests that the scheme converges to a smooth approximate solution of non-linear fourth order boundary value problem. The convergence of the approach is also discussed. Main purpose of this article is to explore and seek the applications of subdivision schemes in the field of physics and engineering.

Keywords: Boundary value problem, Subdivision schemes, Collocation algorithm, Approximation, interpolation

AMS Classification: 30E25; 65D07; 97N50

1 Introduction

Boundary value problems arise in several branches of physics and engineering. In recent years, there has been significant progress in solving problems associated with system of linear and nonlinear partial and ordinary differential equations involving boundary conditions. Two point nonlinear boundary value problems often cannot be solved by analytical methods. With increasing interest in finding solutions to nonlinear boundary value problems has come an increasing need for solution techniques.

In this paper, we consider the following type of nonlinear boundary value problem

$$y^{(iv)} = f(x, y, y') \tag{1.1}$$

with the boundary conditions

$$\begin{cases} y(a) = \alpha_1, & y'(a) = \alpha_2, \\ y(b) = \alpha_3, & y'(b) = \alpha_4, \end{cases} \tag{1.2}$$

where $\alpha_i, i = 1, 2, 3, 4$ are constants. We assume that the problem is well-posed.

A variety of methods have been introduced to solve these problem e.g., shooting methods, splines methods [2, 3, 4, 5, 6], finite difference methods, finite element methods, the collocation methods and other approximation methods. For discrete methods, like shooting and finite differences methods, only discrete approximate values of the unknown $y(x)$ can be obtained. For fitting curve to data we need further data processing techniques. For the case of spline interpolation or approximation methods the unknown function $y(x)$ is assumed to be piecewise polynomial which requires at least piecewise higher order differentiability of the function $f(x, y, y')$. To overcome these disadvantages, Qu and Agarwal

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[8, 9, 10] introduced the subdivision based algorithm for the solution of two point second order boundary value problems. Mustafa and Ejaz [14] solved third order boundary value problem by using subdivision technique. Higher order nonlinear problems have not been solved by subdivision techniques. This motivates us to solve fourth order boundary value problems by subdivision schemes based collocation iterative method. This paper introduces a numerical method based on subdivision technique for the solution of fourth order nonlinear boundary value problem.

In Section 2, some results about subdivision algorithms and basis function are given. In Section 3, a numerical method to solve (1.1) using refinable basis functions is formulated and its convergence properties studied. Error properties are given in Section 4. Numerical examples illustrating the feasibility of our proposed algorithm are given in Section 5.

2 Basis Functions and their Derivatives

Some useful results for the solution of non-linear boundary value problem are discussed in this section. Introduction to the basis functions of subdivision schemes that are used to construct the approximate solutions of proposed problem (1.1) is also part of this section.

2.1 Interpolating subdivision scheme

A mathematical formulation of binary subdivision scheme is defined as:

$$\begin{cases} P_{2i}^{k+1} = \sum\limits_{j\in\mathbb{Z}} a_{-2j} P_{i+j}^{k} \\ \\ P_{2i+1}^{k+1} = \sum\limits_{j\in\mathbb{Z}} a_{i-2j} P_{i+j}^{k} \end{cases} \tag{2.1}$$

The scheme is a stepwise interpolatory scheme if and only if the coefficient a_i satisfy $a_{2i} = \delta_i \quad \forall\ i \in \mathbb{Z}$. We consider the following binary interpolating subdivision scheme [7, 12, 13].

$$\begin{cases} p_{2i}^{k+1} = p_i^k, \\ \\ p_{2i+1}^{k+1} = \frac{35}{65536}\left(p_{i-4}^k + p_{i+5}^k\right) - \frac{405}{65536}\left(p_{i-3}^k + p_{i+4}^k\right) + \frac{567}{16384}\left(p_{i-2}^k + p_{i+3}^k\right) \\ \\ \qquad\qquad -\frac{2205}{16384}\left(p_{i-3}^k + p_{i+4}^k\right) + \frac{19845}{32768}\left(p_i^k + p_{i+1}^k\right). \end{cases} \tag{2.2}$$

The scheme (2.2) is C^4-continuous, having support length $(-9, 9)$ and approximation order is ten.

2.2 Basis functions

The basis functions is the limit function resulting from cardinal data, where all vertices of the polygon have value zero except for one. Let $\phi(x)$, $x \in \mathbb{R}$ be the fundamental solution of (2.2) and satisfies the two scale equations

$$\begin{aligned} \phi(x) = \phi(2x) + \frac{1}{65536}&\left[39690\{\phi(2x-1) + \phi(2x+1)\} - 8820\{\phi(2x-3)\right. \\ &+\phi(2x+3)\} + 2268\{\phi(2x-5) + \phi(2x+5)\} - 405\{\phi(2x-7) \\ &\left.+\phi(2x+7)\} + 35\{\phi(2x-9) + \phi(2x+9)\}\right], \quad x \in \mathbb{R} \end{aligned} \tag{2.3}$$

and

$$\phi(x) \in C^4, \quad \phi(x) = 0, \quad x \in]-8, 8[, \quad \phi(i) = \delta_0, \quad i \in \mathbb{Z}. \tag{2.4}$$

Furthermore, first derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(i)}(0) \quad = 0, & \phi^{(i)}(\pm 1) = \mp\frac{1914621952}{1159104017}, & \phi^{(i)}(\pm 2) = \pm\frac{530452796}{1159104017}, \\ \\ \phi^{(i)}(\pm 3) = \mp\frac{1470464}{13780629}, & \phi^{(i)}(\pm 4) = \pm\frac{17297069}{1159104017}, & \phi^{(i)}(\pm 5) = \mp\frac{2772992}{5795520085}, \\ \\ \phi^{(i)}(\pm 6) = \mp\frac{1127636}{10431936153}, & \phi^{(i)}(\pm 7) = \mp\frac{4096}{8113728119}, & \phi^{(i)}(\pm 8) = \mp\frac{5}{9272832136}. \end{cases} \tag{2.5}$$

Second derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(ii)}(0) = -\frac{2370618501415}{309077185968}, & \phi^{(ii)}(\pm 1) = \frac{3265310153216}{676106344305}, & \phi^{(ii)}(\pm 2) = -\frac{878265102572}{676106344305}, \\ \phi^{(ii)}(\pm 3) = \frac{734063059456}{2028319032915}, & \phi^{(ii)}(\pm 4) = -\frac{80883901277}{1352212688610}, & \phi^{(ii)}(\pm 5) = \frac{214899200}{135221268861}, \\ \phi^{(ii)}(\pm 6) = \frac{297875188}{405663806583}, & \phi^{(ii)}(\pm 7) = \frac{64000}{19317324123}, & \phi^{(ii)}(\pm 8) = \frac{4375}{618154371936}. \end{cases} \tag{2.6}$$

Third derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(iii)}(0) = 0, & \phi^{(iii)}(\pm 1) = \pm\frac{43317515008}{15295995855}, & \phi^{(iii)}(\pm 2) = \mp\frac{121530512357}{61183983420}, \\ \phi^{(iii)}(\pm 3) = \pm\frac{240606976}{566518365}, & \phi^{(iii)}(\pm 4) = \mp\frac{5285889107}{244735933680}, & \phi^{(iii)}(\pm 5) = \mp\frac{37414144}{3059199171}, \\ \phi^{(iii)}(\pm 6) = \pm\frac{1090169}{453214692}, & \phi^{(iii)}(\pm 7) = \mp\frac{21760}{437028453}, & \phi^{(iii)}(\pm 8) = \mp\frac{2975}{13984910496}. \end{cases} \tag{2.7}$$

Fourth derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(iv)}(0) = \frac{33869667}{457408}, & \phi^{(iv)}(\pm 1) = -\frac{5295054752}{89730585}, & \phi^{(iv)}(\pm 2) = \frac{10404741119}{358922340}, \\ \phi^{(iv)}(\pm 3) = -\frac{74879584}{9970065}, & \phi^{(iv)}(\pm 4) = \frac{295020869}{2871378720}, & \phi^{(iv)}(\pm 5) = \frac{9238624}{17946117}, \\ \phi^{(iv)}(\pm 6) = -\frac{900187}{7976052}, & \phi^{(iv)}(\pm 7) = \frac{71840}{17946117}, & \phi^{(iv)}(\pm 8) = \frac{11225}{328157568}. \end{cases} \tag{2.8}$$

The above derivative values are found by using the left eigenvectors of the subdivision process (2.2). The detailed description about these left eigenvectors and derivatives can be found in [8, 11, 14]. The graphical representations of above mentioned derivatives are shown in Figure 1.

3 Description of Iterative Numerical Method

This section describes the method for the numerical solution of nonlinear boundary value problem (1.1). The detail of the method is given below:

3.1 The collocation method

In this subsection, the collocation method is constructed based on the interpolating subdivision scheme (2.2). Our numerical approach for nonlinear fourth order boundary value problem using collocation method based on subdivision scheme is to seek an approximate solution as

$$Z(x) = \sum_{i=-8}^{N+8} z_i \phi\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1 \tag{3.1}$$

where N is the positive integer $N \geqslant 8$, $h = 1/N$ and $x_i = i/N = ih$ and $\{z_i\}$ are the unknowns to be determined for the solution of (1.1). In order to solve the problem, a collocation method $Z(x)$ is considered to be the solution of the above differential equation at $x = x_j$ and we substitute equation (3.1) into equation (1.1). This leads to

$$Z^{(iv)}(x_j) = f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N \tag{3.2}$$

and boundary conditions

$$Z(0) = \alpha_1,\ Z^{'}(0) = \alpha_2,\ Z(N) = \alpha_3,\ Z^{'}(N) = \alpha_4 \tag{3.3}$$

From (3.1), we get

$$Z^{(iv)}(x) = \frac{1}{h^4} \sum_{i=-8}^{N+8} z_i \phi^{(iv)}\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1 \tag{3.4}$$

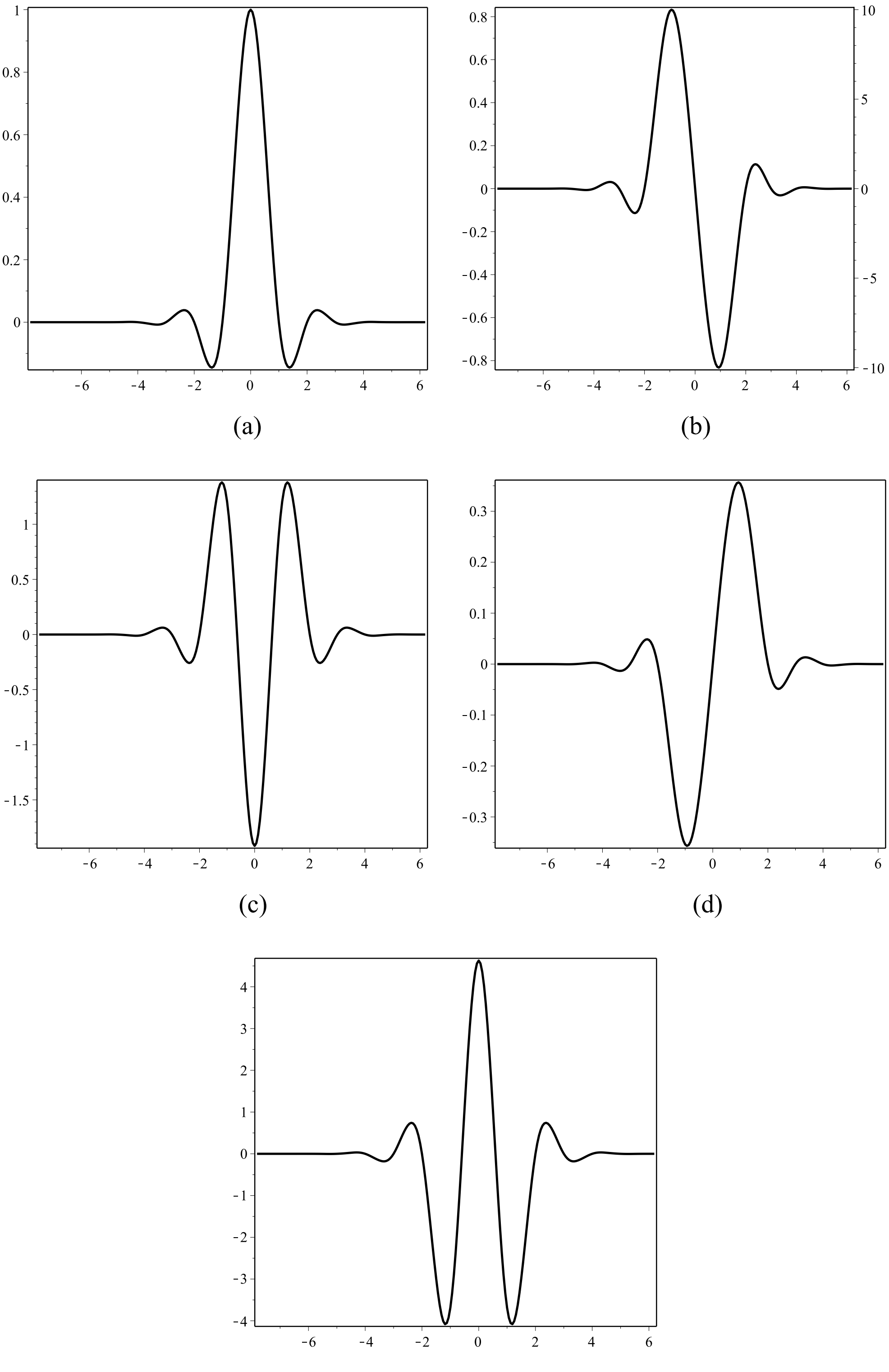

Figure 1: Graphical representation of: (a) basis functions, (b) first, (c) second, (d) third, (e) fourth derivatives of basis function

Substituting (3.4) into (3.2), we obtain

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)}\left(\frac{x_j - x_i}{h}\right) = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N.$$

This can be written as:

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)}_{j-i} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N$$

Since $\phi^{(iv)}_i = \phi^{(iv)}_{-i}$, the above system of equations becomes

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)}_{i-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N. \tag{3.5}$$

The nonlinear system of equations (3.5) can be simply in following Theorems 1 and 2.

Theorem 1. *The nonlinear system of equations (3.5) for $j = 0$ becomes*

$$\sum_{i=-8}^{8} z_i \phi^{(iv)}_i = h^4 f(x_0, Z(x_0), Z^{'}(x_0)). \tag{3.6}$$

Proof. By expanding (3.5) for $j = 0$, we obtain

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)}_i = h^4 f(x_0, Z(x_0), Z^{'}(x_0))$$

$$\begin{aligned} &z_{-8}\phi^{iv}_{-8} + z_{-7}\phi^{iv}_{-7} + z_{-6}\phi^{iv}_{-6} + \cdots + z_7\phi^{iv}_7 + z_8\phi^{iv}_8 + z_9\phi^{iv}_9 + \cdots + z_{N+7}\phi^{iv}_{N+7} \\ &+ z_{N+8}\phi^{iv}_{N+8} = h^4 f(x_0, Z(x_0), Z^{'}(x_0)). \end{aligned}$$

Since $\phi^{iv}(i)$ exists only for the interval for $i \in [-8, 8]$ and outside the interval it will be zero. Then above equation can be written as

$$z_{-8}\phi^{iv}_{-8} + z_{-7}\phi^{iv}_{-7} + z_{-6}\phi^{iv}_{-6} + \cdots + z_7\phi^{iv}_7 + z_8\phi^{iv}_8 = h^4 f(x_0, Z(x_0), Z^{'}(x_0)).$$

□

Theorem 2. *For $j = 1, 2, \cdots, N$, the nonlinear system of equations (3.5) becomes*

$$\sum_{i=-8+j}^{8+j} z_i \phi^{(iv)}_{i-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \tag{3.7}$$

Proof. By expanding (3.5), for $j = 1, 2, 3, \cdots, N$, we get

$$\begin{aligned} &z_{-8}\phi^{iv}_{-8-j} + z_{-7}\phi^{iv}_{-7-j} + z_{-6}\phi^{iv}_{-6-j} + \cdots + z_7\phi^{iv}_{7-j} + z_8\phi^{iv}_{8-j} + z_9\phi^{iv}_{9-j} + z_{10}\phi^{iv}_{10-j} \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+6-j} + z_{N+7}\phi^{iv}_{N+7-j} + z_{N+8}\phi^{iv}_{N+8-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \end{aligned} \tag{3.8}$$

Substituting $j = 1$ in (3.8), it becomes

$$\begin{aligned} &z_{-8}\phi^{iv}_{-8-1} + z_{-7}\phi^{iv}_{-7-1} + z_{-6}\phi^{iv}_{-6-1} + \cdots + z_7\phi^{iv}_{7-1} + z_8\phi^{iv}_{8-1} + z_9\phi^{iv}_{9-1} + z_{10}\phi^{iv}_{10-1} \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+6-1} + z_{N+7}\phi^{iv}_{N+7-1} + z_{N+8}\phi^{iv}_{N+8-1} = h^4 f(x_1, Z(x_1), Z^{'}(x_1)). \end{aligned}$$

This implies

$$\begin{aligned} &z_{-8}\phi^{iv}_{-9} + z_{-7}\phi^{iv}_{-8} + z_{-6}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_6 + z_8\phi^{iv}_7 + z_9\phi^{iv}_8 + z_{10}\phi^{iv}_9 + \cdots \\ &+ z_{N+6}\phi^{iv}_{N+5} + z_{N+7}\phi^{iv}_{N+6} + z_{N+8}\phi^{iv}_{N+7} = h^4 f(x_1, Z(x_1), Z^{'}(x_1)). \end{aligned}$$

Since $\phi^{iv}(i)$ is non-zero only for the interval for $i \in [-8, 8]$ and outside the interval it will be zero. Then above equation becomes

$$z_{-7}\phi^{iv}_{-8} + z_{-6}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_6 + z_8\phi^{iv}_7 + z_9\phi^{iv}_8 = h^4 f(x_1, Z(x_1), Z'(x_1)). \tag{3.9}$$

For j=2, (3.8) becomes

$$\begin{aligned} &z_{-8}\phi^{iv}_{-8-2} + z_{-7}\phi^{iv}_{-7-2} + z_{-6}\phi^{iv}_{-6-2} + \cdots + z_7\phi^{iv}_{7-2} + z_8\phi^{iv}_{8-2} + z_9\phi^{iv}_{9-2} + z_{10}\phi^{iv}_{10-2} \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+6-2} + z_{N+7}\phi^{iv}_{N+7-2} + z_{N+8}\phi^{iv}_{N+8-2} = h^4 f(x_2, Z(x_2), Z'(x_2)). \end{aligned}$$

This implies

$$\begin{aligned} &z_{-8}\phi^{iv}_{-10} + z_{-7}\phi^{iv}_{-9} + z_{-6}\phi^{iv}_{-8} + \cdots + z_7\phi^{iv}_5 + z_8\phi^{iv}_6 + z_9\phi^{iv}_7 + z_{10}\phi^{iv}_8 \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+4} + z_{N+7}\phi^{iv}_{N+5} + z_{N+8}\phi^{iv}_{N+6} = h^4 f(x_2, Z(x_2), Z'(x_2)). \end{aligned}$$

By using the definition of ϕ^{iv}_i given in (2.8), above equation yields

$$z_{-6}\phi^{iv}_{-8} + z_{-5}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_5 + z_8\phi^{iv}_6 + z_9\phi^{iv}_7 + z_{10}\phi^{iv}_8 = h^4 f(x_2, Z(x_2), Z'(x_2)). \tag{3.10}$$

By using the similar pattern for $j = 1, 2$, we can find the expression for $j = 3, 4, \cdots N$

$$\begin{aligned} &z_{-8+j}\phi^{iv}_{-8-j} + z_{-7+j}\phi^{iv}_{-7-j} + z_{-6+j}\phi^{iv}_{-6-j} + \cdots + z_{7+j}\phi^{iv}_{7-j} + z_{8+j}\phi^{iv}_{8-j} + z_{9+j}\phi^{iv}_{9-j} \\ &+ \cdots + z_{N+6+j}\phi^{iv}_{N+6-j} + z_{N+7+j}\phi^{iv}_{N+7-j} + z_{N+8+j}\phi^{iv}_{N+8-j} = h^4 f(x_j, Z(x_j), Z'(x_j)). \end{aligned} \tag{3.11}$$

□

The nonlinear system of equations (3.5) is equivalent to the following non-linear system of $N + 1$ equations with (N+17) unknowns $\{z_i\}$.

$$AZ = F(z) \tag{3.12}$$

where A is banded matrix of order $(N+1) \times (N+17)$, Z is the unknown vector of order $N+17$ and $F(z)$ is the vector of order $N + 1$ depends on z. The matrix A, vectors Z and $F(z)$ are given explicitly by

$$A = [\phi^{iv}_{pq}(q - p - 8)]_{(N+1)\times(N+17)} \tag{3.13}$$

where $p = 1, 2, 3 \cdots, N + 1$ and $q = 1, 2, 3, \cdots, N + 17$ represent the row and column respectively.

$$F(z) = \Big(h^4 f(x_0, Z(x_0), Z'(x_0)), \cdots, h^4 f(x_N, Z(x_N), Z'(x_N))\Big)^T \tag{3.14}$$

$$Z = (z_{-8}, z_{-7}, z_{-6}, \cdots, z_{N+6}, z_{N+7}, z_{N+8})^T \tag{3.15}$$

$$Z'(x_j) = \sum_{i=-8}^{N+8} z_j \phi'\left(\frac{x_j - x_i}{h}\right)$$

where $\phi'(i)$ is already defined in (2.5) with $\phi(i) = \phi_i$.

3.2 Boundary conditions at end points

For unique solution of the nonlinear system (3.5), we need sixteen more conditions. Four conditions can be attained from given boundary conditions for the nonlinear system of equations and remaining conditions are attained by using some extrapolation method. The details of the given boundary conditions and extrapolation method are given below:

3.2.1 Boundary Conditions

The given boundary conditions are

$$Z(0) = \alpha_1,\ Z^{'}(0) = \alpha_2,\ Z(N) = \alpha_3,\ Z^{'}(N) = \alpha_4$$

The approximation of derivative conditions at ends point is defined as:

$$\begin{aligned} Z'(0) = \left(\frac{N}{2520}\right)\{&-7381z_0 + 25200z_1 - 56700z_2 + 100800z_3 - 132300z_4 + 127008z_5 \\ &-88200z_6 + 43200z_7 - 14175z_8 + 28800z_9 - 252z_{10}\} + O(h^{10}) \end{aligned} \tag{3.16}$$

$$\begin{aligned} Z'(N) = \left(\frac{N}{2520}\right)\{&7381z_N - 25200z_{N-1} + 56700z_{N-2} - 100800z_{N-3} + 132300z_{N-4} \\ &-127008z_{N-5} + 88200z_{N-6} - 43200z_{N-7} + 14175z_{N-8} - 28800z_{N-9} \\ &+252z_{N-10}\} + O(h^{10}). \end{aligned} \tag{3.17}$$

3.2.2 Extrapolation Method

The remaining twelve conditions for the nonlinear systems (3.5) to obtain stable systems for the solution of (1.1) are obtained by using the following extrapolation method.
We define six conditions at left end points and six conditions at the right end points. Since subdivision scheme (2.2) reproduces nine degree (i.e. tenth order) polynomials, so we define boundary conditions of order ten for the solution of (3.5). For simplicity only left end points $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ are discussed and the values of right end points $z_{N+2}, z_{N+3}, z_{N+4}, z_{N+5}, z_{N+6}, z_{N+7}$ can be treated similarly.

The values $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ can be determined by the polynomial $q(x)$ interpolating $(x_i, z_i),\ 2 \leq i \leq 7$. Precisely, we have

$$z_{-i} = q(-x_i),\ i = 2, 3, 4, 5, 6, 7$$

where

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} Z(x_{i-j}).$$

From (3.1), $Z_1(x_i) = z_i$, $i = 2, 3, 4, 5, 6, 7$ and replacing x_i by $-x_i$, we have

$$q(-x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{-i+j}.$$

Hence the following boundary conditions can be employed at the left end

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^{j} z_{-i+j} = 0,\ i = 7, 6, 5, 4, 3, 2. \tag{3.18}$$

Similarly for the right end, we can define $z_i = q(-x_i)$, $i = N+2,\ N+3,\ N+4,\ N+5,\ N+6,\ N+7$ and

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{i-j}.$$

So we have the following boundary conditions at the right end

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^{j} z_{i-j} = 0,\ i = N+2, N+3, N+4, N+5, N+6, N+7. \tag{3.19}$$

Finally, we obtain a new system of $(N + 17)$ linear equations with $(N + 17)$ unknowns $\{z_i\}$. The $N+1$ equations are obtained from (3.5), four equations from boundary conditions (3.3) and twelve from boundary conditions (3.18) and (3.19) for the numerical solution of proposed problem.
Hence the stable nonlinear system of equations is defined as:

$$BZ = R(z) \tag{3.20}$$

where the matrix B is given by

$$B = (C_0^T, A^T, C_1^T)^T \tag{3.21}$$

A is defined in (3.13), C_0, C_1 and the vector R(z) is defined as

$$C_0 = \left(\begin{matrix}
0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 \\
0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 \\
0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 \\
0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 \\
0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7381N}{2520} & \frac{25200N}{2520} & -\frac{56700N}{2520} & \frac{100800N}{2520} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{matrix}\right.$$

$$\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-10 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
45 & -10 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-120 & 45 & -10 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
210 & -120 & 45 & -10 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{132300N}{2520} & \frac{127008N}{2520} & -\frac{88200N}{2520} & \frac{43200N}{2520} & -\frac{14175N}{2520} & \frac{2800N}{2520} & -\frac{252N}{2520} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{matrix}\right) \tag{3.22}$$

The first six rows of C_0 are obtained from (3.18), second last row is obtained from (3.16) and last row is taken from given boundary conditions $Z_1(0)$ which is defined in (3.3) and

$$C_1 = \left(\begin{matrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & \frac{N}{10} & -\frac{10N}{9} & \frac{45N}{8} & -\frac{120N}{7} & 35N & -\frac{252N}{5} & \frac{105N}{2} & -40N & \frac{45N}{2} & -10N \\
0 & 0 & \cdots & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45
\end{matrix}\right.$$

$$\left.\begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7381N}{2520} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
45 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-120 & 45 & -10 & 1 & 0 & 0 & 0 & 0 & 0 \\
210 & -120 & 45 & -10 & 1 & 0 & 0 & 0 & 0 \\
-252 & 210 & -120 & 45 & -10 & 1 & 0 & 0 & 0 \\
-120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 & 0
\end{matrix}\right), \tag{3.23}$$

First row of C_1 is obtained from $Z_1(N)$ which is defined in (3.3), second row is obtained from (3.17) and the last six rows are obtained from (3.19), Z which is defined in (3.15) and R_1 is defined as

$$R(z) = (0, 0, 0, 0, 0, 0, Z'(0), Z(1), F^T(z), Z(1), Z'(1), 0, 0, 0, 0, 0, 0)^T, \tag{3.24}$$

where $F(z)$ is defined by (3.14).

3.2.3 Non-singularity of a matrix

We can check the non-singularity of coefficient matrix B defined in (3.21) by different methods. We observe that the determinant of matrix B is non-zero for $N \leq 500$. Hence the non linear system of equations have a solution for $N \leq 500$. We also check the non singularity of matrix by finding eigenvalues up to $N \leq 500$ and we observe that all the eigenvalues are non-zero. Hence by [15] we conclude that the B is non-singular. For large $N > 500$ the matrix may or may not be singular.

3.3 Iterative algorithm and its convergence

An iterative algorithm and its convergence are described in this section.

3.3.1 Iterative algorithm based on basis function

The iterative algorithm based on basis function of the subdivision scheme (2.2) are as defined in the following three steps.

First step: Initial approximation

The initial approximation is important because the numerical solution depends on the initial approximation. We define the process for finding the initial approximation as follows:
Let initial approximate solution Z^0 be the solution of the following linear system

$$BZ^0 = F^0 \tag{3.25}$$

where

$$\begin{cases} F^0 = (0,0,0,0,0,0,y'(a),y(a),f_0,f_1,f_2,\cdots,f_N,y(b),y'(b),0,0,0,0,0,0)^T, \\ f_i = h^4 f(x_i,L_i,D), \quad i = 0,1,2,\cdots N \\ L_i = y(0) + ih\left(\frac{y(b)-y(a)}{b-a}\right) \\ D = y(b) - y(a). \end{cases} \tag{3.26}$$

F^0 is the initial linear approximation of the non-linear vector $R(z)$.

Second step: Numerical solution

The numerical solutions Z^* of the nonlinear system are obtained by using the simple iterative scheme

$$BZ^{(m+1)} = R(Z^m), \quad m = 0,1,2,3,\cdots \tag{3.27}$$

Third step: Stopping condition

The above iterative processes will terminate when the following condition is satisfied

$$||z^{(m)} - z^{(m-1)}|| \leq tol \tag{3.28}$$

where tolerance is supposed value i.e. $tol = 10^{-6}$. The convergence of the above iterative algorithm is guaranteed by the following proposition.

Theorem 3. *The successive solutions $\{Z^{(m)}\}$ generated by the iterative algorithm (3.27) linearly converges to the solution Z^* of the non-linear solution of the system (3.20) provided that the M_0 and M_1 are Lipschitz constants and step size h is small.*
i.e.

$$\|B^{-1}\| \leq \left(M_0 h^4 + \frac{4994220330463}{1460471061420} M_1 h^3\right). \tag{3.29}$$

Proof. Let Z^* and $Z^{(m)}$ be the solutions of the nonlinear system (3.20). Then by definition, for small h we have

$$BZ^* = R(Z^*), \tag{3.30}$$
$$BZ^{m+1} = R(Z^m). \tag{3.31}$$

Let the error vector be defined as $e^{(k)} = Z^k - Z^*$ at kth iteration which satisfies

$$\begin{aligned} BZ^{(m+1)} - BZ^* &= R(Z^k) - R(Z^*), \\ B(Z^{(m+1)} - Z^*) &= R(Z^k) - R(Z^*), \\ Be^{(k+1)} &= R(Z^k) - R(Z^*). \end{aligned} \tag{3.32}$$

For $i = 0, 1, 2, \cdots, N$

$$D_4 e_i^{(k+1)} = (F(Z^k) - F(Z^*))_i.$$

By mean value theorem, which is stated as "If a function $f(x, y, z)$ is continuously differentiable in an open set of $\mathbb{R}^3$ containing points (x_1, y_1, z_1) and (x_2, y_2, z_2) and the line segment connecting them, then an equation

$$f(x_2, y_2, z_2) - f(x_1, y_1, z_1) = f_x^{'}(r, s, t)(x_2 - x_1) + f_y^{'}(r, s, t)(y_2 - y_1) + f_z^{'}(r, s, t)(z_2 - z_1)$$

is valid for the interior point (a, b, c) of the segment.", we have

$$D_4 e_i^{(k+1)} = f(x_i, Z_i^{(k)}, Z'^{(k)}) - f(x_i, Z_i^{(*)}, Z'^{(*)}).$$

The above equation can be written as (by using mean value theorem)

$$D_4 e_i^{(k+1)} = f_x^*(x_i - x_i) + f_y^*(Z_i^{(k)} - Z_i^{(*)}) + f_{y'}^*(Z'^{(k)} - Z'^{(*)})$$

by using the definition of error vector, we have

$$\begin{aligned} D_4 e_i^{(k+1)} &= f_y^* e^{(k)} + f_{y'}^* e'^{(k)}, \\ D_4 e_i^{(k+1)} &= f_y^* e^{(k)} + f_{y'}^* D_1 e^{(k)} \end{aligned}$$

where D_4 and D_1 are the derivative difference operators defined as

$$\begin{aligned} D_1 f_i = \frac{1}{2920942122840h} &[1575(f_{i-8} - f_{i+8}) + 1474560(f_{i-7} - f_{i+7}) \\ &+315738080(f_{i-6} - f_{i+6}) + 1397587968(f_{i-5} - f_{i+5}) \\ &-43588613880(f_{i-4} - f_{i+4}) + 311679549440(f_{i-3} - f_{i+3}) \\ &-1336741045920(f_{i-2} - f_{i+2}) + 4824847319040(f_{i-1} - f_{i+1})] \end{aligned}$$

$$\begin{aligned} D_4 f_i = \frac{1}{183768238080h^4} &[392875(f_{i+8} - f_{i-8}) + 45977600(f_{i+7} - f_{i-7}) \\ &-1296269280(f_{i+6} - f_{i-6}) + 5912719360(f_{i+5} - f_{i-5}) \\ &+1180083476(f_{i+4} - f_{i-4}) - 86261280768(f_{i+3} - f_{i-3}) \\ &+332951715808(f_{i+2} - f_{i-2}) - 677767008256(f_{i+1} - f_{i-1}) \\ &+850467338370 f_i]. \end{aligned}$$

This implies

$$D_4 e_i^{(k+1)} = h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}.$$

Since $e_i = e_{N-i} = 0, \quad i = 0, -1, -2, \cdots, -8$, we have

$$Be_i^{(k+1)} = h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}$$

This can be written as

$$e_i^{(k+1)} = B^{-1}(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}).$$

By taking norm on both sides, we get

$$\|e_i^{(k+1)}\| = \|B^{-1}(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)})\|.$$

This implies

$$\|e_i^{(k+1)}\| = \|B^{-1}\| \|(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)})\|.$$

By using the definition of Lipschitz condition, we get

$$\|e^{(k+1)}\| \leq h^4 M_0 (b-a) \|B^{-1}\| \|e^k\| + h^3 M_1 \|D_1\| \|e^{(k)}\|.$$

This implies

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \leq \|B^{-1}\| \left(h^4 M_0 (b-a) + h^3 M_1 \|D_1\| \right),$$

which is equivalent to

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \approx h^3 M_1 \|B^{-1}\| \|D_1\| \leq h M_1 \|B^{-1}\| \|D_1\|,$$

i-e

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \approx h M_1 \|B^{-1}\| \|D_1\|.$$

The results follows immediately from this inequality and the following fact

$$\|D_1\| = \frac{4994220330463}{1460471061420}. \tag{3.33}$$

A simple approximation of condition by omitting the quatric term is

$$h \leq \frac{1460471061420}{4994220330463} M_1^{-1} \left\|B^{-1}\right\|^{-1}. \tag{3.34}$$

This complete the proof. $\square$

4 Error Estimation

From the approximation properties of the basis function $\phi(x)$, it is shown that the collocation method (3.1) with nonic precision treatments at the end points has at least power of approximation $O(h^3)$. Here we present our main results for error estimation. Proof of these results are similar to the proof of Proposition [14, 8].

Theorem 4. *Suppose the exact solution $y(x) \in C^4[0,1]$ and $\{z_i\}$ are obtained by (3.20) then absolute error by interpolating collocation algorithm is*

$$||err(x)||_\infty = ||Z^{(l)}(x) - y^{(l)}(x)||_\infty = O(h^{3-l}), \; l = 0, 1, 2, 3.$$

where l denotes the order of derivative.

Proof. Since the order of approximation of subdivision scheme (2.2) is ten so by direct calculation (fourth left eigenvector), we can find derivative of smooth function $y(x)$ as

$$y^{iv}(x_j) = \frac{2^4}{183768238080h^4}\{392875y(x_j-8h) + 45977600y(x_j-7h)$$

$$-1296269280y(x_j-6h) + 5912719360y(x_j-5h) + 1180083476y(x_j-4h)$$

$$-86261280786y(x_j-3h) + 332951715808y(x_j-2h) - 677767008256y(x_j-h)$$

$$+850467338370y(x_j) - 677767008256y(x_j+h) + 332951715808y(x_j+2h)$$

$$-86261280786y(x_j+3h) + 1180083476y(x_j+4h) + 5912719360y(x_j+5h)$$

$$-1296269280y(x_j+6h) + 45977600y(x_j+7h) + 392875y(x_j+8h)\} + O(h^{10}).$$

This can be written as

$$y_j^{iv} = \frac{2^4}{183768238080h^4}\{392875y_{j-8} + 45977600y_{j-7} - 1296269280y_{j-6}$$

$$+5912719360y_{j-5} + 1180083476y_{j-4} - 86261280786y_{j-3} + 332951715808y_{j-2}$$

$$-677767008256y_{j-1} + 850467338370y_j - 677767008256y_{j+1} + 332951715808y_{j+2}$$

$$-86261280786y_{j+3} + 1180083476y_{j+4} + 5912719360y_{j+5} - 1296269280y_{j+6}$$

$$+45977600y_{j+7} + 392875y_{j+8}\} + O(h^{10}). \tag{4.1}$$

Similarly, we have

$$Z_j^{iv} = \frac{2^4}{183768238080h^4}\{392875z_{j-8} + 45977600z_{j-7} - 1296269280z_{j-6}$$

$$+5912719360z_{j-5} + 1180083476z_{j-4} - 86261280786z_{j-3} + 332951715808z_{j-2}$$

$$-677767008256z_{j-1} + 850467338370z_j - 677767008256z_{j+1} + 332951715808z_{j+2}$$

$$-86261280786z_{j+3} + 1180083476z_{j+4} + 5912719360z_{j+5} - 1296269280z_{j+6}$$

$$+45977600z_{j+7} + 392875z_{j+8}\} + O(h^{10}). \tag{4.2}$$

If we define error function $e(x) = Z(x) - y(x)$ and error vectors at the nodes by

$$e(x_j) = Z(x_j) - y(x_j + jh),\ -8 \leq j \leq N+8,$$

or equivalently $e_j = Z_j - y_j,\ -8 \leq j \leq N+8$, This implies

$$\begin{cases} e_j' = Z_j' - y_j', \\ e_j'' = Z_j'' - y_j'', \\ e_j''' = Z_j''' - y_j''' \\ e_j^{iv} = Z_j^{iv} - y_j^{iv}. \end{cases} \tag{4.3}$$

By subtracting (4.2) from (4.1), we get

$$y_j^{iv} - Z_j^{iv} = \frac{2^4}{183768238080h^4}\{392875(y_{j-8} - z_{j-8}) + 45977600(y_{j-7} - z_{j-7})$$

$$-1296269280(y_{j-6} - z_{j-6}) + 5912719360(y_{j-5} - z_{j-5}) + 1180083476(y_{j-4} - z_{j-4})$$

$$-86261280786(y_{j-3} - z_{j-3}) + 332951715808(y_{j-2} - z_{j-2}) - 677767008256(y_{j-1} - z_{j-1})$$

$$+850467338370(y_j - z_j) - 677767008256(y_{j+1} - z_{j+1}) + 332951715808(y_{j+2} - z_{j+2})$$

$$-86261280786(y_{j+3} - z_{j+3}) + 1180083476(y_{j+4} - z_{j+4}) + 5912719360(y_{j+5} - z_{j+5})$$

$$-1296269280(y_{j+6} - z_{j+6}) + 45977600(y_{j+7} - z_{j+7}) + 392875(y_{j+8} - z_{j+8})\} + O(h^{10}).$$

This implies

$$e_j^{iv} = \frac{2^4}{183768238080h^4}\{392875e_{j-8} + 45977600e_{j-7} - 1296269280e_{j-6}$$

$$+5912719360e_{j-5} + 1180083476e_{j-4} - 86261280786e_{j-3} + 332951715808e_{j-2}$$

$$-677767008256e_{j-1} + 850467338370e_j - 677767008256e_{j+1} + 332951715808e_{j+2}$$

$$-86261280786e_{j+3} + 1180083476e_{j+4} + 5912719360e_{j+5} - 1296269280e_{j+6}$$

$$+45977600e_{j+7} + 392875e_{j+8}\} + O(h^{10}). \tag{4.4}$$

From (1.1), (3.1), (4.3) and by assuming the tenth order boundary treatments at the end points, we have

$$e_j^{iv} = a_j e_j + b_j e_j^{'}, \quad 0 \le i \le N \tag{4.5}$$

and

$$e_j = \begin{cases} \max\limits_{0\le k\le 7}\{|e_k|\}O(h^{10}), & -8 \le i \le 0 \\ \max\limits_{N-3\le k\le N}\{|e_k|\}O(h^{10}), & N \le i \le N+8 \end{cases} \tag{4.6}$$

where $j = 0, 1, \cdots N$

$$a_j = f_y(t_j, y_j^*, y_j^{'*}), \quad b_j = f_{y'}(t_j, y_j^*, y_j^{'*}),$$

and

$$y_j^* = y_j + \theta_j e_j, \quad y_j^{'*} = y_j' + \theta_j e_j^{'}, \quad 0 \le \theta_j \le 1.$$

Using the results (4.4) and

$$\begin{aligned}&[1575(z_{i-8} - z_{i+8}) + 1474560(z_{i-7} - z_{i+7}) + 315738080(z_{i-6} - z_{i+6}) + 1397587968\\ &(z_{i-5} - z_{i+5}) - 43588613880(z_{i-4} - z_{i+4}) + 311679549440(z_{i-3} - z_{i+3}) - 1336741045920\\ &(z_{i-2} - z_{i+2}) + 4824847319040(z_{i-1} - z_{i+1})] = 2920942122840hZ^{'} + O(h^{10}),\end{aligned} \tag{4.7}$$

It can be conclude that relation (4.5)and (4.6) is equivalent to

$$(B + O(h^8) - O(h^4) - D_1 O(h^3))E = O(h^{10})\|E\|,$$

where $E = (e_{-8}, e_{-7}, \cdots, e_7, e_8)$.

Hence for small h, the coefficient matrix $B + O(h)$, will be invertible, thus using the standard result from algebra and effect of $\|B^{-1}\|$, we have the following estimate

$$\|E\| \le \frac{\|B^{-1}\|}{1 - O(h)} O(h^{10}) = O(h^3). \tag{4.8}$$

□

5 Results and Discussions

In this section, we test the proposed method on some nonlinear problems. Numerical results for each of the problems are presented in the tables. These values are very close to the true solutions and the values of the errors are also given in the table.

Example 1. *Consider the following non-linear boundary value problem [1]*

$$y^{iv} - 6\exp(-4y) = -12(1+x)^{-4}, \tag{5.1}$$

with boundary conditions

$$y(0) = 0, y'(0) = 1, y(1) = ln(2) = y'(1) = 0.5.$$

The exact solution of the problem (5.1) is $y = \ln(1+x)$. *Using the collocation method described in Section 3 for* $N = 10$, $h = 10^{-1}$ *and tol* $= 10^{-6}$ *with tenth order boundary treatment at end points. The numerical results are obtained after third iteration with the condition (3.28). The obtained numerical results for this problem are presented in Table 1. The maximum absolute error obtained by the proposed method is* 1.78×10^{-3}. *The graphical comparison between exact and approximate solutions is shown in Figure 2.*

Table 1: Numerical results of Example 1

x_i	Analytic solution Y_i	Approximate solution Z_i	Error $= \|\|Y_i - Z_i\|\|_\infty$
0.0	0	0	0
0.1	0.0953101798	0.0950147533	0.0002954265
0.2	0.1823215568	0.1814496227	0.0008719341
0.3	0.2623642645	0.2609546573	0.0014096072
0.4	0.3364722366	0.3347370220	0.0017352146
0.5	0.4054651081	0.4036840381	0.0017810699
0.6	0.4700036292	0.4684459279	0.0015577013
0.7	0.5306282511	0.5294932609	0.0011349902
0.8	0.5877866649	0.5871580370	0.0006286279
0.9	0.6418538862	0.6416636708	0.0001902154
1.0	0.6931471806	0.6931471806	0

Example 2. *Consider the non-linear boundary value problem [1]*

$$y^{(iv)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 \tag{5.2}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, y(1) = y'(1) = 1.$$

Using the collocation method described in Section 3 for $N = 10$, $h = 10^{-1}$ *and tol* $= 10^{-6}$ *with tenth order boundary treatment at end points. The numerical results are obtained after third iteration with the condition (3.28). The obtained numerical results for this problem are presented in Table 2. The maximum absolute error obtained by the proposed method is* 1.73×10^{-2}. *The graphical comparison between exact and approximate solutions is shown in Figure 3.*

6 Conclusion

This study has presented a numerical approach based on subdivision collocation algorithm for solving the numerical solution of nonlinear fourth order boundary value problems. The proposed iterative method

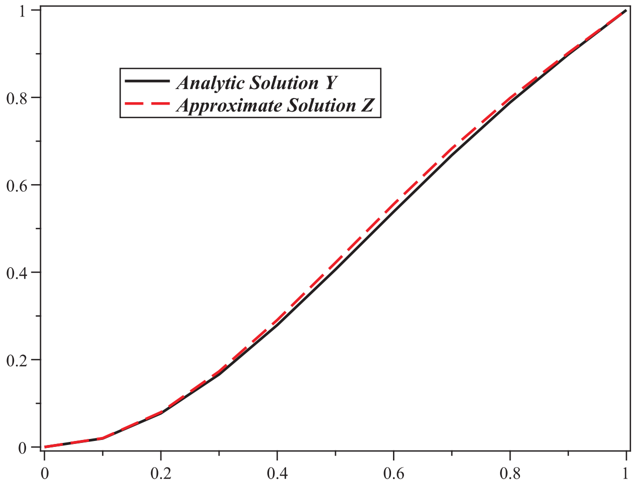

Figure 2: *Comparison of the analytic and approximate solution of Example 1.*

Table 2: Numerical results of Example 2

x_i	Analytic solution Y_i	Approximate solution Z_i	Error $= \|\|Y_i - Z_i\|\|_\infty$
0.0	0	0	0
0.1	0.01981	0.0202195	0.0004095
0.2	0.07712	0.0796952	0.0025752
0.3	0.16623	0.1728732	0.0066432
0.4	0.27904	0.2905995	0.0115595
0.5	0.40625	0.4219208	0.0156708
0.6	0.53856	0.5558846	0.0173246
0.7	0.66787	0.6833406	0.0154706
0.8	0.78848	0.7987412	0.0102612
0.9	0.89829	0.9019417	0.0036517
1.0	1.00000	1.0000000	0

has been applied on different nonlinear fourth order boundary value problems. Numerical results show that the accuracy of the approximate solution is $O(h^3)$. We have also observed that the accuracy of the solution can be improved by choosing different subdivision schemes with the proper adjustment of boundary conditions.

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Competing interests

The authors declare that they have no competing interests.

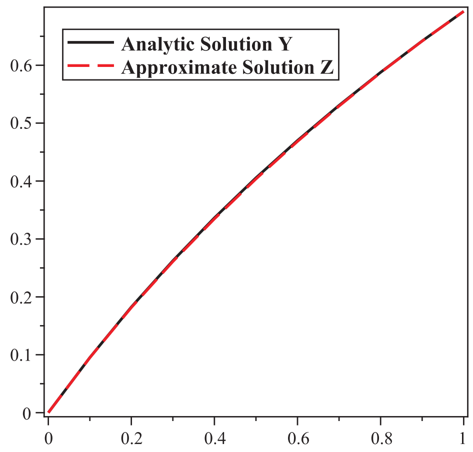

Figure 3: *Comparison of the analytic and approximate solution of Example 2.*

Authorś contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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On Stability of Quintic Functional Equations in Random Normed Spaces

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Abstract. In this paper, using the direct and fixed point methods, we investigate the generalized Hyers-Ulam stability of the quintic functional equation:

$$2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y)=20[f(x+y)+f(x-y)]+90f(x)$$

in random normed spaces under the minimum t-norm.

1. **Introduction**

A classical question in stability of functional equations is as follows:

Under what conditions, is it true that a mapping which approximately satisfies a functional equation (ξ) *must be somehow close to an exact solution of* (ξ)?

We say the functional equation (ξ) is *stable* if any approximate solution of (ξ) is near to a true solution of (ξ).

The study of stability problem for functional equations is related to a question of Ulam [15] concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [9] for linear functional equation of Banach spaces. Subsequently, the result of Hyers theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. Cădariu and Radu [3] applied the *fixed point method* to investigation of the Jensen functional equation. They could present a short and a simple proof (different from the *direct method* initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation and for quadratic functional equation. Their methods are a powerful tool for studying the stability of several functional equations.

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On the other hand, the theory of *random normed spaces* (briefly, RN-spaces) is important as a generalization of deterministic result of normed spaces and also in the study of random operator equations. The notion of an RN-space corresponds to the situations when we do not know exactly the norm of the point and we know only probabilities of passible values of this norm. The RN-spaces may provide us the appropriate tools to study the geometry of nuclear physics and have usefully application in quantum particle physics. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in RN-spaces [5, 6, 10, 11, 16].

In the sequel, we use the definitions and notations of a random normed space as in [1, 13, 14].

A function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ is called a *distribution function* if it is nondecreasing and left-continuous, with $F(0) = 0$ and $F(+\infty) = 1$. The class of all probability distribution functions F with $F(0) = 0$ is denoted by Λ. D^+ is a subset of Λ consisting of all functions $F \in \Lambda$ for which $F(+\infty) = 1$, where $l^- F(x) = \lim_{t \to x^-} F(t)$. For any $a \geq 0$, ϵ_a is the element of D^+, which is defined by

$$\epsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

Definition 1.1. ([13]) A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:
(1) T is commutative and associative;
(2) T is continuous;
(3) $T(a, 1) = a$ for all $a \in [0, 1]$;
(4) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Three typical examples of continuous t-norms are as follows:

$$T_M(a, b) = \min\{a, b\}, \quad T_P(a, b) = ab, \quad T_L(a, b) = \max\{a + b - 1, 0\}.$$

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence of numbers in $[0, 1]$, then $T^n_{i=1} x_i$ is defined recurrently by $T^1_{i=1} x_i = x_1$ and $T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n) = T(x_1, \cdots, x_n)$ for each $n \geq 2$ and $T^\infty_{i=n} x_n$ is defined as $T^\infty_{i=1} x_{n+i}$ ([8]).

Definition 1.2. ([14]) Let X be a real linear space, μ be a mapping from X into D^+ (for any $x \in X$, $\mu(x)$ is denoted by μ_x) and T be a continuous t-norm. The triple (X, μ, T) is called a random normed space (briefly RN-space) if μ satisfies the following conditions:
(RN1) $\mu_x(t) = \epsilon_o(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$ and all $t \geq 0$;
(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Example 1.1. Every normed space $(X, \|\cdot\|)$ defines a RN-space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ and T_M is the minimum t-norm. This space is called the *induced random normed space.*

Definition 1.3. Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x}(t) > 1 - \lambda$$

whenever $n \geq N$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} \mu_{x_n - x} = 1$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x_m}(t) > 1 - \lambda$$

whenever $n \geq m \geq N$.

(3) The RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([13]) *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence of X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently, Cho et. al. [4] was introduced and proved the Hyers-Ulam-Rassias stability of the following quintic functional equations

$$2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) = 20[f(x+y) + f(x-y)] + 90f(x) \tag{1.1}$$

for fixed $k \in \mathbb{Z}^+$ with $k \geq 3$ in quasi-β-normed spaces.

Remark 1.1. (1) If we put $x = y = 0$ in the equation (1.1), then $f(0) = 0$.

(2) $f(2^n x) = 2^{5n} f(x)$ for all $x \in X$ and $n \in \mathbb{Z}^+$.

(3) f is an odd mapping.

Throughout this paper, let X be a real linear space, (Z, μ', T_M) be an RN-space and (Y, μ, T_M) be a complete RN-space. For any mapping $f : X \to Y$, we define

$$\begin{aligned} &Df(x, y) \\ &= 2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) - 20[f(x+y) + f(x-y)] - 90f(x) \end{aligned}$$

for all $x, y \in X$. In this paper, using the direct and fixed point methods, we investigate the generalized Hyers-Ulam stability of the quintic functional equation:

$$2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) = 20[f(x+y) + f(x-y)] + 90f(x)$$

in random normed spaces under the minimum t-norm.

2. Random stability of the functional equation (1.1)

In this section, we investigate the generalized Hyers-Ulam stability problem of the quintic functional equation (1.1) in RN-spaces in the sense of Scherstnev under the minimum t-norm T_M.

Theorem 2.1. *Let $\phi : X^2 \to Z$ be a function such that, for some $0 < \alpha < 2^5$,*

$$\mu'_{\phi(2x,2y)}(t) \geq \mu'_{\alpha\phi(x,y)}(t) \tag{2.1}$$

and $\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = 1$ *for all* $x, y \in X$ *and* $t > 0$. *If* $f : X \to Y$ *is a mapping with* $f(0) = 0$ *such that*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\phi(x,y)}(t) \tag{2.2}$$

for all $x, y \in X$ *and* $t > 0$, *then there exists a unique quintic mapping* $Q : X \to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\Big(2^2(2^5 - \alpha)t\Big) \tag{2.3}$$

for all $x \in X$ *and* $t > 0$.

Proof. Letting $y = 0$ in (2.2), we get

$$\mu_{\frac{f(2x)}{2^5} - f(x)}(t) \geq \mu'_{\phi(x,0)}(128t) \tag{2.4}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^n x$ in (2.4), we get

$$\mu_{\frac{f(2^{n+1}x)}{2^{5(n+1)}} - \frac{f(2^n x)}{2^{5n}}}(t) \geq \mu'_{\phi(x,0)}\Big(\Big(\frac{2^5}{\alpha}\Big)^n 128t\Big)$$

for all $x \in X$ and $t > 0$. Since $\frac{f(2^n x)}{2^{5n}} - f(x) = \sum_{j=0}^{n-1}\Big(\frac{f(2^{j+1}x)}{2^{5(j+1)}} - \frac{f(2^j x)}{2^{5j}}\Big)$,

$$\mu_{\frac{f(2^n x)}{2^{5n}} - f(x)}\Big(\sum_{j=0}^{n-1} \frac{1}{128}\Big(\frac{\alpha}{2^5}\Big)^j t\Big) \geq T_M{}_{j=0}^{n-1}(\mu'_{\phi(x,0)}(t)) = \mu'_{\phi(x,0)}(t) \tag{2.5}$$

for all $x \in X$ and $t > 0$. Substituting x by $2^m x$ in (2.5), we get

$$\mu_{\frac{f(2^{n+m}x)}{2^{5(n+m)}} - \frac{f(2^m x)}{2^{5m}}}(t) \geq \mu'_{\phi(x,0)}\Big(\frac{t}{\sum_{j=m}^{n+m-1}(\frac{\alpha}{2^5})^j}\Big) \tag{2.6}$$

for all $x \in X$ and $m, n \in \mathbb{Z}$ with $n > m \geq 0$. Since $\alpha < k^3$, the sequence $\{\frac{f(2^n x)}{2^{5n}}\}$ is a Cauchy sequence in the complete RN-space (Y, μ, T_M) and so it converges to some point $Q(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (2.6). Then we get

$$\mu_{\frac{f(2^n x)}{2^{5n}} - f(x)}(t) \geq \mu'_{\phi(x,0)}\left(\frac{128t}{\sum_{j=0}^{n-1}(\frac{\alpha}{2^5})^j}\right),$$

and so, for any $\delta > 0$,

$$\begin{aligned} &\mu_{Q(x)-f(x)}(\delta + t) \\ &\geq T_M\Big(\mu_{Q(x)-\frac{f(2^n x)}{2^{5n}}}(\delta), \mu_{\frac{f(2^n x)}{2^{5n}} - f(x)}(t)\Big) \\ &\geq T_M\left(\mu_{Q(x)-\frac{f(2^n x)}{2^{5n}}}(\delta), \mu'_{\phi(x,0)}\left(\frac{128t}{\sum_{j=0}^{n-1}(\frac{\alpha}{2^5})^j}\right)\right) \end{aligned} \tag{2.7}$$

for all $x \in X$ and $t > 0$. Taking the limit as $n \to \infty$ in (2.7), we get

$$\mu_{Q(x)-f(x)}(\delta + t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5 - \alpha)t\right) \tag{2.8}$$

Since δ is arbitrary, by taking $\delta \to 0$ in (2.8), we have

$$\mu_{Q(x)-f(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5 - \alpha)t\right) \tag{2.9}$$

for all $x \in X$ and $t > 0$. Therefore, we conclude that the condition (2.3) holds.

Also, replacing x and y by $2^n x$ and $2^n y$ in (2.2), respectively, we have

$$\mu_{\frac{Df(2^n x, 2^n y)}{2^{5n}}}(t) \geq \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t)$$

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for all $x, y \in X$ and $t > 0$. It follows from $\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = 1$ that Q satisfies the equation (1.1), which implies that Q is a quintic mapping.

To prove the uniqueness of the quintic mapping Q, let us assume that there exists another mapping $\widetilde{Q} : X \to Y$ which satisfies (2.3). Fix $x \in X$. Then $Q(2^n x) = 2^{5n}Q(x)$ and $\widetilde{Q}(2^n x) = 2^{5n}\widetilde{Q}(x)$ for all $n \in \mathbb{Z}^+$. Thus it follows from (2.3) that

$$\begin{aligned} &\mu_{Q(x)-\widetilde{Q}(x)}(t) \\ &= \mu_{\frac{Q(2^n x)}{2^{5n}} - \frac{\widetilde{Q}(2^n x)}{2^{5n}}}(t) \\ &\geq T_M\Big(\mu_{\frac{Q(2^n x)}{2^{5n}} - \frac{f(2^n x)}{2^{5n}}}\Big(\frac{t}{2}\Big), \mu_{\frac{f(2^n x)}{2^{5n}} - \frac{\widetilde{Q}(2^n x)}{2^{5n}}}\Big(\frac{t}{2}\Big)\Big) \\ &\geq \mu'_{\phi(x,0)}\Big(2^2(2^5-\alpha)\Big(\frac{2^5}{\alpha}\Big)^n t\Big). \end{aligned} \tag{2.10}$$

Since $\lim_{n\to\infty}\Big(2^2(2^5-\alpha)\Big(\frac{2^5}{\alpha}\Big)^n t\Big) = \infty$, we have $\mu_{Q(x)-\widetilde{Q}(x)}(t) = 1$ for all $t > 0$. Thus the quintic mapping Q is unique. This completes the proof. □

Theorem 2.2. *Let $\phi : X^2 \to Z$ be a function such that, for some $2^5 < \alpha$,*

$$\mu'_{\phi(\frac{x}{2},\frac{y}{2})}(t) \geq \mu'_{\phi(x,y)}(\alpha t) \tag{2.11}$$

and $\lim_{n\to\infty} \mu'_{2^{5n}\phi(\frac{x}{2^n},\frac{y}{2^n})}(t) = 1$ for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (2.2), then there exists a unique cubic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(\alpha - 2^5)t\right) \tag{2.12}$$

for all $x \in X$ and $t > 0$.

Proof. It follows from (2.2) that

$$\mu_{f(x)-2^5 f(\frac{x}{2})}(t) \geq \mu'_{\phi(x,0)}\left(2^2\alpha t\right) \tag{2.13}$$

for all $x \in X$. Applying the triangle inequality and (2.13), we have

$$\mu_{f(x)-2^{5n}f(\frac{x}{2^n})}(t) \geq \mu'_{\phi(x,0)}\left(\frac{2^2\alpha t}{\sum_{j=m}^{n+m-1}\left(\frac{2^5}{\alpha}\right)^j}\right) \tag{2.14}$$

for all $x \in X$ and $m, n \in \mathbb{Z}$ with $n > m \geq 0$. Then the sequence $\{2^{5n}f(\frac{x}{2^n})\}$ is a Cauchy sequence in the complete RN-space (Y, μ, T_M) and so it converges to some point $Q(x) \in Y$. We can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 2^{5n} f\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$. Then the mapping Q satisfies (1.1) and (2.12). The remaining assertion follows the similar proof method in Theorem 2.1. This complete the proof. □

Corollary 2.3. *Let θ be a nonnegative real number and z_0 be a fixed unit point of Z. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\theta z_0}(t) \tag{2.15}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $C : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\theta z_0}(124t) \tag{2.16}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\phi : X^2 \to Z$ be defined by $\phi(x, y) = \theta z_0$. Then, the proof follows from Theorem 2.1 by $\alpha = 1$. This completes the proof. □

Corollary 2.4. *Let $p, q \in \mathbb{R}$ be positive real numbers with $p, q < 5$ and z_0 be a fixed unit point of Z. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{Df(x,y)}(t) \geq \mu'_{(\|x\|^p+\|y\|^q)z_0}(t) \tag{2.17}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\|x\|^p z_0}\Big(2^2(2^5 - 2^p)t\Big) \tag{2.18}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\phi : X^2 \to Z$ be defined by $\phi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the proof follows from Theorem 2.1 by $\alpha = 2^p$. This completes the proof. □

Now, we give an example to illustrate that the quintic functional equation (1.1) is not stable for $r = 5$ in Corollary 2.4

Example 2.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} x^5, & \text{for} \quad |x| < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^{5n}}$$

for all $x \in \mathbb{R}$. Then f satisfies the functional inequality

$$\begin{aligned} &|2f(2x + y) + 2f(2x - y) + f(x + 2y) + f(x - 2y) - 20[f(x + y) + f(x - y)] - 90f(x)| \\ &\leq \frac{136 \cdot 32^2}{31}\Big(|x|^5 + |y|^5\Big) \end{aligned} \tag{2.19}$$

for all $x, y \in X$, but there do not exist a quintic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $d > 0$ such that

$$|f(x) - Q(x)| \leq d|x|^5$$

for all $x \in \mathbb{R}$. In fact, it is clear that f is bounded by $\frac{32}{31}$ on $\mathbb{R}$. If $|x|^5 + |y|^5 = 0$, then (2.19) is trivial. If $|x|^5 + |y|^5 \geq \frac{1}{32}$, then

$$|Df(x, y)| \leq \frac{136 \cdot 32}{31} \leq \frac{136 \cdot 32^2}{31}\Big(|x|^5 + |y|^5\Big).$$

Now, suppose that $0 < |x|^5 + |y|^5 < \frac{1}{32}$. Then there exists a positive integer $k \in Z^+$ such that

$$\frac{1}{32^{k+2}} \leq |x|^5 + |y|^5 < \frac{1}{32^{k+1}}$$

and so

$$32^k|x|^5 < \frac{1}{32}, \quad 32^k|y|^5 < \frac{1}{32},$$
$$2^n(2x + y),\ 2^n(2x - y),\ 2^n(x + 2y),\ 2^n(x - 2y),\ 2^n(x - y),\ 2^n x \in (-1, 1)$$

and

$$\begin{aligned}&\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\quad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)\\&=0\end{aligned}$$

for all $n=0,1,\cdots,k-1$. Thus we obtain

$$\begin{aligned}&|Df(x,y)|\\&\le\sum_{n=0}^{\infty}\frac{1}{2^{5n}}|\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\qquad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)|\\&\le\sum_{n=k}^{\infty}\frac{1}{2^{5n}}|\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\qquad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)|\\&\le\frac{136\cdot 32^2}{31}\Big(|x|^5+|y|^5\Big).\end{aligned}$$

Therefore, f satisfies (2.19).

Now, we claim that the quintic functional equation (1.1) is not stable for $r=5$ in Corollary 2.4. Suppose on the contrary that there exists a quintic mapping $Q:\mathbb{R}\to\mathbb{R}$ and constant $d>0$ such that

$$|f(x)-Q(x)|\le d|x|^5$$

for all $x\in\mathbb{R}$. Since f is bounded and continuous for all $x\in\mathbb{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.1, Q must have $Q(x)=cx^5$ for all $x\in\mathbb{R}$. So, we obtain

$$|f(x)|\le(d+|c|)|x|^5 \tag{2.20}$$

for all $x\in\mathbb{R}$. Let $m\in\mathbb{Z}^+$ such that $m+1>d+|c|$.

If x is in $(0,2^{-m})$, then $2^nx\in(0,1)$ for $n=0,1,\cdots,m$. For this x, we have

$$f(x)=\sum_{n=0}^{\infty}\frac{\phi(2^n)}{2^{5n}}\ge\sum_{n=0}^{m}\frac{(2^nx)^5}{2^{5n}}=(m+1)x^5>(d+|c|)|x|^5,$$

which contradiction (2.20).

Remark 2.1. In Corollary 2.4, if we assume that

$$\phi(x,y)=\|x\|^r\|y\|^rz_0$$

or

$$\phi(x,y)=(\|x\|^r\|y\|^s+\|x\|^{r+s}+\|y\|^{r+s})z_0,$$

then we have Ulam-Gavuta-Rassias product stability and JMRassias mixed product-sum stability, respectively.

Next, we apply a fixed point method for the generalized Hyer-Ulam stability of the functional equation (1.1) in RN-spaces. The following Theorem will be used in the proof of Theorem 2.6.

Theorem 2.5. ([7]) *Suppose that (Ω, d) is a complete generalized metric space and $J : \Omega \to \Omega$ is a strictly contractive mapping with Lipshitz constant $L < 1$. Then, for each $x \in \Omega$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers $n \geq 0$ or there exists a natural number n_0 such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point $y*$ of J;*
(3) *$y*$ is the unique fixed point of J in the set $\Lambda = \{y \in \Omega : d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Lambda$.*

Theorem 2.6. *Let $\phi : X^2 \to D^+$ be a function such that, for some $0 < \alpha < 2^5$,*

$$\mu'_{\phi(x,y)}(t) \leq \mu'_{\phi(2x,2y)}(\alpha t) \tag{2.21}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that

$$\mu_{D(x,y)}(t) \geq \mu'_{\phi(x,y)}(t) \tag{2.22}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,y)}\big(2^2(2^5 - \alpha)t\big) \tag{2.23}$$

for all $x \in X$ and $t > 0$.

Proof. It follows from (2.22) that

$$\mu_{f(x)-\frac{f(2x)}{2^5}}(t) \geq \mu'_{\phi(x,0)}(128t) \tag{2.24}$$

for all $x \in X$ and $t > 0$. Let $\Omega = \{g : X \to Y, g(x) = 0\}$ and the mapping d defined on Ω by

$$d(g, h) = \inf\{c \in [0, \infty) : \mu_{g(x)-h(x)}(ct) \geq \mu'_{\phi(x,0)}(t),\ \forall x \in X\}$$

where, as usual, $\inf \emptyset = -\infty$. Then (Ω, d) is a generalized complete metric space (see [10]). Now, let us consider the mapping $J : \Omega \to \Omega$ defined by

$$Jg(x) = \frac{1}{2^5} g(2x)$$

for all $g \in \Omega$ and $x \in X$. Let g, h in Ω and $c \in [0, \infty)$ be an arbitrary constant with $d(g, h) < c$. Then $\mu_{g(x)-h(x)}(ct) \geq \mu'_{\phi(x,0)}(t)$ for all $x \in X$ and $t > 0$ and so

$$\mu_{Jg(x)-Jh(x)}\Big(\frac{\alpha ct}{2^5}\Big) = \mu_{g(2x)-h(2x)}(\alpha ct) \geq \mu'_{\phi(x,0)}(t) \tag{2.25}$$

for all $x \in X$ and $t > 0$. Hence we have

$$d(Jg, Jh) \leq \frac{\alpha c}{2^5} \leq \frac{\alpha}{2^5} d(g, h)$$

for all $g, h \in \Omega$. Then J is a contractive mapping on Ω with the Lipschitz constant $L = \frac{\alpha}{2^5} < 1$. Thus it follows from Theorem 2.5 that there exists a mapping $Q : X \to Y$, which is a unique fixed point of J in the set $\Omega_1 = \{g \in \Omega : d(f, g) < \infty\}$, such that

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{5n}}$$

for all $x \in X$ since $\lim_{n\to\infty} d(J^n f, Q) = 0$. Also, from $\mu_{f(x)-\frac{f(2x)}{2^5}}(t) \geq \mu'_{\phi(x,0)}(128t)$, it follows that $d(f, Jf) \leq \frac{1}{128}$. Therefore, using Theorem 2.5 again, we get

$$d(f, Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{2^2(2^5 - \alpha)}.$$

This means that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)t\right)$$

for all $x \in X$ and $t > 0$.

Also, replacing x and y by $2^n x$ and $2^n y$ in (2.22), respectively, we have

$$\mu_{DQ(x,y)}(t) \geq \lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = \lim_{n\to\infty} \mu'_{\phi(x,y)}\left(\left(\frac{2^5}{\alpha}\right)^n t\right) = 1$$

for all $x, y \in X$ and $t > 0$. By (RN1), the mapping Q is quintic.

To prove the uniqueness, let us assume that there exists a quintic mapping $Q' : X \to Y$ which satisfies (2.23). Then Q' is a fixed point of J in Ω_1. However, it follows from Theorem 2.5 that J has only one fixed point in Ω_1. Hence $Q = Q'$. This completes the proof. □

Theorem 2.7. *Let $\phi : X^2 \to D^+$ be a function such that, for some $0 < 2^5 < \alpha$,*

$$\mu'_{\phi(x,y)}(t) \leq \mu'_{\phi(\frac{x}{2},\frac{y}{2})}(\alpha t) \tag{2.26}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (2.22), then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(\alpha - 2^5)t\right) \tag{2.27}$$

for all $x \in X$ and $t > 0$.

Proof. By a modification in the proofs of Theorem 2.2 and 2.6, we can easily obtain the desired results. This completes the proof. □

Now, we present a corollary that is an application of Theorem 2.6 and 2.7 in the classical case.

Corollary 2.8. *Let X be a Banach space, ϵ and p be positive real numbers with $p \neq 5$. Assume that $f : X \to X$ is a mapping with $f(0) = 0$ which satisfies*

$$\|Df(x,y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\|Q(x) - f(x)\| \leq \frac{\epsilon\|x\|^p}{2^2|2^5 - 2^p|}$$

for all $x \in X$ and $t > 0$.

Proof. Define $\mu : X \times \mathbb{R} \to \mathbb{R}$ by

$$\mu_x(t) = \begin{cases} \frac{t}{t+\|x\|}, & \text{if } t > 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then (X, μ, T_M) is a complete RN-space. Denote $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and $t > 0$. It follows from $\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$ that

$$\mu_{Df(x,y)}(t) \geq \mu'_{\phi(x,y)}(t)$$

for all $x, y \in X$ and $t > 0$, where $\mu' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\mu'_x(t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is a random norm on $\mathbb{R}$. Then all the conditions of Theorems 2.6 and 2.7 hold and so there exists a unique quintic mapping $Q : X \to X$ such that

$$\begin{aligned} \frac{t}{t + \|Q(x) - f(x)\|} &= \mu_{Q(x)-f(x)}(t) \\ &\geq \mu'_{\phi(x,0)}\Big(2^2|2^5 - \alpha|t\Big) = \frac{2^2|2^5 - \alpha|t}{2^2|2^5 - \alpha|t + \epsilon\|x\|^p}. \end{aligned}$$

Therefore, we obtain the desired result, where $\alpha = 2^p$. This completes the proof. □

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Generalized composition operators on Zygmund type spaces and Bloch type spaces

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Abstract. In this paper, we investigate the boundedness and compactness of generalized composition operators on Zygmund type spaces and Bloch type spaces with normal weight.

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Keywords: Generalized composition operator, Bloch type space, Zygmund type space.

1 Introduction

Let k be a positive continuous function on $[0,1)$. k is called normal, if there exist positive numbers a and b, $0<a<b$, and $\delta\in[0,1)$ such that(see [12]),

$$\frac{k(r)}{(1-r)^a}\ \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to1}\frac{k(r)}{(1-r)^a}=0; \tag{1}$$

$$\frac{k(r)}{(1-r)^b}\ \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to1}\frac{k(r)}{(1-r)^b}=\infty. \tag{2}$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let ω be normal on $[0,1)$. An $f\in H(\mathbb{D})$ is said to belong to the Bloch type space, denoted by $\mathcal{B}_\omega$, if

$$\|f\|_{\mathcal{B}_\omega}=|f(0)|+\sup_{z\in\mathbb{D}}\omega(|z|)|f'(z)|<\infty.$$

It is easy to see that $\mathcal{B}_\omega$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}_\omega}$. When $\omega(t)=1-t^2$, we get the Bloch space, denoted by $\mathcal{B}=\mathcal{B}(\mathbb{D})$. See [19] for more information of the Bloch space.

Suppose μ is normal on $[0,1)$. The Zygmund type space, denoted by $\mathcal{Z}_\mu$, is the space of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}_\mu}=|f(0)|+|f'(0)|+\sup_{z\in\mathbb{D}}\mu(|z|)|f''(z)|<\infty.$$

It is also easy to see that $\mathcal{Z}_\mu$ is a Banach space with the norm $\|\cdot\|_{\mathcal{Z}_\mu}$. When $\mu(t)=1-t^2$, we get the Zygmund space (see [2, 8]).

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Throughout the paper, $S(\mathbb{D})$ denotes the set of analytic self-map of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ, which is defined by $(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D})$. We refer the books [1, 19] for the theory of composition operators. Composition operators mapping into the Bloch space on $\mathbb{D}$ were studied in, for example, [1, 4, 11, 14, 15, 18]. See [5, 6, 9, 10] for some results of the composition operator mapping into the Zygmund space.

Motivated by the fact that weighted composition operators naturally come from isometries of some function spaces, for $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, Li and Stević [9] defined the generalized composition operator, denoted by C_φ^g, as follows.

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

They characterized the boundedness and compactness of C_φ^g on the Zygmund space and the Bloch space in [9]. See, for example, [7, 13, 16] for the study of the operator C_φ^g.

In this paper, motivated by [9], we investigate the boundedness and compactness of the generalized composition operator C_φ^g on Zygmund type spaces and Bloch type spaces with normal weight.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2 Proof of main results

In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas which follow.

Lemma 1. [3] *Suppose μ is normal on $[0, 1)$. Then there exists $\mu_* \in H(\mathbb{D})$, such that*

(i) For any $t \in [0, 1)$, $\mu_*(t) \in R^+$, $\mu_*(t)$ is increasing on $[0, 1)$;

(ii) $\inf\limits_{t\in[0,1)} \mu(t)\mu_*(t) > 0$; $\quad \sup\limits_{z\in\mathbb{D}} \mu(|z|)|\mu_*(z)| < \infty$.

In the rest of the paper, we will always use μ_* to denote the analytic function related to μ in Lemma 1. By a calculation, we get the following lemma.

Lemma 2. *Suppose μ is normal on $[0, 1)$. Then the following statements hold.*

(i) There exists a $\delta \in (0, 1)$, such that μ is decreasing on $[\delta, 1)$, $\lim\limits_{t\to 1} \mu(t) = 0$.

(ii) For all $\alpha > 1, \beta \in (0, 1)$, when $t \in (0, 1)$, $s \in (\beta, 1)$,

$$\mu(t) \approx \mu(t^\alpha) \approx \frac{1}{\mu_*(t)}, \quad \int_0^{s^\alpha} \frac{1}{\mu(t)}dt \approx \int_0^s \frac{1}{\mu_(t)}dt.$$

(iii) For any $z \in \mathbb{D}$, $\left|\int_0^z \mu_*(\eta)d\eta\right| \lesssim \int_0^{|z|} \mu_*(t)dt$. If $|\eta| \leq |z|$, $\mu(|z|)|\mu_*(\eta)| < C$.

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Proof. (i). By the definition of normal function, there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that (1) and (2) hold. Since $\mu(t) = \frac{\mu(t)}{(1-t)^a}(1-t)^a$, we see that μ is decreasing on $[\delta, 1)$ and $\lim_{t\to 1}\mu(t) = 0$.

(ii). From $\lim_{t\to 1} \frac{1-t}{1-t^\alpha} = \frac{1}{\alpha} > 0$, for any $t \in [\delta^{\frac{1}{\alpha}}, 1)$,

$$1 > \frac{\mu(t)}{\mu(t^\alpha)} = \frac{\frac{\mu(t)}{(1-t)^b}}{\frac{\mu(t^\alpha)}{(1-t^\alpha)^b}} \frac{(1-t)^b}{(1-t^\alpha)^b} > \frac{(1-t)^b}{(1-t^\alpha)^b} > C.$$

So when $t \in (0, 1)$, $\mu(t) \approx \mu(t^\alpha)$. By Lemma 1, when $t \in (0, 1)$, $\mu(t) \approx \frac{1}{\mu_*(t)}$ is obvious.

When $s \in (\beta, 1)$,

$$\begin{aligned}\int_0^{s^\alpha} \frac{1}{\mu(t)}dt &= \int_0^{\beta^\alpha} \frac{1}{\mu(t)}dt + \int_{\beta^\alpha}^{s^a} \frac{1}{\mu(t)}dt = C + \int_\beta^s \frac{\alpha t^{\alpha-1}}{\mu(t^\alpha)}dt \\ &\approx \int_0^\beta \frac{1}{\mu(t)}dt + \int_\beta^s \frac{1}{\mu(t)}dt = \int_0^s \frac{1}{\mu(t)}dt.\end{aligned}$$

(iii). Since μ_* is analytic, we see that (iii) holds. The proof is completed. □

Lemma 3.[17] *Suppose μ is normal on $[0, 1)$. Then for all $z \in \mathbb{D}$ and $f \in \mathcal{B}_\mu$,*

$$|f(z)| < G_\mu(z)\|f\|_{\mathcal{B}_\mu}, \text{ where } G_\mu(z) = 1 + \int_0^{|z|} \frac{1}{\mu(t)}dt.$$

Remark 1. From the definitions of $\mathcal{Z}_\mu$ and $\mathcal{B}_\mu$, for all $z \in \mathbb{D}$ and $f \in \mathcal{Z}_\mu$,

$$|f'(z)| \leq G_\mu(z)\|f'\|_{\mathcal{B}_\mu} \leq G_\mu(z)\|f\|_{\mathcal{Z}_\mu}.$$

Lemma 4. [17] *Suppose that μ is normal on $[0, 1)$ such that $\int_0^1 \frac{1}{\mu(t)}dt < \infty$. If $\{f_n\}$ is bounded in $\mathcal{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, then*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n(z)| = 0.$$

The relationship between Zygmund type spaces and Bloch type spaces was established as follows.

Lemma 5. *Suppose that μ is normal on $[0, 1)$. Let $\mu_+(t) = (1-t)\mu(t)$. Then*

(i) μ_+ *is normal on* $[0, 1)$, $\lim_{|z|\to 1} G_{\mu_+}(z) = \infty$.

(ii) $\mathcal{B}_\mu = \mathcal{Z}_{\mu_+}$ *and* $\|\cdot\|_{\mathcal{B}_\mu} \approx \|\cdot\|_{\mathcal{Z}_{\mu_+}}$.

Proof. (i) Obviously, μ_+ is normal on $[0, 1)$. Since μ is normal,there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that (1) and (2) holds. Then

$$\int_0^1 \frac{1}{\mu_+(t)}dt > \int_\delta^1 \frac{1}{(1-t)^{a+1}} \frac{(1-t)^a}{\mu(t)}dt > \frac{(1-\delta)^a}{\mu(\delta)} \int_\delta^1 \frac{1}{(1-t)^{1+a}}dt = +\infty,$$

as desired.

(ii) First we prove that $\mathscr{Z}_{\mu_+} \subseteq \mathcal{B}_\mu$. For all $f \in \mathscr{Z}_{\mu_+}$, we have

$$\mu(|z|)|f'(z) - f'(0)| = \mu(|z|)\left|\int_0^z f''(\eta)d\eta\right| \leq \|f\|_{\mathscr{Z}_{\mu+}} \int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt. \tag{3}$$

If $|z| \leq \delta$, $\int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt < C$. If $|z| > \delta$,

$$\begin{aligned}\int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt &= \left(\int_0^{\delta} \frac{\mu(|z|)}{\mu(t)(1-t)}dt + \int_\delta^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt\right)\\ &\leq \left(C + \int_\delta^{|z|} \frac{\frac{\mu(|z|)}{(1-|z|)^a}}{\frac{\mu(t)}{(1-t)^a}} \frac{(1-|z|)^a}{(1-t)^{a+1}}dt\right)\\ &\leq \left(C + \int_\delta^{|z|} \frac{(1-|z|)^a}{(1-t)^{a+1}}dt\right) \leq C.\end{aligned}$$

From Lemma 2, $\mu(t)$ is bounded on $[0, 1)$. By (3),

$$\mu(|z|)|f'(z)| \leq C\|f\|_{\mathscr{Z}_{\mu+}} + \mu(|z|)|f'(0)| \lesssim \|f\|_{\mathscr{Z}_{\mu+}} + |f'(0)| \leq 2\|f\|_{\mathscr{Z}_{\mu+}}.$$

Therefore $\|f\|_{\mathcal{B}_\mu} \lesssim \|f\|_{\mathscr{Z}_{\mu+}}$ and $\mathscr{Z}_{\mu_+} \subseteq \mathcal{B}_\mu$.

Next we prove that $\mathcal{B}_\mu \subseteq \mathscr{Z}_{\mu_+}$. For any $f \in \mathcal{B}_\mu$, by Cauchy's formula,

$$|f''(z)| \leq \frac{2}{1-|z|} \max_{|\eta-z|=\frac{1-|z|}{2}} |f'(\eta)| \leq \frac{2}{1-|z|} \max_{|\eta|=\frac{1+|z|}{2}} |f'(\eta)| \leq \frac{2\|f\|_{\mathcal{B}_\mu}}{\mu(\frac{1+|z|}{2})(1-|z|)}.$$

If $|z| \leq \delta$, $\frac{\mu(|z|)}{\mu(\frac{1+|z|}{2})} < C$ is obvious. When $\delta < |z| < 1$,

$$\frac{\mu(|z|)}{\mu(\frac{1+|z|}{2})} = 2^b \frac{\frac{\mu(|z|)}{(1-|z|)^b}}{\frac{\mu(\frac{1+|z|}{2})}{(1-\frac{1+|z|}{2})^b}} < 2^b.$$

So $\|f\|_{\mathscr{Z}_{\mu_+}} \lesssim \|f\|_{\mathcal{B}_\mu}$ and hence $\mathcal{B}_\mu \subseteq \mathscr{Z}_{\mu_+}$. The proof is completed. □

To study the compactness, we need the following lemma, which can be proved in a standard way (see, for example, Proposition 3.11 in [1]).

Lemma 6. *Suppose that $g \in H(\mathbb{D}), \varphi \in S(\mathbb{D})$, X, Y are Bloch type spaces or Zygmund type spaces. If $C_\varphi^g : X \to Y$ is bounded, then $C_\varphi^g : X \to Y$ is a compact operator if and only if whenever $\{f_n\}$ is bounded in X and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, $\lim_{n\to\infty} \|C_\varphi^g f_n\|_Y = 0$.*

3 The boundness and compactness of $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega(\mathcal{B}_\omega)$

Theorem 1. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$. Then $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty \qquad \textit{and} \qquad \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty. \tag{4}$$

Proof. Suppose that (4) holds. For any $f \in \mathscr{Z}_\mu$, by Lemma 3 and Remark 1, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} \omega(|z|)|(C_\varphi^g f)''(z)| &\le \sup_{z\in\mathbb{D}} \omega(|z|)\left|f''(\varphi(z))\varphi'(z)g(z)\right| + \sup_{z\in\mathbb{D}} \omega(|z|)\left|f'(\varphi(z))g'(z)\right| \\ &\le \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}\|f\|_{\mathscr{Z}_\mu} + \sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_\mu(\varphi(z))\|f\|_{\mathscr{Z}_\mu} \\ &< \infty,\end{aligned}$$

and $|(c_\varphi^g f)(0)| + |(C_\varphi^g f)'(0)| = |f'(\varphi(0))g(0)| \le |g(0)|G_\mu(\varphi(0))\|f\|_{\mathscr{Z}_\mu} < \infty$. Hence $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded.

Conversely, suppose $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded. From $z, z^2 \in \mathscr{Z}_\mu$, we see that

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)| < \infty \qquad \text{and} \qquad \sup_{z\in\mathbb{D}} \omega(|z|)|\varphi'(z)g(z)| < \infty. \tag{5}$$

Therefore

$$\sup_{|\varphi(z)|\le\frac{1}{2}} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty \qquad \text{and} \qquad \sup_{|\varphi(z)|\le\frac{1}{2}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty. \tag{6}$$

For any $\xi \in \mathbb{D}$, if $|\varphi(\xi)| > \frac{1}{2}$, let $a = \varphi(\xi)$ and

$$p_a(z) = \int_0^{\overline{a}z}\left(\int_0^{t^2} \mu_*(\eta)d\eta\right)^2 dt - \int_0^{\overline{a}z}\left(\int_0^{\frac{t^3}{|a|^2}} \mu_*(\eta)d\eta\right)^2 dt,$$

$$q_a(z) = \frac{p_a(z)}{\int_0^{|a|}\mu_*(\eta)d\eta}. \tag{7}$$

Then

$$p_a'(z) = \overline{a}\left(\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta\right)^2 - \overline{a}\left(\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta\right)^2,$$

$$p_a''(z) = 4\overline{a}^3 z\mu_*(\overline{a}^2z^2)\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta - \frac{6\overline{a}^4z^2}{|a|^2}\mu_*\left(\frac{(\overline{a}z)^3}{|a|^2}\right)\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta.$$

By Lemmas 1 and 2,

$$\mu(|z|)|p_a''(z)| \lesssim \left|\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta\right| + \left|\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta\right| \lesssim \int_0^{|a|} \mu_*(\eta)d\eta$$

So

$$\|q_a\|_{\mathscr{Z}_\mu} = q_a(0) + q_a'(0) + \sup_{z\in\mathbb{D}} \mu(|z|)|q_a''(z)| < C. \tag{8}$$

Hence, when $|\varphi(\xi)| > \frac{1}{2}$,

$$\frac{\omega(|\xi|)|\varphi'(\xi)g(\xi)|}{\mu(|\varphi(\xi)|)} \approx \omega(|\xi|)|(C_\varphi^g q_a)''(\xi)| \le \|C_\varphi^g q_a\|_{\mathscr{Z}_\omega} < \|q_a\|_{\mathscr{Z}_\mu}\|C_\varphi^g\| < \infty. \tag{9}$$

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From (6) and (9), we see that the second inequality in (4) holds.

Let $f_a(z) = \int_0^{\overline{a}z} \int_0^{\eta} \mu_*(w)dwd\eta$. Then

$$f_a'(z) = \overline{a} \int_0^{\overline{a}z} \mu_*(w)dw, \;\; f_a''(z) = \overline{a}^2 \mu_*(\overline{a}z), \; \|f_a\|_{\mathscr{Z}_\mu} \le C.$$

By Lemma 2, when $|\varphi(\xi)| > \frac{1}{2}$,

$$\begin{aligned} \omega(|\xi|)|g'(\xi)| \int_0^{|a|} \frac{1}{\mu(t)} dt \approx & \omega(|\xi|) \left|(C_\varphi^g f_a)''(\xi) - f_a''(\varphi(\xi))\varphi'(\xi)g(\xi)\right| \\ \le & \|C_\varphi^g f_a\|_{\mathscr{Z}_\omega} + \sup_{\xi\in\mathbb{D}} \omega(|\xi|)\mu_*(|\varphi(\xi)|^2)|\varphi'(\xi)g(\xi)| \\ \lesssim & \|f_a\|_{\mathscr{Z}_\mu}\|C_\varphi^g\| + \sup_{\xi\in\mathbb{D}} \frac{\omega(|\xi|)|\varphi'(\xi)g(\xi)|}{\mu(|\varphi(\xi)|)}. \end{aligned} \tag{10}$$

From (6) and (10), we see that the first inequality in (4) holds. The proof is completed. □

Theorem 2. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$ such that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded. Then the following statements hold:*

(i) When $\lim_{|z|\to 1} G_\mu(z) < \infty$, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0. \tag{11}$$

(ii) When $\lim_{|z|\to 1} G_\mu(z) = \infty$, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) = 0 \quad \textit{and} \quad \lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0. \tag{12}$$

Proof. Because $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded, (5) holds.

(i). Suppose (11) holds. For any $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \text{ when } |\varphi(z)| > \delta. \tag{13}$$

Let $\{f_n\} \subset \mathscr{Z}_\mu$ be bounded and converge to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 4 and Cauchy estimate,

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n'(z)| = 0, \quad \text{and} \quad \lim_{n\to\infty} \sup_{|z|\le\delta} |f_n''(z)| = 0. \tag{14}$$

From Remark 1, (5) and $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathscr{Z}_\mu} < \infty$,

$$\begin{aligned} \|C_\varphi^g f_n\|_{\mathscr{Z}_\omega} &= |(C_\varphi^g f_n)'(0)| + \sup_{z\in\mathbb{D}} \omega(|z|) \left|f_n''(\varphi(z))\varphi'(z)g(z) + f_n'(\varphi(z))g'(z)\right| \\ &\lesssim |f_n'(\varphi(0))| + \sup_{|\varphi(z)|\le\delta} |f_n''(\varphi(z))| + \sup_{|\varphi(z)|>\delta} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} + \sup_{z\in\mathbb{D}} |f_n'(z)|. \end{aligned}$$

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By (13) and (14), $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathscr{Z}_\mu} = 0$. Using Lemma 6, we see that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact.

Conversely, assume that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. Suppose $\{z_n\} \subset \mathbb{D}$ is a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and

$$r_n(z) = \mu(|a_n|) \int_0^{\overline{a_n}z} \int_0^t \mu_*^2(\eta) d\eta dt.$$

From Lemma 2, $\{r_n\}$ is bounded in $\mathscr{Z}_\mu$ and $r_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$ when $n \to \infty$. By Lemmas 4 and 6, we have

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |r_n'(z)| = 0 \text{ and } \lim_{n\to\infty} \|C_\varphi^g r_n\|_{\mathscr{Z}_\omega} = 0. \tag{15}$$

Using Lemma 2, (5) and (15),

$$\begin{aligned} \lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} &\approx \lim_{n\to\infty} \omega(|z_n|) \left|(C_\varphi^g r_n)''(z_n) - r_n'(a_n)g'(z_n)\right| \\ &\le \lim_{n\to\infty} \|C_\varphi^g r_n\|_{\mathscr{Z}_\omega} + \lim_{n\to\infty} \omega(|z_n|) \left|r_n'(a_n)g'(z_n)\right| = 0, \end{aligned}$$

which implies that $\lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0$.

(ii). Suppose (12) holds. For any $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \varepsilon \quad \text{and} \quad \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \tag{16}$$

when $|\varphi(z)| > \delta$. Let $\{f_n\}$ be a bounded sequence in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Cauchy estimate,

$$\lim_{n\to\infty} \sup_{|\varphi(w)|\le\delta} |f_n'(\varphi(w))| = 0, \ \lim_{n\to\infty} \sup_{|\varphi(w)|\le\delta} |f_n''(\varphi(w))| = 0. \tag{17}$$

From Lemma 3, Remark 1 and (5),

$$\begin{aligned} \|C_\varphi^g f_n\|_{\mathscr{Z}_\omega} &= |(C_\varphi^g f_n)'(0)| + \sup_{z\in\mathbb{D}} \omega(|z|) \left|f_n''(\varphi(z))\varphi'(z)g(z) + f_n'(\varphi(z))g'(z)\right| \\ &\lesssim |f_n'(\varphi(0))| + \sup_{|\varphi(z)|\le\delta} |f_n''(\varphi(z))| + \sup_{|\varphi(z)|\le\delta} |f_n'(\varphi(z))| + \\ &\quad \sup_{|\varphi(z)|>\delta} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} \|f_n\|_{\mathscr{Z}_\mu} + \sup_{|\varphi(z)|>\delta} \omega(|z|)|g'(z)|G_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu} \end{aligned}$$

By (16) and (17), we see that $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathscr{Z}_\mu} = 0$. From Lemma 6, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact.

Conversely, suppose that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and $q_n = q_{a_n}$, where q_a is defined in (7). By (8), $\{q_n\}$ is bounded in $\mathscr{Z}_\mu$. Obviously, $q_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 6, $\lim_{n\to\infty} \|C_\varphi^g q_n\|_{\mathscr{Z}_\omega} = 0$. By (9),

$$\lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} \approx \lim_{n\to\infty} \omega(|z_n|)|(C_\varphi^g q_n)''(z_n)| \le \lim_{n\to\infty} \|C_\varphi^g q_n\|_{\mathscr{Z}_\omega} = 0,$$

which implies that $\lim\limits_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0$.

Let

$$k_n(z) = \frac{\int_0^{\overline{a_n}z} \left(\int_0^t \mu_*(s)ds\right)^2 dt}{\int_0^{|a_n|} \mu_*(s)ds}. \tag{18}$$

Then

$$k_n'(z) = \frac{\overline{a_n}\left(\int_0^{\overline{a_n}z} \mu_*(s)ds\right)^2}{\int_0^{|a_n|} \mu_*(s)ds}, k_n''(z) = \frac{2(\overline{a_n})^2\mu_*(\overline{a_n}z)\int_0^{\overline{a_n}z} \mu_*(s)ds}{\int_0^{|a_n|} \mu_*(s)ds}.$$

By Lemma 2, $\{k_n\}$ is bounded in $\mathscr{Z}_\mu$ and $k_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Lemma 6, $\lim\limits_{n\to\infty} \|C_\varphi^g k_n\|_{\mathscr{Z}_\omega} = 0$. By Lemma 2,

$$\begin{aligned} & \lim_{n\to\infty} \omega(|z_n|)|g'(z_n)| \int_0^{|\varphi(z_n)|} \mu_*(s)ds \\ \lesssim & \lim_{n\to\infty} \|C_\varphi^g k_n\|_{\mathscr{Z}_\omega} + 2\lim_{n\to\infty} \left|\varphi'(z_n)g(z_n)\omega(|z_n|)\mu_*(|\varphi(z_n)|^2)\right| \\ \lesssim & \lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} = 0, \end{aligned}$$

which implies that $\lim\limits_{|\varphi(z)|\to 1} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) = 0$. The proof is completed. □

Theorem 3. *Suppose $g \in H(\mathbb{D}), \varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is bounded.*

(ii) $\sup\limits_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_\mu(\varphi(z)) < \infty$.

(iii) $\sup\limits_{z\in\mathbb{D}} \omega_+(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty$ *and* $\sup\limits_{z\in\mathbb{D}} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty$.

Proof. *(ii)⇒(i).* Suppose that *(ii)* holds. For any $f \in \mathscr{Z}_\mu$, using Remark 1,

$$\|C_\varphi^g f\|_{\mathcal{B}_\omega} = \sup_{z\in\mathbb{D}} \omega(|z|)|g(z)f'(\varphi(z))| \le \sup_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_\mu(\varphi(z))\|f\|_{\mathscr{Z}_\mu} \lesssim \|f\|_{\mathscr{Z}_\mu} < \infty.$$

So $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded.

(ii)⇒(i). Suppose $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded. Then

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g(z)| = \|C_\varphi^g z\|_{\mathcal{B}_\omega} < \infty. \tag{19}$$

For all $\eta \in \mathbb{D}$, let $u_a(z) = \int_0^{\overline{a}z}\int_0^t \mu_*(s)dsdt$, where $a = \varphi(\eta)$. By Lemma 2, $\sup_{\eta\in\mathbb{D}} \|u_a\|_{\mathscr{Z}_\mu} < \infty$. Thus $\sup_{\eta\in\mathbb{D}} \|C_\varphi^g u_a\|_{\mathcal{B}_\omega} < \infty$. When $|\varphi(\eta)| > \frac{1}{2}$,

$$\omega(|\eta|)|g(\eta)| \int_0^{|\varphi(\eta)|} \frac{1}{\mu(s)}ds \approx \omega(|\eta|)|(C_\varphi^g u_a)'(\eta)| \le \|C_\varphi^g u_a\|_{\mathcal{B}_\omega} < C. \tag{20}$$

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By (19) and (20),

$$\sup_{\eta\in\mathbb{D}} \omega(|\eta|)|g(\eta)|G_\mu(\varphi(\eta)) < \infty.$$

By Lemma 5 and Theorem 1, *(i)*⇔ *(iii)*. The proof is completed. □

Theorem 4. *Suppose* $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω *and* μ *are normal on* $[0,1)$ *such that* $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is bounded. Then the following statements hold.*

(i) If $\lim_{|z|\to 1} G_\mu(z) < \infty$, $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is compact.*

(ii) if $\lim_{|z|\to 1} G_\mu(z) = \infty$, *then the following statements are equivalent.*

(a) $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is compact.*

(b) $\lim_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0.$

(c) $\lim_{|\varphi(z)|\to 1} \omega_+(|z|)|g'(z)|G_\mu(\varphi(z)) = 0$ *and* $\lim_{|\varphi(z)|\to 1} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0.$

Proof. Since $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded, (19) holds.

(i). Suppose $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then $\{f_n'\}$ is also bounded in $\mathcal{B}_\mu$ and $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Lemma 4, $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n'(z)| = 0$. Using (19),

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} \omega(|z|)|(C_\varphi^g f_n)'(z)| = \lim_{n\to\infty} \sup_{z\in\mathbb{D}} \omega(|z|)|g(z) f_n'(\varphi(z))| \lesssim \lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n'(\varphi(z))| = 0.$$

Thus $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact.

(ii). *(b)*⇒*(a)*. Assume that $\lim_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0$. Then for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$, such that

$$\omega(|z|)|g(z)|G_\mu(\varphi(z)) < \varepsilon, \text{ when } \delta < |\varphi(z)| < 1.$$

Suppose that $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. By (19) and Remark 1,

$$\begin{aligned}
\|C_\varphi^g f_n\|_{\mathcal{B}_\omega} &= \sup_{z\in\mathbb{D}} \omega(|z|)|g(z) f_n'(\varphi(z))| \\
&\le \sup_{|\varphi(z)|\le\delta} \omega(|z|)|g(z) f_n'(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|g(z) f_n'(\varphi(z))| \\
&\lesssim \sup_{|\varphi(z)|\le\delta} |f_n'(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|g(z)|G_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu} \\
&\lesssim \sup_{|\varphi(z)|\le\delta} |f_n'(\varphi(z))| + \varepsilon,
\end{aligned}$$

which implies that $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact.

(a)⇒(b). Assume that $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. From the proof of Theorem 2, we see that $\{k_n\}$ is bounded in $\mathscr{Z}_\mu$ and $k_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemmas 1, 2 and 6,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\omega(|z_n|)|g(z_n)|\left(\int_0^{|a_n|^2} \frac{1}{\mu(s)}ds\right)^2}{\int_0^{|a_n|} \frac{1}{\mu(s)}ds} \\
\approx\ & \lim_{n\to\infty} \omega(|z_n|)\left|(C_\varphi^g k_n)'(z_n)\right| \le \lim_{n\to\infty} \|C_\varphi^g k_n\|_{\mathcal{B}_\omega} = 0,
\end{aligned}$$

which implies $\lim_{n\to\infty} \omega(|z_n|)|g(z_n)G_\mu(\varphi(z_n))| = 0$. So $\lim_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0$.

Using Lemma 5 and Theorem 2, we see that *(a)⇔(c)*. The proof is completed. □

4 The boundness and compactness of $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega(\mathcal{B}_\omega)$

From Lemma 5, Theorems 1 and 2, notice that $\lim_{|z|\to 1} G_{\mu_+}(z) = \infty$, we have the following theorems.

Theorem 5. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. Then $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) < \infty \quad \textit{and} \qquad \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} < \infty.$$

Theorem 6. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$ such that $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega$ is bounded. Then C_φ^g is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \omega(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) = 0 \quad \textit{and} \qquad \lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} = 0.$$

Theorem 7. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ *is bounded.*

(ii) $\sup_{z\in\mathbb{D}} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} < \infty$.

(iii) $\sup_{z\in\mathbb{D}} \omega_+(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) < \infty$ *and* $\quad \sup_{z\in\mathbb{D}} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} < \infty$.

(iv) $\sup_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_{\mu_+}(\varphi(z)) < \infty$.

Proof. (ii)⇔(i). By Lemma 3 and taking the function $f(z) = \int_0^z \mu_*(\eta)d\eta \in \mathcal{B}_\mu$, we can get the desired result. Since the proof is similar to the proof of Theorem 1, we omit the details.

By Lemma 5, Theorems 1 and 3, we see that *(i)⇔ (iii)* and *(i)⇔ (iv)* hold. The proof is completed. □

Theorem 8. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$ such that $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is bounded. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ *is compact.*

(ii) $\lim\limits_{|\varphi(z)|\to 1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} = 0.$

(iii) $\lim\limits_{|\varphi(z)|\to 1} \omega_+(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) = 0,\ \lim\limits_{|\varphi(z)|\to 1} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} = 0.$

(iv) $\lim\limits_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_{\mu_+}(\varphi(z)) = 0.$

Proof. Since $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is bounded, we get that $\sup\limits_{z\in\mathbb{D}} \omega(|z|)|g(z)| = \|C_\varphi^g z\|_{\mathcal{B}_\omega} < \infty$.

(ii)⇒(i). Suppose $\lim\limits_{|\varphi(z)|\to 1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} = 0$. For any $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$\frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \text{ when } \delta < |\varphi(z)| < 1.$$

Let $\{f_n\}$ be bounded in $\mathcal{B}_\mu$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Cauchy estimate, $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 3,

$$\begin{aligned}
\|C_\varphi^g f_n\|_{\mathcal{B}_\omega} &= \sup_{z\in\mathbb{D}} \omega(|z|)|f_n'(\varphi(z))g(z)| \\
&\le \sup_{|\varphi(z)|\le\delta} \omega(|z|)|f_n'(\varphi(z))g(z)| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|f_n'(\varphi(z))g(z)| \\
&\lesssim \sup_{|\varphi(z)|\le\delta} |f_n'(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)}\|f_n\|_{\mathcal{B}_\mu} \\
&\lesssim \sup_{|\varphi(z)|\le\delta} |f_n'(\varphi(z))| + \varepsilon,
\end{aligned}$$

which implies that $\lim\limits_{n\to\infty} \|C_\varphi^g f_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is compact.

(i)⇒(ii). Suppose that $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim\limits_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and $f_n(z) = \mu(|a_n|)\int_0^{\overline{a_n}z} \mu_*^2(\eta)d\eta$. Then $\{f_n\}$ is bounded in $\mathcal{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 6, $\lim\limits_{n\to\infty} \|C_\varphi^g f_n\|_{\mathcal{B}_\omega} = 0$. Therefore

$$\lim_{n\to\infty} \frac{\omega(|z_n|)|g(z_n)|}{\mu(|\varphi(z_n)|)} = \lim_{n\to\infty} \omega(|z_n|)|(C_\varphi^g f_n)'(z_n))| = 0,$$

which implies that (ii) holds.

By Lemma 5, Theorems 2 and 4, *(i)⇔(iii)* and *(i)⇔(iv)* hold. The proof is completed. □

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Convergence and error estimates for the series solutions of higher-order differential equations

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Abstract

Little work on the convergence and error estimates of approximate series solutions exists in the literature. For general nth-order linear differential equations with initial conditions, a rigorous proof of convergence for the series solutions given by the homotopy analysis method is first presented in this paper. Furthermore, an upper bound for the absolute error of these approximations is obtained.

1 Introduction

Higher-order differential equations arise in various branches of science and engineering. However, unlike numerical solutions, little work on the convergence and error estimates of the approximate series solutions to these equations can be found in the literature.

Consider general nth-order linear differential equations with initial conditions

$$\begin{cases} L[u(x)] = f(x), \\ u^{(i)}(x_0) = A_i, \quad i = 0, \cdots, n-1 \end{cases} \tag{1}$$

where

$$L := \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x),$$

$p_i(x)$, $i = 0, 1, \cdots, n-1$ and $f(x)$ are continuous in some neighborhood $[x_0 - \delta, x_0 + \delta]$ of x_0. The main purpose of this paper is to present a rigorous proof of convergence for the series solutions given by the homotopy analysis method and to establish an upper bound for the absolute error of these approximations.

The homotopy analysis method (or HAM) [1, 2] is a popular analytic approach for seeking series solutions to differential equations and related problems. It has been applied to solve many problems in different fields of science and engineering [3, 4, 5, 6, 7, 8, 9, 10].

For the sake of easy reference, the homotopy analysis method is briefly described as follows. Given a (usually nonlinear) problem

$$\mathcal{N}[u(x)] = 0, \ x \in \Omega, \tag{2}$$

one first constructs a zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(x;q) - u_0(x)] = q\, c_0\, \mathcal{N}[\phi(x;q)], \tag{3}$$

where $\mathcal{L}$ is an auxiliary linear operator, $u_0(x)$ an initial guess satisfying the given initial/boundary conditions, and $c_0 \neq 0$ the convergence-control parameter. At $q = 0$ and $q = 1$, one has

$$\phi(x;0) = u_0(x), \ \phi(x;1) = u(x), \tag{4}$$

respectively. To seek a series solution, one expands $\phi(x;q)$ into a Taylor series at $q = 0$

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x;c_0) q^m. \tag{5}$$

Assuming that c_0 is properly chosen so that the series (5) converges at $q = 1$, then

$$u(x) = \phi(x;1) = \sum_{m=0}^{+\infty} u_m(x;c_0) \tag{6}$$

must be one of the solutions to the given problem as shown in [1], where $u_m(x;c_0)$ is governed by the mth-order deformation equation

$$\mathcal{L}[u_m(x;c_0) - \chi_m u_{m-1}(x;c_0)] = \frac{c_0}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\phi(x;q)]}{\partial q^{m-1}} \right|_{q=0}, \tag{7}$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

In practice, one can only calculate an Nth-order approximation

$$\psi_N(x;c_0) = \sum_{m=0}^{N} u_m(x,c_0) \tag{8}$$

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of which the higher-order terms $u_m(x;c_0), m \geq 1$ are calculated via (7).

To obtain an accurate approximation (8), the optimal value c_0 is determined by minimizing the average residual error

$$\mathbf{E}(c_0) = \frac{1}{M}\sum_{j=1}^{M}\left(\mathcal{N}[\psi_N(x_j;c_0)]\right)^2, \tag{9}$$

where $x_1, x_2 \cdots, x_M \in \Omega$ are sample points.

For the problem (1), a rigorous proof of convergence for the series solutions given by the HAM is presented in Section 2. Moreover, an approach is also given for determining the valid region of c_0 that ensures the convergence of (6), and for obtaining an upper bound for the absolute error of the Nth-order approximation (8). In Section 3, two examples are given to illustrate the procedure and to demonstrate how accurate and convergent series solutions can be obtained. Some concluding remarks are given in the last section.

2 Convergence and error estimates

Let $F(x)$ be continuous on $[x_0-\delta, x_0+\delta]$. Denote

$$\|F(\cdot)\| := \max_{x\in[x_0-\delta,x_0+\delta]} |F(x)|.$$

For the initial value problem (1), it is assumed for the rest of the paper that the neighborhood $[x_0-\delta, x_0+\delta]$ of x_0 is sufficiently small so that

$$1-\sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\| > 0. \tag{10}$$

First, one constructs the zeroth-order deformation equation (3) with an initial guess $u_0(x)$ satisfying the initial conditions in (1)

$$u^{(i)}(x_0) = A_i, \qquad i = 0, \cdots, n-1. \tag{11}$$

The linear operator and nonlinear operator are set out below

$$\mathcal{L}[\phi(x;q)] = \frac{\partial^n\phi(x;q)}{\partial x^n}, \qquad \mathcal{N}[\phi(x;q)] = L[\phi(x;q)] - f(x).$$

Following the procedure outlined in Section 1, one obtains a series solution

$$u(x;c_0) = \sum_{m=0}^{+\infty} u_m(x;c_0) \tag{12}$$

and an Nth-order approximation

$$\psi_N(x; c_0) = \sum_{m=0}^{N} u_m(x; c_0) \tag{13}$$

to the initial value problem (1). For these solutions, one has the following convergence theorem:

Theorem 1. *For* $c_0 \in \left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}, -\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right]$, *the series solution (12) converges to the true solution* $u(x)$ *on* $[x_0-\delta, x_0+\delta]$, *where* ϵ *is a small number satisfying* $0 < \epsilon < 1 - \sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|$. $K\|E_N(\cdot; c_0)\|$ *is an upper bound for the absolute error of the* N*th-order approximation (13), where* $K = \frac{\delta^n}{(n-1)!\left[1-\sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\right]}$ *and* $E_N(x; c_0) = \frac{u_{N+1}^{(n)}(x;c_0)}{c_0}$.

Proof: Let

$$u(x) = \sum_{m=0}^{N} u_m(x; c_0) + R_N(x; c_0). \tag{14}$$

Then the series (12) converges to $u(x)$ on $[x_0-\delta, x_0+\delta]$ if and only if

$$\lim_{N\to\infty} \|R_N(\cdot; c_0)\| = 0. \tag{15}$$

To achieve this goal, we substitute (14) into the differential equation in (1), which gives a differential equation satisfied by $R_N(x; c_0)$,

$$L[R_N(x; c_0)] = -S_N(x; c_0) + f(x), \tag{16}$$

where

$$S_N(x; c_0) = \sum_{m=0}^{N} L[u_m(x; c_0)].$$

Since the initial guess $u_0(x)$ satisfies the initial conditions in (1), one sets

$$u_m(x_0; c_0) = 0, u_m'(x_0; c_0) = 0, \cdots, u_m^{(n-1)}(x_0; c_0) = 0 \quad \text{for } m \geq 1. \tag{17}$$

Consequently the initial conditions for the remainder $R_N(x; c_0)$ are

$$R_N(x_0; c_0) = R_N'(x_0; c_0) = \cdots = R_N^{(n-1)}(x_0; c_0) = 0. \tag{18}$$

Next, we want to simplify $S_N(x; c_0)$ in (16). Note that the mth-order deformation equation (7) becomes

$$u_m^{(n)}(x; c_0) - \chi_m u_{m-1}^{(n)}(x; c_0) = c_0[L(u_{m-1}(x; c_0)) - (1-\chi_m)f(x)]. \tag{19}$$

Adding both sides of (19) from $m=1$ to $N+1$ yields

$$\frac{1}{c_0}u_{N+1}^{(n)}(x;c_0)=S_N(x;c_0)-f(x). \tag{20}$$

Consequently (16) becomes

$$L[R_N(x;c_0)]=-\frac{1}{c_0}u_{N+1}^{(n)}(x;c_0). \tag{21}$$

Our next goal is to estimate $\|R_N(\cdot;c_0)\|$. Noticing (18), one has

$$\begin{aligned}R_N(x;c_0)&=\int_{x_0}^{x}R_N'(t_1;c_0)dt_1=\int_{x_0}^{x}\int_{x_0}^{t_2}R_N''(t_1;c_0)dt_1dt_2=\cdots\\&=\int_{x_0}^{x}\int_{x_0}^{t_{n-1}}\cdots\int_{x_0}^{t_2}R_N^{(n-1)}(t_1;c_0)dt_1\cdots dt_{n-2}dt_{n-1},\end{aligned}$$

which implies

$$\left\|R_N^{(k)}(\cdot;c_0)\right\|\le\frac{\delta^{n-1-k}}{(n-1-k)!}\left\|R_N^{(n-1)}(\cdot;c_0)\right\|,\quad k=0,1,\cdots,n-2. \tag{22}$$

Using the initial condition $R_N^{(n-1)}(x_0;c_0)=0$ and integrating (21) from x_0 to x give

$$\begin{aligned}R_N^{(n-1)}(x;c_0)=-\Bigg(&\int_{x_0}^{x}p_{n-1}(t)R_N^{(n-1)}(t;c_0)dt+\cdots+\int_{x_0}^{x}p_0(t)R_N(t;c_0)dt\\&+\frac{1}{c_0}\int_{x_0}^{x}u_{N+1}^{(n)}(t;c_0)dt\Bigg).\end{aligned}$$

Following a similar reasoning as above, one arrives at

$$\left\|R_N^{(n-1)}(\cdot;c_0)\right\|\le\sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\cdot\left\|R_N^{(n-1)}(\cdot;c_0)\right\|+\frac{\delta\left\|u_{N+1}^{(n)}(\cdot;c_0)\right\|}{|c_0|}.$$

In view of (10), one has

$$\left\|R_N^{(n-1)}(\cdot;c_0)\right\|\le\frac{\delta}{1-\sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|}\frac{\left\|u_{N+1}^{(n)}(\cdot;c_0)\right\|}{|c_0|}. \tag{23}$$

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Combining (22) and (23), one finally achieves

$$\|R_N(\cdot;c_0)\| \le \frac{\delta^n}{(n-1)!\left[1-\sum_{k=1}^n \frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\right]}\frac{\left\|u_{N+1}^{(n)}(\cdot;c_0)\right\|}{|c_0|}.$$

Consequently

$$\|R_N(\cdot;c_0)\| \le K\,\|E_N(\cdot;c_0)\|\,, \tag{24}$$

where

$$K=\frac{\delta^n}{(n-1)!\left[1-\sum_{k=1}^n \frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\right]} \quad \text{and} \quad E_N(x;c_0)=\frac{u_{N+1}^{(n)}(x;c_0)}{c_0}.$$

Therefore, we have proved that $K\,\|E_N(\cdot;c_0)\|$ is an upper bound for the absolute error of the Nth-order approximation (13) on $[x_0-\delta, x_0+\delta]$. Our next goal is to prove that, if

$$c_0 \in \left[-\frac{2-\epsilon}{1+\sum_{k=1}^n \frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}, -\frac{\epsilon}{1-\sum_{k=1}^n \frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right],$$

then

$$\lim_{N\to\infty}\|R_N(\cdot;c_0)\|=0. \tag{25}$$

First we figure out an expression for $E_N(x;c_0)$. The following lemma can be proved by mathematical induction.

Lemma 2. *Noticing* $E_N(x;c_0)=\frac{u_{N+1}^{(n)}(x;c_0)}{c_0}$, *one has*

$$E_N(x;c_0)=\sum_{k=0}^N \binom{N}{k} a_k(x)c_0^k, \tag{26}$$

where $a_0(x)=L[u_0(x;c_0)]-f(x)$, *and for* $0\le k\le N-1$,

$$a_{k+1}(x)=a_k(x)+p_{n-1}(x)\int_{x_0}^x a_k(t_1)dt_1+p_{n-2}(x)\int_{x_0}^x\int_{x_0}^{t_2} a_k(t_1)dt_1dt_2 + \cdots + p_0(x)\int_{x_0}^x\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_2} a_k(t_1)dt_1\cdots dt_{n-1}dt_n. \tag{27}$$

Based on Lemma 2, one can determine the relation between $E_N(x;c_0)$ and $E_{N+1}(x;c_0)$. In view of (17), (26) and (27), one has

$$\begin{aligned}
&E_{N+1}(x;c_0)\\
&=(1+c_0)\frac{u_{N+1}^{(n)}(x;c_0)}{c_0}+p_{n-1}(x)u_{N+1}^{(n-1)}(x;c_0)+\cdots+p_0(x)u_{N+1}(x;c_0)\\
&=(1+c_0)E_N(x;c_0)+c_0p_{n-1}(x)\int_{x_0}^{x}E_N(t_1;c_0)dt_1+\cdots\\
&+c_0p_0(x)\int_{x_0}^{x}\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_2}E_N(t_1;c_0)dt_1\cdots dt_{n-1}dt_n,
\end{aligned}$$

which implies

$$\|E_{N+1}(\cdot;c_0)\|\le\left[|1+c_0|+|c_0|\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right]\|E_N(\cdot;c_0)\|.$$

Next, let

$$w(\delta,c_0)=|1+c_0|+|c_0|\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|. \tag{28}$$

Case I. If $c_0\in\left[-1,-\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right]$, then (28) becomes

$$w(\delta,c_0)=1+c_0\left(1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right)\le 1-\epsilon.$$

Case II. If $c_0\in\left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|},-1\right)$, then (28) becomes

$$w(\delta,c_0)=-1-c_0\left(1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right)\le 1-\epsilon.$$

One thus concludes that if

$$c_0\in\left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|},-\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right], \tag{29}$$

then

$$w(\delta,c_0)\le 1-\epsilon<1.$$

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Thus $E_N(x; c_0)$ is a contraction mapping

$$\|E_{N+1}(\cdot; c_0)\| \leq w(\delta, c_0) \|E_N(\cdot; c_0)\| \leq (1-\epsilon) \|E_N(\cdot; c_0)\| .$$

Iteration then yields

$$\|E_N(\cdot; c_0)\| \leq (1-\epsilon)^N \|E_0(\cdot; c_0)\| ,$$

Consequently

$$\lim_{N\to\infty} \|E_N(\cdot; c_0)\| = 0 \quad \text{and} \quad \lim_{N\to\infty} \|R_N(\cdot; c_0)\| = 0,$$

and the theorem has finally been proved. ♠

3 Examples

In this section, we apply the approach to investigate some initial value problems.

3.1 Buckling of a cantilever bar

A cantilever bar of length l is free at the upper end and is built-in at the bottom. The axial load P is supposed to be uniformly distributed along the bar axis. The deflection w satisfies the differential equation

$$\frac{d^3w}{dx^3} + \frac{P}{EI}(l-x)\frac{dw}{dx} = 0, \tag{30}$$

where EI is the bending rigidity. The initial conditions are

$$w(0) = 0, \qquad \left.\frac{dw}{dx}\right|_{x=0} = 0, \qquad \left.\frac{d^2w}{dx^2}\right|_{x=0} = 1. \tag{31}$$

A simple way to reduce the order of the equation is to set $u(x) = dw/dx$, but a more interesting way is as follows.

We make a change of variable

$$z = \frac{2}{3}\sqrt{\frac{P}{EI}}(l-x)^{\frac{3}{2}}. \tag{32}$$

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Table 1: Upper bound for the absolute error of the Nth-order approximation.

N	2nd order	3rd order	4th order	5th order	6th order
$K \Vert E_N(\cdot;-1)\Vert$	1.190E-2	1.146E-3	8.188E-5	5.794E-6	3.800E-7

Step by step differentiation yields

$$\begin{aligned} \frac{dw}{dx} &= -\frac{dw}{dz}\sqrt[3]{\frac{3P}{2EI}}z, \\ \frac{d^2w}{dx^2} &= \left(\frac{3P}{2EI}\right)^{\frac{2}{3}}\left(\frac{1}{3}z^{-\frac{1}{3}}\frac{dw}{dz} + z^{\frac{2}{3}}\frac{d^2w}{dz^2}\right), \\ \frac{d^3w}{dx^3} &= \frac{3P}{2EI}\left(\frac{1}{9}z^{-1}\frac{dw}{dz} - \frac{d^2w}{dz^2} - z\frac{d^3w}{dz^3}\right). \end{aligned} \tag{33}$$

Introducing this in (30) and using the notation

$$\frac{dw}{dz} = u, \tag{34}$$

we get

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{1}{9z^2}\right)u = 0. \tag{35}$$

It turns out that (35) is a differential equation of Bessel type. For computational purposes, one sets $l = 1, EI = 1$, and $P = \left(\frac{1000\pi}{1122}\right)^2$, the buckling critical load (see [11]). Then the initial conditions become

$$u(z_0) = 0, \quad u'(z_0) = \frac{1}{P}, \tag{36}$$

where $z_0 = \frac{1000\pi}{1683}$. Now let us solve the initial value problem (35)-(36).

Following the procedure outlined at the beginning of Section 2, one obtains a series solution (12) and an Nth-order approximation (13) to the problem (35)-(36).

The radius $\delta = \frac{3}{5}$ of the neighborhood of $z_0 = \frac{1000\pi}{1683}$ is determined by the condition (10). Then a valid region $[-1.211828346 + 0.6059141731\,\epsilon, -2.860401756\,\epsilon]$ of c_0 is obtained by means of (29), where $0 < \epsilon < 0.3496012397$.

It follows from Theorem 1 that, for $c_0 \in [-1.211828346 + 0.6059141731\,\epsilon, -2.860401756\,\epsilon]$, the series solution (12) converges on $\left[\frac{1000\pi}{1683} - \frac{3}{5}, \frac{1000\pi}{1683} + \frac{3}{5}\right]$.

To get an accurate approximation, the optimal value of c_0 is determined by minimizing the averaged residual error (9) of the 5th-order approximation (13). It turns out that $c_0 = -1$. Notice that $-1 \in [-1.211828346 +$

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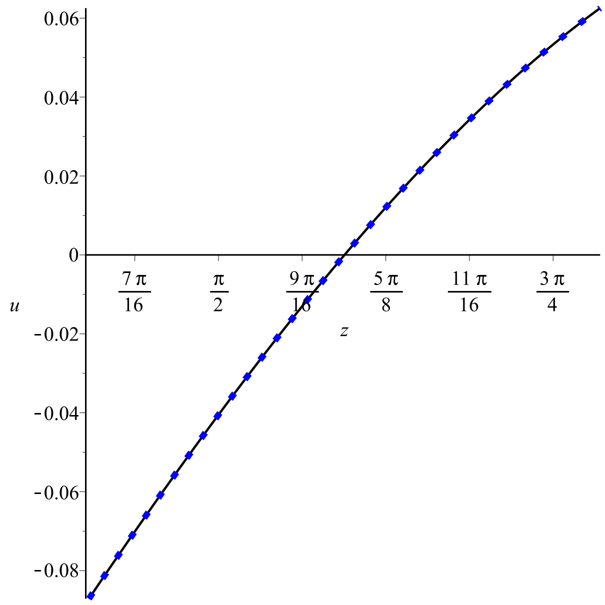

Figure 1: The solid line: numerical solution; the dot line: the 5th-order HAM approximation.

$0.6059141731\,\epsilon, -2.860401756\,\epsilon]$. So the corresponding series solution (12) does converge.

The upper bounds $K\,\|E_N(\cdot;-1)\|$ for the absolute error of the Nth-order approximation (13) when $N = 2, 3, 4, 5$, and 6 on $\left[\frac{1000\pi}{1683} - \frac{3}{5}, \frac{1000\pi}{1683} + \frac{3}{5}\right]$ are calculated, as shown in Table 1. It is very accurate as shown in Figure 1.

3.2 Third-order equation with variable coefficients

Consider the third-order initial value problem (see [12])

$$\begin{aligned} &u^{(3)}(x) + xu''(x) + x^{\frac{2}{3}}u'(x) + x^{\frac{1}{3}}u(x) = 0, \\ &u(1) = 1,\ u'(1) = 0,\ u''(1) = 1. \end{aligned} \tag{37}$$

This initial value problem does not have a closed-form solution.

Following the procedure outlined at the beginning of Section 2, one obtains a series solution (12) and an Nth-order approximation (13) to the problem (37).

The radius $\delta = \frac{9}{20}$ of the neighborhood of $x_0 = 1$ is determined by the condition (10). Then a valid region $[-1.111481567{+}0.5557407833\,\epsilon, -4.985046404\,\epsilon]$ of c_0 is obtained by means of (29), where $0 < \epsilon < 0.2005999381$.

Table 2: Upper bound for the absolute error of the Nth-order approximation.

N	5th order	10th order	15th order	20th order
$K\,\|E_N(\cdot;-1)\|$	4.000E-4	1.116E-8	1.347E-13	1.017E-18

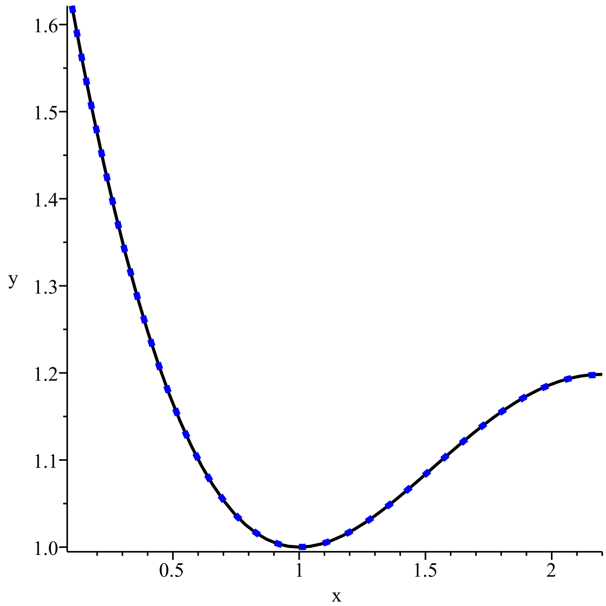

Figure 2: The solid line: numerical solution; the dot line: the 5th-order HAM approximation.

It follows from Theorem 1 that, for $c_0 \in [-1.111481567 + 0.5557407833\,\epsilon, -4.985046404\,\epsilon]$, the series solution (12) converges on $\left[\frac{11}{20}, \frac{29}{20}\right]$.

To get an accurate approximation, the optimal value of c_0 is determined by minimizing the averaged residual error (9) of the 5th-order approximation (13). It turns out that $c_0 = -1$. Notice that $-1 \in [-1.111481567 + 0.5557407833\,\epsilon, -4.985046404\,\epsilon]$. So the corresponding series solution (12) does converge.

The upper bounds $K\,\|E_N(\cdot;-1)\|$ for the absolute error of the Nth-order approximation (13) when $N = 5, 10, 15$ and 20 on $\left[\frac{11}{20}, \frac{29}{20}\right]$ are calculated, as shown in Table 2. It is very accurate as shown in Figure 2.

4 Conclusion

For general nth-order linear differential equations with initial conditions, we have presented a rigorous proof of convergence for the series solutions given

by the HAM for the first time. Furthermore, we have proposed an approach for seeking convergent series solutions to these problems. Some outstanding features of the approach include the determination of a valid region of the convergence-control parameter for ensuring convergence, and the calculation of an upper bound for the absolute error of an approximation.

Some issues still deserve further investigations. For example, how can one enlarge the neighborhood $[x_0 - \delta, x_0 + \delta]$ of x_0 so that Theorem 1 is still valid? How can one extend the techniques to investigate the convergence of nonlinear problems? It is believed that substantial work has to be done for dealing with these issues.

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ON THE GENERALIZED Z-ALGORITHM FOR THE NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. For any d-dimensional neutral stochastic functional differential equation with infinite delay and m-dimensional Brownian motion, we introduce a sequence of approximate equations and offer sufficient conditions such that the approximate solutions converge with probability one to the solution of the given equation. This iterative method called the generalized Z-algorithm is a generalization to many well-known analytic iterative method.

1. Introduction

The neutral stochastic functional differential equation, abbreviated as NSFDE, which was introduced by Kolmanovskii and Nosov [5], has been received much more attention in recent years. The existence and uniqueness theorems of the solution to the NSFDE with finite delay can be seen in Mao's books [6]. Recently, the existence of the solution to NSFDEs with infinite delay has been established in [1, 8, 9] by the classic Picard iteration argument. In 2010, S. Janković, M. Vasilova and M. Krstić [4] utilized successfully a general analytic method called the Z-algorithm to verify the existence of the solutions to NSFDEs with finite delay.

Actually, the Z-algorithm method could be backed to works [10, 11] in which Zuber posed an analytic iterative method for solving the Cauchy problem of the ordinary differential equation $X' = f(t, X)$ with $X(t_0) = X_0$. In fact, Zuber considered the approximate equations $X'_{n+1} = f_n(t; X_{n+1})$ with $X_{n+1}(t_0) = X_0$ for $n = 0, 1, \cdots$. It was showed in [10] that if $\sum_{n=1}^{\infty} \sup_{|t-t_0|<\varepsilon} |f(t, X_n(t)) - f_n(t, X_n(t))| < 1$, then the sequence of the solutions $\{X_n\}$ converges to the solution X of the initial equation, uniformly in an interval around t_0. Here we remark that, by the advantages of the Z-algorithm method, if we choose the functions $\{f_n\}$ good enough so that the approximate equations can be effectively solved, then the solution X of the initial equation can be effectively approximated. Later, Janković

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[2, 3] applied an analogous analytic method to the following stochastic differential equation of the Itô type,

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t), \quad t \in [0, T] \tag{1.1}$$

$$X(0) = X_0, \tag{1.2}$$

by comparing its solution with the solutions to the related equations

$$dX_{n+1}(t) = a_n(t, X_{n+1}(t))dt + b_n(t, X_{n+1}(t))dB(t), \quad t \in [0, T] \tag{1.3}$$

$$X_{n+1}(0) = X_0, \tag{1.4}$$

for $n = 0, 1, ...$ and some suitable functions $\{a_n\}$ and $\{b_n\}$. It was shown in [2] that, if

$$\sum_{n=1}^{\infty} \sup_{t,X}\{|a(t, X) - a_n(t, X)| + |b(t, X) - b_n(t, X)|\} < \infty,$$

then the solutions $X_{n+1}(t)$ of (1.3) converge to the solution $X(t)$ of (1.1) a.s. uniformly in $[0, T]$ as $n \to \infty$.

Motivated by the works mentioned above, we will discuss in this paper the existence of the Z-algorithm approximate solutions to NSFDEs with infinite delay. Our main theorem will be stated and shown in the next section, see Theorem 2.4; and some comments will be given in section 3. Here we need to give the notations and definitions which will be used in the paper.

Let $|\cdot|$ denote the Euclidean norm in $\mathbf{R}^d$. If A is a vector or a matrix, its transpose is denoted by A^T; The trace norm of a matrix A is represented by $|A| = \sqrt{trace(A^T A)}$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq t_0}$ satisfying the usual conditions, i.e., it is right continuous and $\mathscr{F}_{t_0}$ contains all P-null sets. Assume that $B(t)$ is a m-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, P)$, that is $B(t) = \big(B_1(t), B_2(t), ..., B_m(t)\big)^T$ and each $B_i(t)$ is a standard Brownian motion for $i = 1, 2, \cdots, n$. Let $BC((-\infty, 0]; \mathbf{R}^d)$ denote the family of bounded continuous $\mathbf{R}^d$-value functions φ defined on $(-\infty, 0]$ with the norm $||\varphi|| = \sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$.

In this paper, we consider the following d-dimensional NSFDEs with infinite delay,

$$d\,[X(t) - D(X_t)] = f(t, X_t)dt + g(t, X_t)dB(t), \quad t_0 \leq t \leq T, \tag{1.5}$$

where

$$X_t = \big\{X(t + \theta) : -\infty < \theta \leq 0\big\}$$

can be regarded as a $BC((-\infty, 0]; \mathbf{R}^d)$-valued stochastic process; and we assume that D is a vector-value function from $BC((-\infty, 0]; \mathbf{R}^d)$ to $\mathbf{R}^d$, and f is a Borel measurable function from $[t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d)$ to $\mathbf{R}^d$, and g is a matrix-value Borel measurable function from $[t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d)$ to $\mathbf{R}^{md}$. Here one notes that the last term, $g(t, X_t)dB(t)$, in equation (1.5) can be rewritten as

$$g_1(t, X_t)dB_1(t) + g_2(t, X_t)dB_2(t) + \cdots + g_m(t, X_t)dB_m(t).$$

The initial value is assumed to be

$$X_{t_0} = \{\xi(\theta) : -\infty < \theta \leq 0\} \tag{1.6}$$

where $\xi(\theta)$ is an $\mathscr{F}_{t_0}$-measurable $BC((-\infty, 0]; \mathbf{R}^d)$-value random variable such that $\mathbf{E}||\xi||^2 < \infty$.

Definition 1.1. $\mathbf{R}^d$-value stochastic process $X(t)$ defined on $(-\infty, T]$ is called the solution of (1.5) with initial value (1.6), if $X(t)$ has the following properties:

(1) $X(t)$ is continuous and $\{X(t): -\infty \leq t \leq T\}$ is $\mathscr{F}_t$-adapted;
(2) $f(t, X_t) \in \mathscr{L}^1([t_0, T]; \mathbf{R}^d)$ and $g(t, X_t) \in \mathscr{L}^2([t_0, T]; \mathbf{R}^{d\times m})$;
(3) $X_{t_0} = \xi$, and for each $t_0 \leq t \leq T$,

$$X(t) = \xi(0) + D(X_t) - D(\xi) + \int_{t_0}^t f(s, X_s)ds + \int_{t_0}^t g(s, X_s)dB(s), \quad a.s. \tag{1.7}$$

We say that the solution of (1.5) with initial value (1.6), $X(t)$, is the unique solution, if for any other solution $\bar{X}(t)$ distinguishable with $X(t)$ we have

$$P\left\{X(t) = \bar{X}(t): -\infty < t \leq T\right\} = 1$$

Here we remarked that the existence and uniqueness of the solution to the equation (1.5) were established by the well-known Picard iteration in [1, 8, 9] under the following assumptions:

(M1) There exists a positive constant K such that, for all $\varphi, \psi \in BC((-\infty, 0]; \mathbf{R}^d)$ and $t \in [t_0, T]$,

$$|f(t,\varphi) - f(t,\psi)| + |g(t,\varphi) - g(t,\psi)| \leq K||\varphi - \psi||$$

(M2) There exists a positive constant $\bar{K}$ such that, for all $\varphi \in BC((-\infty, 0]; \mathbf{R}^d)$ and $t \in [t_0, T]$,

$$|f(t,\varphi)| + |g(t,\varphi)| \leq \bar{K}(1 + ||\varphi||)$$

(M3) There exists a constant $k \in (0, 1)$ such that, for all $\varphi, \psi \in BC((-\infty, 0]; \mathbf{R}^d)$,

$$|D(\varphi) - D(\psi)| \leq k||\varphi - \psi||$$

Recall the stochastic integral equation (1.7), we introduce the following sequence of the related equations:

$$\begin{aligned} X^{n+1}(t) - D_n(X_t^{n+1}) &= \xi^{n+1}(0) - D_n(\xi^{n+1}) + \int_{t_0}^t f_n(s, X_s^{n+1})ds \\ &\quad + \int_{t_0}^t g_n(s, X_s^{n+1})dB(s) \end{aligned} \tag{1.8}$$

for $t \in [t_0, T]$ and $n = 0, 1, \cdots$, where $X_t^{n+1} = \{X^{n+1}(t+\theta): -\infty < \theta \leq 0\}$ are $BC((-\infty, 0]; \mathbf{R}^d)$-value stochastic processes, $\xi^{n+1} = X_{t_0}^{n+1}$ are the initial conditions. The functions D_n, f_n, g_n will be chosen late such that

$$D_n(X) \to D(X),\ f_n(t, X) \to f(t, X),\ g_n(t, X) \to g(t, X),\ \text{as } n \to \infty,$$

uniformly in $[t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d)$.

Let $X(t)$ be the unique $\mathscr{F}_t$-adapted solution to the equation (1.7) satisfying $\mathbf{E}\sup_{t\in(-\infty,T]} |X(t)|^p < \infty$. Let $X^{n+1}(t)$ be the unique $\mathscr{F}_t$-adapted solution to the equation (1.8) satisfying $\mathbf{E}\sup_{t\in(-\infty,T]} |X^{n+1}(t)|^p < \infty$ for $n = 0, 1, \cdots$.

We will use the following notations in this paper,

$$\begin{aligned} \gamma_n &= \mathbf{E}||\xi - \xi^n||^p, \\ \delta_n &= \mathbf{E}\sup_{t\in[t_0,T]} |D(X_t^n) - D_n(X_t^n)|^p, \\ \varepsilon_n &= \mathbf{E}\sup_{t\in[t_0,T]} \left[|f(t, X_t^n) - f_n(t, X_t^n)|^p + |g(t, X_t^n) - g_n(t, X_t^n)|^p\right]. \end{aligned}$$

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We take the initial iteration to be $X^0(t) = \xi(0)$ a.s., for $t \in [t_0, T]$, and $X^0_{t_0} = \xi$. We will use the following conditions:

$$\sum_{n=0}^{\infty} \gamma_n < \infty, \quad \sum_{n=0}^{\infty} \delta_n < \infty, \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty \tag{1.9}$$

2. The main theorem and its proof

At first we give three lemmas, which are useful for our investigation.

Lemma 2.1. *For any $X, Y \in \mathbf{R}^d$ and $\theta \in (0,1)$, we have*

$$|X+Y|^p \leq \frac{|X|^p}{(1-\theta)^{p-1}} + \frac{|Y|^p}{\theta^{p-1}}, \quad p \geq 1$$

The proof of Lemma 2.1 can be found in [6]

Lemma 2.2. *Let $u, v : [a,b] \to \mathbf{R}_+$ be continuous functions and L be a positive constant, if*

$$u(t) \leq v(t) + L\int_a^t u(s)ds,$$

then for all $t \in [a,b]$ we have

$$u(t) \leq v(t) + L\int_a^t e^{L(t-s)} v(s)ds.$$

Especially, if $v(t) = M$ is a constant, then $u(t) \leq Me^{L(t-a)}$.

Lemma 2.2 is a special case of the Gronwall lemma, so we omit its proof here.

Lemma 2.3. *Let $p \geq 2$, $t \in [t_0, T]$, and let $X_t, X_t^n \in BC((-\infty, 0]; \mathbf{R}^d)$ be the stochastic process mentioned above, then for any $r \in [t_0, t]$ we have*

$$||X_r - X_r^n||^p \leq ||\xi - \xi^n||^p + \sup_{u\in[t_0,r]} |X(u) - X^n(u)|^p.$$

Proof. From the definition of the norm in $BC((-\infty, 0]; \mathbf{R}^d)$, we have

$$\begin{aligned} ||X_r - X_r^n||^p &= \sup_{\theta\in(-\infty,0]} |X(r+\theta) - X^n(r+\theta)|^p \\ &= \sup_{u\in(-\infty,r]} |X(u) - X^n(u)|^p \\ &\leq \sup_{u\in(-\infty,t_0]} |X(u) - X^n(u)|^p + \sup_{u\in[t_0,r]} |X(u) - X^n(u)|^p \\ &= ||\xi - \xi^n||^p + \sup_{u\in[t_0,r]} |X(u) - X^n(u)|^p. \end{aligned}$$

The lemma is proved. □

Now we state our main theorem.

Theorem 2.4. *Let $p \geq 2$ and $\mathbf{E}||\xi||^p < \infty$, $\mathbf{E}||\xi^n||^p < \infty$, $n = 0, 1, \ldots$. Assume that the functions D, f, g, D_n, f_n, g_n satisfy the Lipschitz conditions (M1) and (M3) with constants $K > 0$ and $k \in \left(0, 1/(3 \cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$ for any $n = 0, 1, \ldots$. If the condition (1.9) is vilid. Then, the sequence of solutions $\left\{X^{n+1}(t) : t \in (-\infty, T], n = 0, 1, \ldots\right\}$ of the equations (1.8) converges uniformly in $[t_0, T]$, with probability one as $n \to \infty$, to the solution $\left\{X(t), t \in (-\infty, T]\right\}$ of equation (1.7).*

Proof. From (1.7) and (1.8), for all $r \in [t_0, T]$, we have

$$\begin{aligned} X(r) - X^{n+1}(r) &= \xi(0) - \xi^{n+1}(0) - D(\xi) + D_n(\xi^{n+1}) \\ &\quad + D(X_r) - D_n(X_r^{n+1}) \\ &\quad + \int_{t_0}^{r} [f(s, X_s) - f_n(s, X_s^{n+1})]ds \\ &\quad + \int_{t_0}^{r} [g(s, X_s) - g_n(s, X_s^{n+1})]dB(s). \end{aligned} \tag{2.1}$$

Since $k \in \left(0, 1/(3 \cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$ and $p > 1$, we can choose θ such that

$$(3k^p)^{\frac{1}{4(p-1)}} \le \theta < \frac{1}{2}. \tag{2.2}$$

Hence, for $t_0 \le r \le t \le T$, we see from Lemma 2.1 that

$$\mathbf{E} \sup_{r \in [t_0, t]} |X(r) - X^{n+1}(r)|^p \le \frac{I_1}{(1-\theta)^{p-1}} + \frac{J_1(t)}{\theta^{2(p-1)}} + \frac{J_2(t)}{(\theta(1-\theta))^{p-1}}, \tag{2.3}$$

where

$$\begin{aligned} I_1 &= \mathbf{E}|\xi(0) - \xi^{n+1}(0) - D(\xi) + D_n(\xi^{n+1})|^p \\ J_1(t) &= \mathbf{E} \sup_{r \in [t_0, t]} \left|D(X_r) - D_n(X_r^{n+1})\right|^p \\ J_2(t) &= \mathbf{E} \sup_{r \in [t_0, t]} \left| \int_{t_0}^{r} \left[f(s, X_s) - f_n(s, X_s^{n+1})\right] ds \right. \\ &\qquad \left. + \int_{t_0}^{r} \left[g(s, X_s) - g_n(s, X_s^{n+1})\right] dB(s) \right|^p \end{aligned}$$

Next we will give the estimates for I_1, J_1 and J_2, respectively. According to the condition (M3), we can deduce that

$$\begin{aligned} I_1 &\le 2^{p-1}[\mathbf{E}|\xi(0) - \xi^{n+1}(0)|^p + \mathbf{E}|D(\xi) - D_n(\xi^{n+1})|^p] \\ &\le 2^{p-1}\gamma_{n+1} + 8^{p-1}\mathbf{E}\big[|D(\xi) - D(\xi^n)|^p + |D(\xi^n) - D_n(\xi^n)|^p \\ &\qquad + |D_n(\xi^n) - D_n(\xi)|^p + |D_n(\xi) - D_n(\xi^{n+1})|^p\big] \\ &\le 2^{p-1}\gamma_{n+1} + 8^{p-1}k^p\mathbf{E}||\xi - \xi^n||^p + 8^{p-1}\delta_n + 8^{p-1}k^p\mathbf{E}||\xi - \xi^n||^p \\ &\qquad + 8^{p-1}k^p\mathbf{E}||\xi - \xi^{n+1}||^p \\ &= 2^{p-1}\gamma_{n+1} + 8^{p-1}(2k^p\gamma_n + k^p\gamma_{n+1} + \delta_n). \end{aligned} \tag{2.4}$$

For $J_1(t)$, from Lemma 2.1 and the condition (M3), we get

$$\begin{aligned} &|D(X_r) - D_n(X_r^{n+1})|^p \\ &\le \frac{1}{(1-\theta)^{p-1}}\Big[\frac{1}{(1-\theta)^{p-1}}|D(X_r) - D(X_r^n)|^p + \frac{1}{\theta^{p-1}}|D(X_r^n) - D_n(X_r^n)|^p\Big] \\ &\quad + \frac{1}{\theta^{p-1}}\Big[\frac{1}{(1-\theta)^{p-1}}|D_n(X_r^n) - D_n(X_r)|^p + \frac{1}{\theta^{p-1}}|D_n(X_r) - D_n(X_r^{n+1})|^p\Big] \\ &\le k^p\Big[\frac{1}{(1-\theta)^{2(p-1)}} + \frac{1}{(\theta(1-\theta))^{p-1}}\Big]||X_r - X_r^n||^p + \frac{\delta_n}{(\theta(1-\theta))^{p-1}} \\ &\quad + \frac{k^p}{\theta^{2(p-1)}}||X_r - X_r^{n+1}||^p. \end{aligned}$$

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Lemma 2.3 yields that

$$\mathbf{E}\sup_{r\in[t_0,t]}||X_r - X_r^n||^p \le \gamma_n + \mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^n(r)|^p$$

Therefore,

$$\begin{aligned} J_1(t) &\le k^p\Big[\frac{1}{(1-\theta)^{2(p-1)}} + \frac{1}{(\theta(1-\theta))^{p-1}}\Big]\Big[\gamma_n + \mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^n(r)|^p\Big] \\ &+ \frac{k^p}{\theta^{2(p-1)}}\Big[\gamma_{n+1} + \mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^{n+1}(r)|^p\Big] + \frac{\delta_n}{(\theta(1-\theta))^{p-1}}. \end{aligned} \tag{2.5}$$

For $J_2(t)$, we can decompose it into two party

$$\begin{aligned} J_2(t) &\le 2^{p-1}\left[\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r [f(s,X_s) - f_n(s,X_s^{n+1})]ds\Big|^p\right. \\ &\quad \left. + \mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r [g(s,X_s) - g_n(s,X_s^{n+1})]dB(s)\Big|^p\right] \\ &=: 2^{p-1}[J_{21}(t) + J_{22}(t)]. \end{aligned}$$

To estimate $J_{21}(t)$, using the Hölder inequality and the condition (M1), we get that

$$\begin{aligned} &J_{21}(t) \\ &\le 4^{p-1}\mathbf{E}\left[\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r [f(s,X_s) - f(s,X_s^n)]ds\Big|^p + \sup_{r\in[t_0,t]}\Big|\int_{t_0}^r [f(s,X_s^n) - f_n(s,X_s^n)]ds\Big|^p\right. \\ &\left. + \sup_{r\in[t_0,t]}\Big|\int_{t_0}^r [f_n(s,X_s^n) - f_n(s,X_s)]ds\Big|^p + \sup_{r\in[t_0,t]}\Big|\int_{t_0}^r [f_n(s,X_s) - f_n(s,X_s^{n+1})]ds\Big|^p\right] \\ &\le 4^{p-1}\Big[K^p(t-t_0)^{p-1}\int_{t_0}^t \mathbf{E}||X_s - X_s^n||^p ds + (t-t_0)^p\varepsilon_n \\ &\quad + K^p(t-t_0)^{p-1}\int_{t_0}^t \mathbf{E}||X_s - X_s^n||^p ds + K^p(t-t_0)^{p-1}\int_{t_0}^t \mathbf{E}||X_s - X_s^{n+1}||^p ds\Big] \\ &\le 4^{p-1}(T-t_0)^{p-1}\left[2K^p\int_{t_0}^t \mathbf{E}||X_s - X_s^n||^p ds\right. \\ &\qquad \left. + K^p\int_{t_0}^t \mathbf{E}||X_s - X_s^{n+1}||^p ds + (t-t_0)\varepsilon_n\right]. \end{aligned}$$

Hence by lemma 2.3 we obtain that

$$\begin{aligned} J_{21}(t) &\le 4^{p-1}(T-t_0)^{p-1}\Big[2K^p\int_{t_0}^t \big(\gamma_n + \mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^p\big)ds \\ &\quad + K^p\int_{t_0}^t \big(\gamma_{n+1} + \mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^p\big)ds + (t-t_0)\varepsilon_n\Big] \\ &= 4^{p-1}(T-t_0)^{p-1}\Big[2K^p\int_{t_0}^t \mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^p ds \\ &\quad + K^p\int_{t_0}^t \mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^p ds + (t-t_0)(2K^p\gamma_n + K^p\gamma_{n+1} + \varepsilon_n)\Big]. \end{aligned}$$

In order to estimate $J_{22}(t)$, we use Cauchy inequality to get that

$$\begin{aligned}
J_{22}(t) \quad \leq \quad & 4^{p-1}\Big[\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g(s,X_s)-g(s,X_s^n)]dB(s)\Big|^p \\
& +\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g(s,X_s^n)-g_n(s,X_s^n)]dB(s)\Big|^p \\
& +\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g_n(s,X_s^n)-g_n(s,X_s)]dB(s)\Big|^p \\
& +\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g_n(s,X_s)-g_n(s,X_s^{n+1})]dB(s)\Big|^p\Big]
\end{aligned}$$

Applying the Burkhölder-Davis-Gundy inequality and the Hölder inequality to the first Itô integral, we obtain that

$$\begin{aligned}
&\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g(s,X_s)-g(s,X_s^n)]dB(s)\Big|^p \leq c_p\mathbf{E}\Big[\int_{t_0}^{t}|g(s,X_s)-g(s,X_s^n)|^2ds\Big]^{\frac{p}{2}} \\
&\leq c_p(t-t_0)^{\frac{p}{2}-1}\int_{t_0}^{t}\mathbf{E}\big|g(s,X_s)-g(s,X_s^n)\big|^p ds
\end{aligned}$$

where c_p is a universal constant, more precisely, $c_p=[p^{p+1}/2(p-1)^{p-1}]^{\frac{p}{2}}$ for $p>2$ and $c_p=4$ for $p=2$. The other Itô integrals can be estimated analogously, Thus according to Lemma 2.3 and the condition (M1), we get that

$$\begin{aligned}
&J_{22}(t) \\
&\leq 4^{p-1}c_p(t-t_0)^{\frac{p}{2}-1}\Big[\int_{t_0}^{t}\mathbf{E}|g(s,X_s)-g(s,X_s^n)|^pds+\int_{t_0}^{t}\mathbf{E}|g(s,X_s^n)-g_n(s,X_s^n)|^pds \\
&\quad +\int_{t_0}^{t}\mathbf{E}|g_n(s,X_s^n)-g_n(s,X_s)|^pds+\int_{t_0}^{t}\mathbf{E}|g_n(s,X_s)-g_n(s,X_s^{n+1})|^pds\Big] \\
&\leq 4^{p-1}c_p(t-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}\mathbf{E}||X_s-X_s^n||^pds+K^p\int_{t_0}^{t}\mathbf{E}||X_s-X_s^{n+1}||^pds+(t-t_0)\varepsilon_n\Big] \\
&\leq 4^{p-1}c_p(T-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}(\gamma_n+\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^p)ds \\
&\quad +K^p\int_{t_0}^{t}(\gamma_{n+1}+\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^p)ds+(t-t_0)\varepsilon_n\Big] \\
&= 4^{p-1}c_p(T-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^pds \\
&\quad +K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^pds+(t-t_0)(2K^p\gamma_n+K^p\gamma_{n+1}+\varepsilon_n)\Big]
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_2(t) \quad \leq \quad & C\Bigg[2K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^pds \\
& +K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^pds+(t-t_0)(2K^p\gamma_n+K^p\gamma_{n+1}+\varepsilon_n)\Bigg]
\end{aligned} \tag{2.6}$$

with the positive constant $C = 8^{p-1}\Big[(T-t_0)^{p-1} + c_p(T-t_0)^{\frac{p}{2}-1}\Big]$.

Introduce the following notations

$$u_n(t) = \mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^n(r)|^p, \quad t\in[t_0,T], \quad n=0,1,\cdots$$

Then from the inequalities (2.4), (2.5) and (2.6), we can easily obtain

$$u_{n+1}(t) \le \frac{k^p}{\theta^{4(p-1)}}u_{n+1}(t)+\bar\alpha\int_{t_0}^t u_{n+1}(s)ds+2\bar\alpha\int_{t_0}^t u_n(s)ds+\bar\beta u_n(t)+(t-t_0)\mu_n+\nu_n$$

where $\bar\beta = \frac{k^p}{\theta^{2(p-1)}}[\frac{1}{(1-\theta)^{2(p-1)}} + \frac{1}{(\theta(1-\theta))^{p-1}}]$, $\bar\alpha$ is a positive constant, and $\mu_n = a_1\gamma_n + a_2\gamma_{n+1} + a_3\varepsilon_n$, $\nu_n = b_1\gamma_n + b_2\gamma_{n+1} + b_3\delta_n$, here a_i, b_i (i=1,2,3) are generic positive constants. Recall the inequality (2.2) we have $k^p < 3k^p \le \theta^{4(p-1)}$ and so $1 - \frac{k^p}{\theta^{4(p-1)}} > 0$, then one can see that

$$u_{n+1}(t) \le \alpha\int_{t_0}^t u_{n+1}(s)ds + 2\alpha\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \lambda(t-t_0)\mu_n + \lambda\nu_n$$

where α, λ are generic positive constants and $\beta = \frac{k^p}{\theta^{4(p-1)}-k^p}[(\frac{\theta}{1-\theta})^{2(p-1)}+(\frac{\theta}{1-\theta})^{(p-1)}]$. This and Lemma 2.2 yield that

$$\begin{aligned}
u_{n+1}(t) \quad &\le \quad 2\alpha\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \lambda(t-t_0)\mu_n + \lambda\nu_n \\
&\quad +\alpha\int_{t_0}^t e^{\alpha(t-s)}\Big[2\alpha\int_{t_0}^s u_n(r)dr + \beta u_n(s) + \lambda(s-t_0)\mu_n + \lambda\nu_n\Big]ds \\
&= \quad 2\alpha\int_{t_0}^t u_n(s)ds + 2\alpha^2\int_{t_0}^t e^{\alpha(t-s)}\int_{t_0}^s u_n(r)drds + \alpha\beta\int_{t_0}^t e^{\alpha(t-s)}u_n(s)ds \\
&\quad +\alpha\lambda\mu_n\int_{t_0}^t (s-t_0)e^{\alpha(t-s)}ds + \alpha\lambda\nu_n\int_{t_0}^t e^{\alpha(t-s)}ds + \beta u_n(t) \\
&\quad +\lambda(t-t_0)\mu_n + \lambda\nu_n.
\end{aligned}$$

By direct computation we note that

$$\begin{aligned}
&2\alpha^2\int_{t_0}^t e^{\alpha(t-s)}\int_{t_0}^s u_n(r)drds = 2\alpha\int_{t_0}^t (e^{\alpha(t-s)}-1)u_n(s)ds, \\
&\alpha\lambda\mu_n\int_{t_0}^t (s-t_0)e^{\alpha(t-s)}ds = -\lambda t\mu_n - \frac{\lambda}{\alpha}\mu_n + \frac{\lambda e^{\alpha(t-t_0)}}{\alpha}\mu_n + \lambda t_0\mu_n, \\
&\alpha\lambda\nu_n\int_{t_0}^t e^{\alpha(t-s)}ds = -\lambda\nu_n + \lambda e^{\alpha(t-t_0)}\nu_n.
\end{aligned}$$

Therefor we have

$$\begin{aligned}
u_{n+1}(t) \quad &\le \quad \alpha(2+\beta)\int_{t_0}^t e^{\alpha(t-s)}u_n(s)ds + \beta u_n(t) + \frac{\lambda e^{\alpha(t-t_0)}-\lambda}{\alpha}\mu_n + \lambda e^{\alpha(t-t_0)}\nu_n \\
&\le \quad \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\mu_n + \lambda e^{\alpha(T-t_0)}\nu_n.
\end{aligned}$$

Thus, for every $m = 1, 2, ...$, we have

$$\sum_{n=0}^m u_n(t) - u_0(t) \quad \le \quad \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^t\sum_{n=0}^m u_n(s)ds + \beta\sum_{n=0}^m u_n(t)$$

$$+\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n.$$

Hence,

$$\begin{aligned}(1-\beta)\sum_{n=0}^{m}u_n(t)\le \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^{t}\sum_{n=0}^{m}u_n(s)ds\\ +\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n+u_0(T)\end{aligned}$$

Noting $0<\frac{\theta}{1-\theta}<1$ and $\theta^{4(p-1)}-k^p\ge 2k^p$, one has that $\beta<\frac{2k^p}{\theta^{4(p-1)}-k^p}\le 1$. From this and by Lemma 2.2, we get that

$$\begin{aligned}\sum_{n=0}^{m}u_n(t) \quad\le\quad & \frac{1}{1-\beta}\Big[\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n+u_0(T)\Big]\\ &\cdot e^{\frac{\alpha(2+\beta)e^{\alpha(T-t_0)}}{1-\beta}(t-t_0)}\end{aligned}$$

The condition (1.9) implies that $\sum_{n=0}^{\infty}\mu_n<\infty$ and $\sum_{n=0}^{\infty}\nu_n<\infty$. Therefore,

$$\sum_{n=0}^{\infty}u_n(T)<\infty.$$

Recall that

$$\begin{aligned}\mathbf{E}\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|^p &\le \mathbf{E}||\xi-\xi^n||^p+\mathbf{E}\sup_{t\in[t_0,T]}|X(t)-X^n(t)|^p\\ &= \gamma_n+u_n(T),\end{aligned}$$

then from the condition (1.9) and Doob's martingale inequality [7], we find for an arbitrary $\varepsilon>0$ that,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{P}\Big\{\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|\ge\varepsilon\Big\} &\le \frac{1}{\varepsilon^p}\sum_{n=0}^{\infty}\mathbf{E}|X(T)-X^n(T)|^p\\ &\le \frac{1}{\varepsilon^p}\Big[\sum_{n=0}^{\infty}\gamma_n+\sum_{n=0}^{\infty}u_n(T)\Big]\\ &< \infty.\end{aligned}$$

Hence, by applying the Borel-Cantelli lemma, we conclude that for an arbitrary small $\varepsilon>0$, there exists a positive integer $n_0=n_0(\omega)$ such that

$$\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|\le\varepsilon,\qquad n\ge n_0.$$

This shows that the sequence $\{X^n(t),\ t\in(-\infty,T],\ n=0,1,...\}$ converges with probability one to the solution $\{X(t),\ t\in(-\infty,T]\}$. The proof is complete. □

3. Comments and examples

In [1, 8, 9], the proof of the existence of the solution to the equation (1.5) is based on the well-known Picard iteration, which establishes the iteration on the solution. However, the Z-algorithm method iterates for the whole equation. The Z-algorithm is a more general algorithm and can be applied to discuss more equations. Many historically and well-known analytic methods are its special cases, for example,

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Picard iteration, Newton-Kantorovich method and Chaplygin methods of chords and tangents.

Next, we give some examples to illustrate Theorem 2.4.

Example 1: Let $\{\xi^n, n = 0, 1, ...\}$ satisfy $\sum_{n=0}^{\infty} \mathbf{E}||\xi - \xi^n||^p < \infty$ and let D_n, f_n, g_n be defined in the following way: for $n = 0, 1, ...,$ $X \in BC((-\infty, 0]; \mathbf{R}^d)$ and for every fixed $t \in [t_0, T]$,

$$D_n(X; X_t^n) := \varphi_n(X - X_t^n) + D(X_t^n) \tag{3.1}$$

$$f_n(t, X; X_t^n) := \psi_n(X - X_t^n) + f(t, X_t^n) \tag{3.2}$$

$$g_n(t, X; X_t^n) := \theta_n(X - X_t^n) + g(t, X_t^n), \tag{3.3}$$

where

$$\begin{aligned} &\varphi_n : BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^d,\ \varphi_n(0) = 0 \\ &\psi_n : [t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^d,\ \psi_n(t, 0) \equiv 0 \\ &\theta_n : [t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^{d+m},\ \theta_n(t, 0) \equiv 0. \end{aligned}$$

The functions φ_n, ψ_n, θ_n satisfy the conditions (M1)-(M3) with constants $K > 0$ and $k \in \left(0, 1/(3 \cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$. Obviously,

$$\begin{aligned} &D_n(X_t^n; X_t^n) - D(X_t^n) \equiv 0 \\ &f_n(t, X_t^n; X_t^n) - f(t, X_t^n) \equiv 0 \\ &g_n(t, X_t^n; X_t^n) - g(t, X_t^n) \equiv 0. \end{aligned}$$

This shows that the condition (1.9) is satisfied. Thus theorem 2.4 is obtained.

Example 2: In particular, we linearize the equation (3.1) by: for $n = 0, 1, ...,$

$$D_n(X; X_t^n) := (X - X_t^n) \cdot \varphi_n + D(X_t^n) \tag{3.4}$$

$$f_n(t, X; X_t^n) := (X - X_t^n) \cdot \psi_n + f(t, X_t^n) \tag{3.5}$$

$$g_n(t, X; X_t^n) := (X - X_t^n) \cdot \theta_n + g(t, X_t^n), \tag{3.6}$$

where $\theta_n = (\theta_{1n}, \theta_{2n}, ..., \theta_{mn})$ and φ_n, ψ_n, $\theta_{in}(i = 1, 2, ..., m)$ are scalar sequences. We can easily see that all conditions of Theorem 2.4 are satisfied. So Theorem 2.4 succeed.

Example 3: More specifically, we assume that $\xi^n = \xi$ a.s. and $\varphi_n = \psi_n = \theta_n = 0$ in equation (3.4) for all $n = 0, 1, ...,$ then we obtain the Picard iteration. Of course, in this case, Theorem 2.4 is successful. Therefore, the Picard iteration is a special Z-algorithm.

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Conflict of interest:

The authors declared that they have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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An improved generalized parameterized inexact Uzawa method for singular saddle point problems *

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Abstract

In this paper, based on the generalized parameterized inexact Uzawa method (GPIU) presented by Zhang and Wang [*Applied Mathematics and Computation*, 219(2013) 4225-4231], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems. Moreover, theoretical analysis shows that the semi-convergence of the IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, numerical experiments are carried out, which show that the improved generalized parameterized inexact Uzawa method (IGPIU) with appropriate parameters improve the convergence of iteration method efficiently when solving singular saddle point problems from the classic incompressible steady state Stokes problems.

Key words: Krylov subspace methods; Generalized saddle point matrices; Minimal polynomial; Preconditioners.

MSC: 65F10; 65F15

1 Introduction

Consider a singular saddle point problem

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1}$$

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where $A \in R^{m\times m}, B \in R^{m\times n}, m \geq n$. The matrix A is symmetric positive matrix and the matrix and B is a rank-deficient matrix. Systems of the form (1) arise in a variety of scientific and engineering applications and have attracted a lot of research, see [1-7] for a comprehensive survey. When A is symmetric positive definite and B is of full column rank, we refer the reader to [2,7-18] for many efficient iterative methods and [19] for a survey.

For large, sparse or structure matrices, iterative method is an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1}b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [20].

Generally speaking, the matrix B is full column rank, but not always. If B is rank-deficient, how to effectively solve the singular saddle point problem (1) is important in both scientific computing and engineering applications. For solving the rank-deficient saddle point problems, Ma and Zheng et al. [17,21] presented the parameterized Uzawa method. Bai et al. [22-23] studied the PHSS iteration method. Fischer et al. [24] considered the preconditioned minimum residual (PMINRES) method. Wu et al. [7] discussed the preconditioned conjugate gradient (PCG) method. Zhang and Wang [17] introduced the generalized parameterized inexact Uzawa (GPIU) method.

In this paper, based on the generalized parameterized inexact Uzawa (GPIU) method presented by Zhang and Wang [17], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems (1). Similar to the proving process of section 3 in [17], theoretical analysis shows that the semi-convergence of IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, one numerical example presented shows correctness and availability of our theory about the improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems.

This paper is organized as follows. In Section 2, we will present the improved generalized parameterized inexact uzawa method (IGPIU) for singular saddle point problems (1). The semi-convergence of the IGPIU method are discussed in Section 3. Moreover, our methods are the generalization of known literature. Some numerical examples are given to demonstrate the efficiency of the IGPIU method in Section 4. Finally, conclusions are made in Section 5.

2 An improved generalized parameterized inexact uzawa method (IGPIU)

Recently, for singular saddle point problems (1), Zhang and Wang [17] make the following splitting

$$\mathcal{A} := \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \mathcal{M} - \mathcal{N},$$

where

$$\mathcal{M} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & Q_2 \end{pmatrix}, \mathcal{N} = \begin{pmatrix} P - A & -B \\ Q_1 & Q_2 \end{pmatrix}$$

2

$P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ is an arbitrary matrix.

To construct the improved generalized parameterized inexact Uzawa method (IGPIU), if we can add one parameter in the above splitting, then we may change the parameter to improve the performance of presented method. Hence, we propose the following splitting

$$\mathcal{A} := \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \mathcal{L} - \mathcal{U},$$

where

$$\mathcal{L} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix}, \mathcal{U} = \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix}$$

$P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ is an arbitrary matrix. Based the generalized parameterized inexact Uzawa (GPIU) iteration method presented by Zhang and Wang [17], we consider an improved generalized parameterized inexact uzawa method (IGPIU) for solving the singular saddle point (1).

$$\begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{2}$$

or equivalently,

$$\begin{cases} x^{(k+1)} = x^{(k)} + P^{-1}\left[p - Ax^{(k)} - By^{(k)}\right], \\ y^{(k+1)} = y^{(k)} + \frac{1}{\omega}Q_2^{-1}\left[B^T x^{(k+1)} - q\right] - \frac{1}{\omega}Q_2^{-1}Q_1\left[x^{(k+1)} - x^{(k)}\right]. \end{cases} \tag{3}$$

The iteration matrix of the IGPIU method (2) or (3) is given by

$$\mathcal{T} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix}^{-1} \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} = I - \mathcal{L}^{-1}\mathcal{A}. \tag{4}$$

The IGPIU method: Let $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ be an arbitrary matrix. Given initial vector $x^{(0)} \in R^m$ and $y^{(0)} \in R^n$ and the relaxation parameter ω with $\omega \neq 0$. For $k = 0, 1, 2, ...$ until the iteration sequence $\{(x^{(k)^T}, y^{(k)^T})^T\}$ is convergent, compute

$$\begin{cases} x^{(k+1)} = x^{(k)} + P^{-1}\left[p - Ax^{(k)} - By^{(k)}\right], \\ y^{(k+1)} = y^{(k)} + \frac{1}{\omega}Q_2^{-1}\left[B^T x^{(k+1)} - q\right] - \frac{1}{\omega}Q_2^{-1}Q_1\left[x^{(k+1)} - x^{(k)}\right]. \end{cases} \tag{5}$$

Remark 2.1. It is obvious that when choosing $\omega = 1$, then the IGPIU method reduces to the GPIU method [17]. Hence, we may change the parameter to improve the performance of presented method.

3 The semi-convergence of IGPIU method

In this section, we discuss the semi-convergence of the IGPIU method for solving the singular saddle point problem (1). We first reveal some basic concepts and notations.

Denote $\sigma(\mathcal{A})$ and $\rho(\mathcal{A})$ as the spectrum and spectral radius of the matrix $\mathcal{A}$, respectively. The *rank* and *index* of the matrix $\mathcal{A}$ are denoted by $rank(\mathcal{A})$ and $index(\mathcal{A})$, respectively.

Assume that the matrix $\mathcal{A}$ can be split into $\mathcal{A} = \mathcal{M} - \mathcal{N}$ with $\mathcal{M}$ nonsingular. Then we can construct a splitting iteration method:

$$x^{(k+1)} = \mathcal{T}x^{(k)} + \mathcal{M}^{-1}c, k = 0, 1, 2, ... \tag{6}$$

where $\mathcal{T} = \mathcal{M}^{-1}\mathcal{N}$ is the iteration matrix.

It is well known that any of the following three conditions is necessary and sufficient for guaranteeing the semi-convergence of the iteration method (6) for the singular linear systems $\mathcal{A}X = c$ (see [17,22]):

(a) The spectral radius of the iteration matrix $\mathcal{T}$ is equal to one , i.e., $\rho(\mathcal{T}) = 1$;

(b) The elementary divisor associated with $\lambda \in \sigma(\mathcal{A})$ is linear when $\lambda = 1$, i.e., $rank((I-\mathcal{T})^2)$ $=rank(I-\mathcal{T})$, or equivalently, $index(I-\mathcal{T}) = 1$;

(c) If $\lambda \in \sigma(\mathcal{T})$ with $|\lambda| = 1$, then $\lambda = 1$, i.e., $\mathcal{V}(\mathcal{T}) \equiv \max\{|\lambda| : \lambda \in \sigma(\mathcal{T}), \lambda \neq 1\} < 1$.

When iteration scheme (6) is semi-convergent, $\mathcal{V}(\mathcal{T})$ is said to be the semi-convergence factor. As usual, the splitting $\mathcal{A} = \mathcal{M} - \mathcal{N}$ and the corresponding iteration matrix $\mathcal{T}$ are called as semi-convergent if the iteration (6) is semi-convergent. Next we study the semi-convergence of the IGPIU iteration (5). To get the semi-convergence conditions, the following lemmas are used.

Lemma 3.1. **[25]** *Consider the quadratic equation $x^2 - \delta x + \eta = 0$, where δ and η are real numbers. Both roots of the equation are less than one in modulus if and only if $|\eta| < 1$ and $|\delta| < 1 + \eta$.*

Lemma 3.2. **[17]** *Let $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be symmetric positive definite and $B \in R^{m\times n}$ be of column rank-deficient, with $m \geq n$. Suppose that λ is an eigenvalue of the iteration matrix $\mathcal{T}$ and $(u^T, v^T)^T \in R^{m+n}$ is the corresponding eigenvector. Then $\lambda = 1$ if and only if $u = 0$.*

Theorem 3.3. *Let $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be symmetric positive definite and $B \in R^{m\times n}$ be of column rank-deficient, with $m \geq n$. Suppose that $\lambda \neq 1$ is an eigenvalue of the iteration matrix $\mathcal{T}$ and $(u^T, v^T)^T \in R^{m+n}$ is the corresponding eigenvector. Then λ satisfies the following quadratic equation:*

$$\lambda^2 + \frac{\beta + \gamma - 2\omega\alpha - \tau}{\alpha}\lambda + \frac{\alpha + \tau - \omega\beta}{\alpha} = 0,$$

where

$$\alpha = \frac{u^*Pu}{u^*u} > 0, \beta = \frac{u^*Au}{u^*u} > 0, \gamma = \frac{u^*BQ_2^{-1}B^Tu}{u^*u} \geq 0, \tau = \frac{u^*BQ_2^{-1}Q_1u}{u^*u}.$$

Proof. Firstly, since $\lambda \neq 1$, we know $u \neq 0$ from Lemma 3.2. By (4) we have

$$\begin{pmatrix} P-A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{7}$$

or equivalently

$$\begin{cases} [(1-\lambda)P - A]\, u = Bv, \\ \left[(1-\lambda)Q_1 + \lambda B^T\right] u = \omega(\lambda - 1)Q_2 v. \end{cases} \tag{8}$$

Because Q_2 is symmetric positive definite and $\lambda \neq 1$, from the second equation in (8), we obtain that $v = \frac{1}{\omega}(-Q_2^{-1}Q_1 + \frac{\lambda}{\lambda-1}Q_2^{-1}B^T)u$, which together with the first equation in (8), result in

$$\omega\lambda^2 Pu + \lambda\omega(Au + BQ_2^{-1}B^Tu - 2\omega Pu - BQ_2^{-1}Q_1u) + \omega(Pu + BQ_2^{-1}Q_1u - \omega Au) = 0. \tag{9}$$

4

Since $u \neq 0$, by left multiplying u^* and with the positive definiteness of $P(u^*Pu \neq 0)$, we have

$$\lambda^2 + \tfrac{\beta+\gamma-2\omega\alpha-\tau}{\alpha}\lambda + \tfrac{\alpha+\tau-\omega\beta}{\alpha} = 0. \tag{10}$$

Thus, the proof is completed.

Theorem 3.4. *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric. Then $\sigma(\mathcal{T}) < 1$ holds if and only if one of the following conditions hold:*

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Proof. Since P is symmetric positive definite and $BQ_2^{-1}Q_1$ is symmetric. By Lemma 3.1 and Eq. (10), we know that the spectral radius of the IGPIU iteration matrix is less than one in modulus if and only if

$$\begin{cases} \left|\frac{\alpha+\tau-\omega\beta}{\alpha}\right| < 1, \\ \left|\frac{\beta+\gamma-2\omega\alpha-\tau}{\alpha}\right| < 1 + \frac{\alpha+\tau-\omega\beta}{\alpha}. \end{cases} \tag{11}$$

or equivalently,

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Thus, the proof is completed. ⋄

Theorem 3.5. **[17]** *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric and $\mathcal{N}(B) \cap \mathcal{R}(Q_2^{-1}B^TA^{-1}B) = \{0\}$, then index$(I - \mathcal{T}) = 1$. Here and in the sequel, $\mathcal{N}(\bullet)$ and $\mathcal{R}(\bullet)$ is used to represent the null space and range space of the corresponding matrix, respectively.*

Combining Theorem 3.4 and Theorem 3.5 , we immediately obtain the following sufficient conditions for the convergence result of the IGPIU method for solving singular saddle point problem (1).

Theorem 3.6. *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric and $\mathcal{N}(B) \cap \mathcal{R}(Q_2^{-1}B^TA^{-1}B) = \{0\}$. Then the IGPIU method for solving singular saddle point problem (1) is semi-convergent if $\omega, \tau, \beta, \gamma, \alpha$ satisfy one of the following conditions*

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Remark 3.1. From the Theorem 3.5, it is not difficult to find that when $\omega > 1, 2\alpha > \beta$ or $0 < \omega < 1, 2\alpha < \beta$, then $(1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta > 2\alpha + \tau - \beta$. Hence, under these conditions, the range of γ is wider and we will have more space of parameters range.

Remark 3.2. It is obvious that when choosing $\omega = 1, P = \frac{1}{\xi}A, Q_1 = 0,$ and $\mathrm{Q}_2 = \frac{1}{\zeta}\mathrm{Q}$, Q is an approximate matrix to the Schur complement $B^TA^{-1}B$, then the IGPIU method reduces to the PIU method in [10,17].

Remark 3.3. Some choices of the parameter matrices P, Q_1 and Q_2 are given in Table 1 [17]. When choosing different parameter matrices P, Q_1 and Q_2, we may immediately obtain a series of iterative methods for solving singular saddle problem (1).

Table 1: Some choices of the parameter matrices P, Q_1 and Q_2.

Case	P	Q_1	Q_2
I	$\frac{1}{\xi}A$	0	$\frac{1}{\zeta}I_n$
II	$\frac{1}{\xi}diag(A)$	0	$\frac{1}{\zeta}I_n$
III	$\frac{1}{\xi}tridiag(A)$	0	$\frac{1}{\zeta}I_n$
IV	$\frac{1}{\xi}A$	$-\frac{\theta}{\zeta}B^T$	$\frac{1}{\zeta}I_n$
V	$\frac{1}{\xi}A$	$-\frac{\theta}{\zeta}B^T$	$\frac{1}{\zeta}diag(\hat{B}^T P^{-1}\hat{B}, \tilde{B}^T\tilde{B})$
VI	$\frac{1}{\xi}A$	$-\theta Q_2 B^T$	$\frac{1}{\zeta}tridiag(\hat{B}^T P^{-1}\hat{B}, \tilde{B}^T\tilde{B})$

4 Numerical examples

In this section, we give numerical experiments to demonstrate the conclusions drawn above. The numerical experiments were done by using MATLAB 7.1 and the matrix of the numerical experiments were generated by IFISS software. In all our runs we used as a zero initial guess and stopped the iteration when the relative residual had been reduced by at least seven orders of magnitude (i.e, when $\|b - \mathcal{A}x^k\|_2 \leq 10^{-7}\|b\|_2$).

We consider the classic incompressible steady state Stokes problems:

$$\begin{cases} -\Delta u + gradp = f, \text{ in } \Omega, \\ -divu = 0, \text{ in } \Omega, \end{cases}$$

with suitable boundary condition on $\partial\Omega$. It is known that many discretization schemes for the above Stokes problems will lead to generalized saddle point problems of the form (1). Here, we get the test problem (leak-lid driven cavity) by using IFISS software written by David Silvester, Howard Elman and Alison Ramage. We take a finite element subdivision based on 32×32 uniform grids of square elements. The mixed finite element used is the bilinear-constant velocity-pressure: $Q_1 - P_0$ pair with stabilization. $Q_1 - P_0$ finite element subdivision is shown in Figure 1. The stabilization parameter is chosen to $\frac{1}{4}$. We get the (1,1) block A of the coefficient matrix corresponding to the discretization of the conservative term. Since the matrix B produced by the software is rank deficient, so $\mathcal{A}$ is singular matrix. In our experiment, we choose uniform grids 8×8, 16×16.

In Tables 2, when choosing different parameters, we show iteration counts, relative residual and computing time about the GPIU and the IGPIU methods for solving singular saddle problem (1), where IT, RES and CPU are the iteration numbers, relative residual and computing time about the GPIU and the IGPIU methods, respectively. Moreover, we also show the corresponding reduction of residual 2-norm and eigenvalues distributions about two methods for different parameters. Figures 2 $\sim$ 5 show the reduction of residual 2-norm with Case I, II, III and IV of Table 2. Figures 6 $\sim$ 9 show the eigenvalues distributions with Case I, II, III and IV of Table 2. Figures 10 and 11 show the reduction of residual 2-norm with uniform grids 16×16 and Cases I, II. Figures 12 and 13 show the eigenvalues distributions with uniform grids 16×16 and Cases I, II.

Remark 3.1. From Table 2, Figures 2 $\sim$ 5, 10 and 11 , it is very easy to get that the IGPIU method is in general better than the GPIU method when choosing suitable parameters. By numerical experiments for many times, we can find that, when $0.75 \leq \omega \leq 1.05$ the IGPIU method is very efficient. For Case II, when $\omega = 1.05$ the IGPIU method is little efficient. Hence, we suggest that, the selection range of the parameters may be $0.75 \leq \omega \leq 1$.

Remark 3.2. From Figures 6 $\sim$ 9,12 and 13, we may find that the eigenvalue distribution about the GPIU method has the same spectral clustering compared with the IGPIU method when choosing suitable parameters.

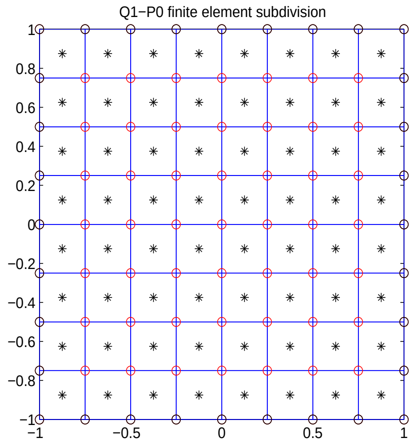

Figure 1: $Q_1 - P_0$ finite element subdivision

Table 2: numerical results of different parameters about GPIU and IGPIU methods for solving singular saddle problem (1). Here, uniform grids are 8×8.

Case	$(\xi, \theta, \zeta, \omega)$	IT	RES	CPU
Case I	$(0.8, 0, 10, 1)$	124	9.6146×10^{-8}	1.776
	$(0.8, 0, 10, 0.85)$	100	9.9265×10^{-8}	1.437
	$(0.8, 0, 10, 0.75)$	103	9.9106×10^{-8}	1.468
	$(0.8, 0, 10, 1.05)$	79	9.8442×10^{-8}	1.156
Case II	$(0.8, 0, 10, 1)$	451	8.8682×10^{-8}	0.813
	$(0.8, 0, 10, 0.85)$	435	9.6783×10^{-8}	0.812
	$(0.8, 0, 10, 0.75)$	439	9.6727×10^{-8}	0.797
	$(0.8, 0, 10, 1.05)$	462	7.7405×10^{-8}	0.844
Case III	$(0.8, 0, 10, 1)$	375	7.2718×10^{-8}	2.797
	$(0.8, 0, 10, 0.85)$	356	9.7272×10^{-8}	2.672
	$(0.8, 0, 10, 0.75)$	354	8.3278×10^{-8}	2.703
	$(0.8, 0, 10, 1.05)$	373	7.3365×10^{-8}	2.829
Case IV	$(0.8, 0, 10, 1)$	141	9.3439×10^{-8}	1.984
	$(0.8, 0, 10, 0.85)$	124	9.3478×10^{-8}	1.734
	$(0.8, 0, 10, 0.75)$	118	9.91×10^{-8}	1.625
	$(0.8, 0, 10, 1.05)$	139	9.913×10^{-8}	1.907

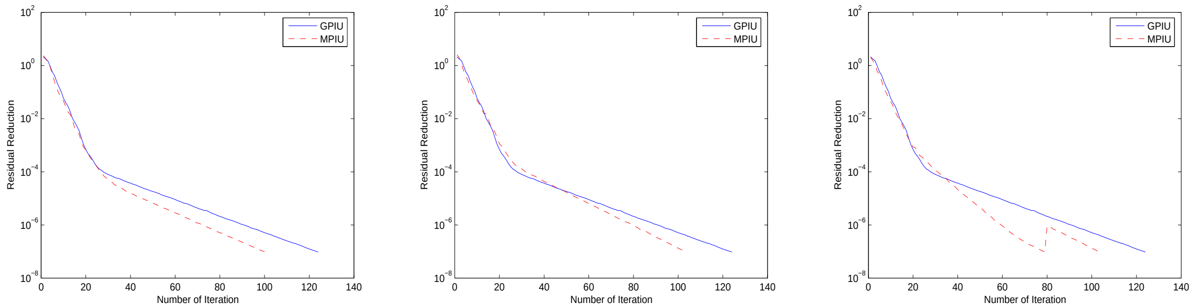

Figure 2: Reduction of residual 2-norm with Case I of Table 2. The left figure shows that the first line parameters (GPIU) of Case I compare with the second line parameters (IGPIU) of Case I; The middle figure shows that the first line parameters (GPIU) of Case I compare with the third line parameters (IGPIU) of Case I; The right figure shows that the first line parameters (GPIU) of Case I compare with the forth line parameters (IGPIU) of Case I.

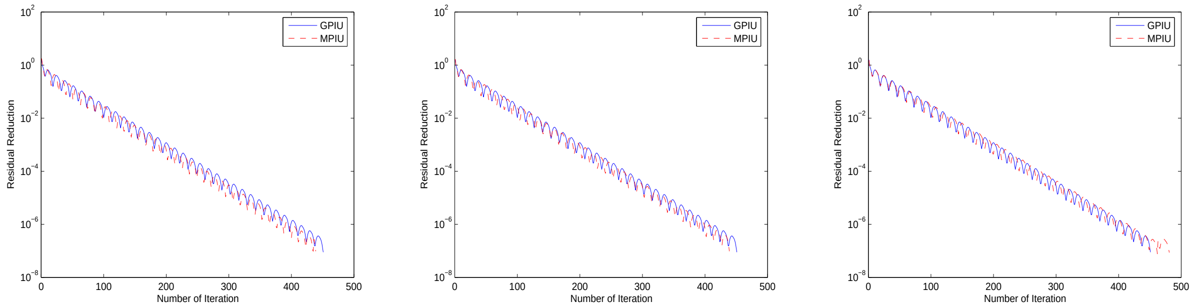

Figure 3: Reduction of residual 2-norm with Case II of Table 2. The left figure shows that the first line parameters (GPIU) of Case II compare with the second line parameters (IGPIU) of Case II; The middle figure shows that the first line parameters (GPIU) of Case II compare with the third line parameters (IGPIU) of Case II; The right figure shows that the first line parameters (GPIU) of Case II compare with the forth line parameters (IGPIU) of Case II.

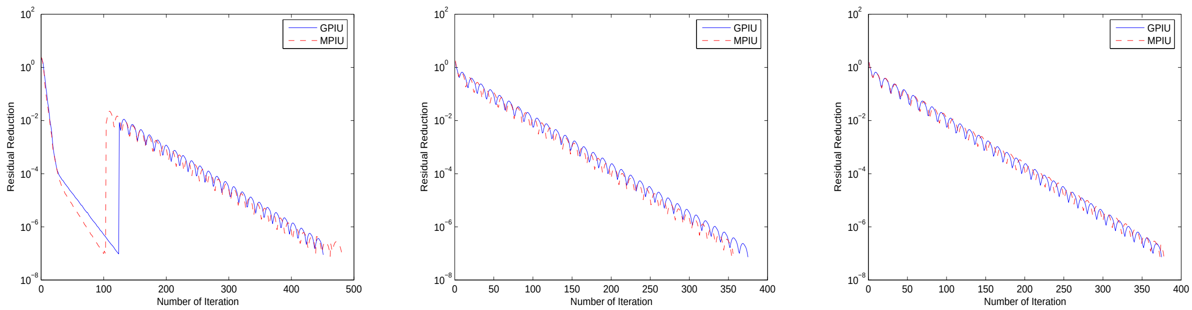

Figure 4: Reduction of residual 2-norm with Case III of Table 2. The left figure shows that the first line parameters (GPIU) of Case III compare with the second line parameters (IGPIU) of Case III; The middle figure shows that the first line parameters (GPIU) of Case III compare with the third line parameters (IGPIU) of Case III; The right figure shows that the first line parameters (GPIU) of Case III compare with the forth line parameters (IGPIU) of Case III.

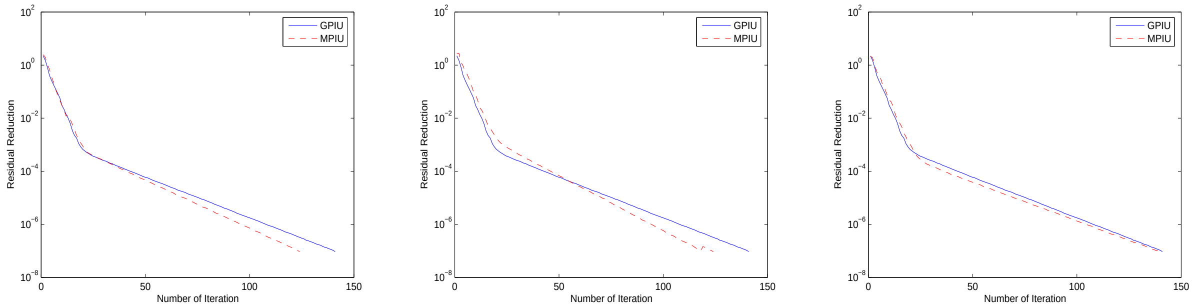

Figure 5: Reduction of residual 2-norm with Case IV of Table 2. The left figure shows that the first line parameters (GPIU) of Case IV compare with the second line parameters (IGPIU) of Case IV; The middle figure shows that the first line parameters (GPIU) of Case IV compare with the third line parameters (IGPIU) of Case IV; The right figure shows that the first line parameters (GPIU) of Case IV compare with the forth line parameters (IGPIU) of Case IV.

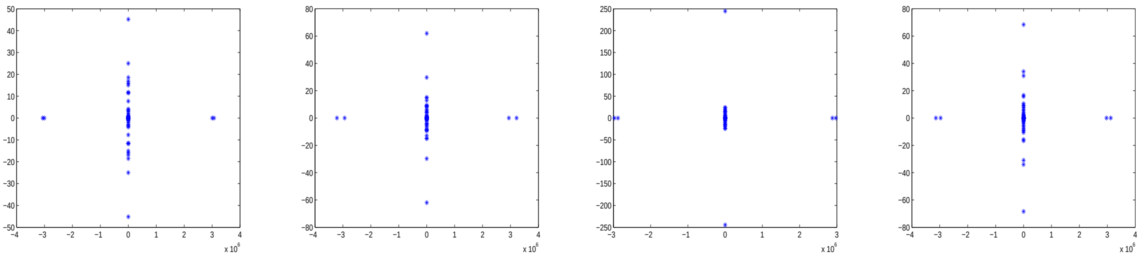

Figure 6: Eigenvalues distributions with Case I of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case I; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case I; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case I; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case I.

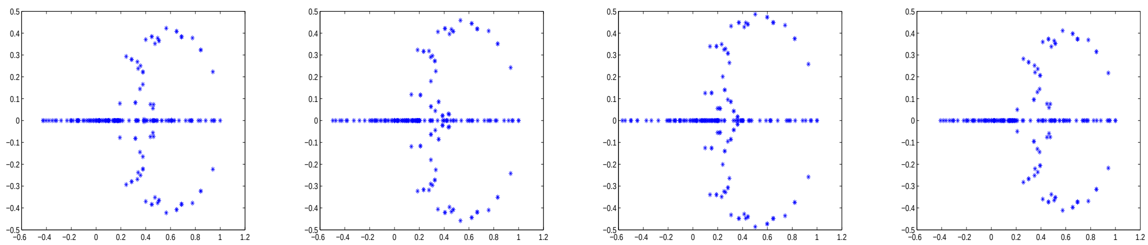

Figure 7: Eigenvalues distributions with Case II of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case II; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case II; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case II; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case II.

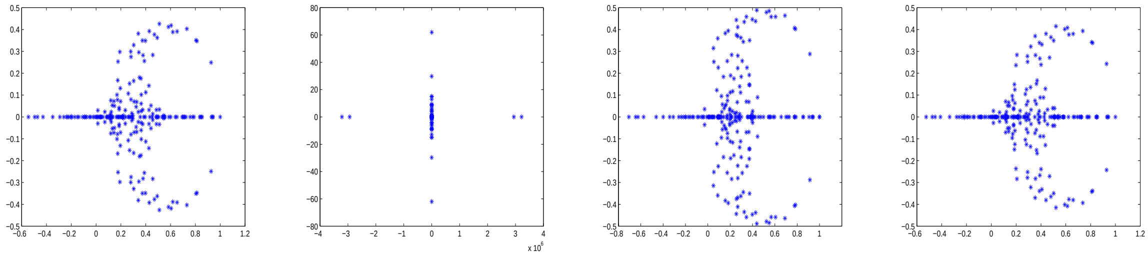

Figure 8: Eigenvalues distributions with Case III of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case III; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case III; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case III; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case III.

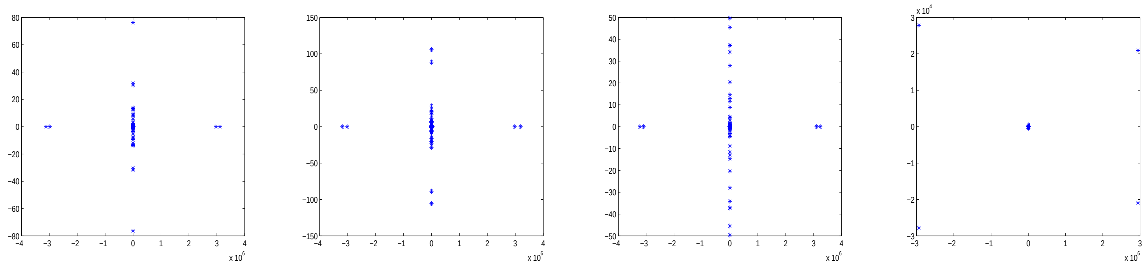

Figure 9: Eigenvalues distributions with Case IV of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case IV; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case IV; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case IV; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case IV.

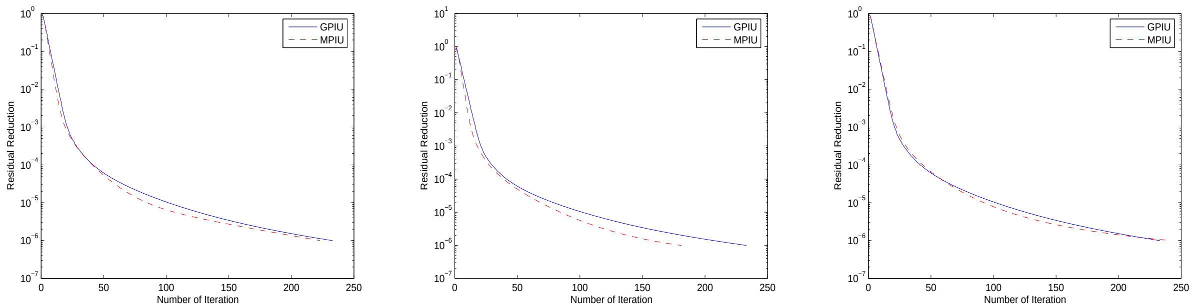

Figure 10: Reduction of residual 2-norm with uniform grids 16×16 and Case I. The left figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case I; The middle figure shows that parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case I; The right figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case I.

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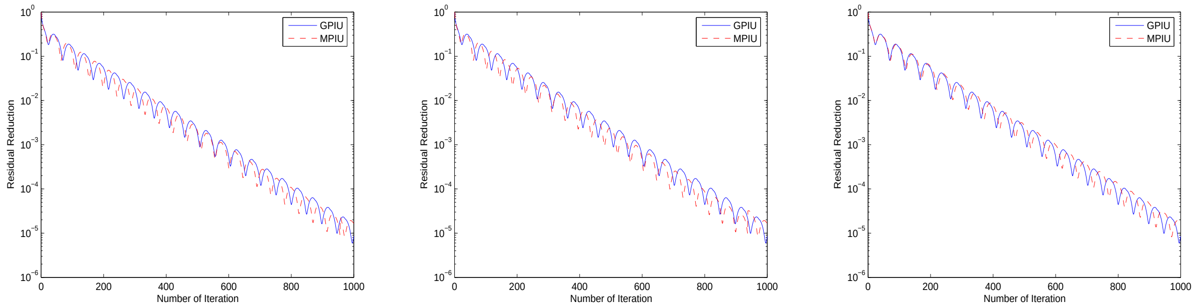

Figure 11: Reduction of residual 2-norm with uniform grids 16×16 and Case II. The left figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case II; The middle figure shows that parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case II; The right figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case II.

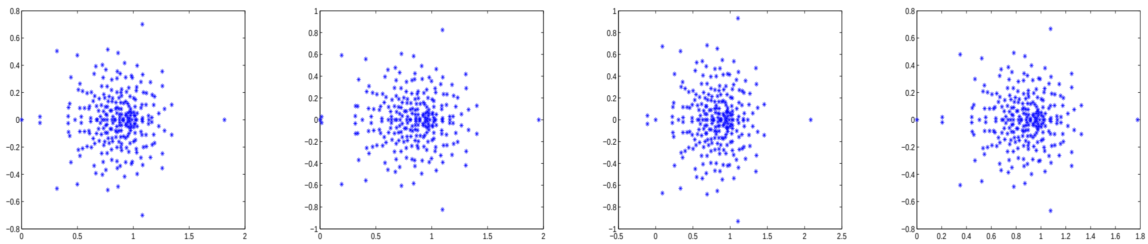

Figure 12: Eigenvalues distributions with uniform grids 16×16 and Case I. The first figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1)$ (GPIU) of Case I; The second figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.85)$ (IGPIU) if Case I; The third figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case I; The forth figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case I.

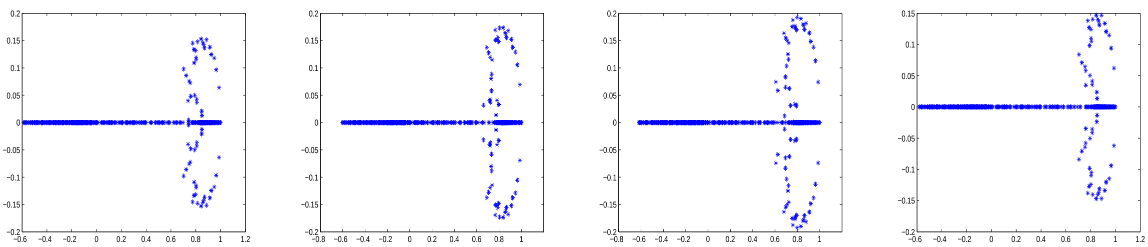

Figure 13: Eigenvalues distributions with uniform grids 16×16 and Case II. The first figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1)$ (GPIU) of Case II; The second figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case II; The third figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case II; The forth figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case II.

5 Conclusions

Based on the generalized parameterized inexact Uzawa method (GPIU) presented by Zhang and wang [17], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems (1). Moreover, theoretical analysis shows that the semi-convergence of IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, numerical experiments are carried out, which show that the IGPIU method is in general better than the GPIU method when choosing suitable parameters. Moreover, we also may find that the eigenvalue distribution about GPIU method has the same spectral clustering compared with IGPIU method when choosing suitable parameters.

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IDENTITIES INVOLVING BESSEL POLYNOMIALS ARISING FROM LINEAR DIFFERENTIAL EQUATIONS

TAEKYUN KIM AND DAE SAN KIM

Abstract. In this paper, we study linear differential equations arising from Bessel polynomials and their applications. From these linear differential equations, we give some new and explicit identities for Bessel polynomials.

1. Introduction

As is well known, the Bessel differential equation is given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\alpha^2\right)y=0,\quad (\text{see [17]}). \tag{1.1}$$

for an arbitrary complex number α.

The Bessel functions of the first kind $J_\alpha(x)$ are defined by the solution of (1.1).

For $n\in\mathbb{Z}$, $J_n(x)$ are sometimes also called cylinder function or cylindrical harmonics.

It is known that

$$J_n(x)=\sum_{l=0}^{\infty}\frac{(-1)^l}{l!\,(n+l)!}\left(\frac{x}{2}\right)^{2l+n},\quad (\text{see [1, 16, 17]}). \tag{1.2}$$

The generating function of Bessel functions is given by

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}J_n(x)\,t^n, \tag{1.3}$$

and $J_n(x)$ can be also represented by the contour integral as

$$J_n(x)=\frac{1}{2\pi i}\oint e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}t^{-n-1}dt,\quad (\text{see [17]}), \tag{1.4}$$

where the contour encloses the origin and is traversed in a counterclockwise direction.

The Bessel polynomials are defined by the solution of the differential equation

$$x^2\frac{d^2y}{dx^2}+2(x+1)\frac{dy}{dx}-n(n+1)y=0,\quad (\text{see [1–6, 15, 16]}). \tag{1.5}$$

Indeed, the solutions of (1.5) are given by

$$\begin{aligned} y_n(x)&=\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k \\ &=\sqrt{\frac{2}{\pi x}}e^{\frac{1}{x}}K_{-n-\frac{1}{2}}\left(\frac{1}{x}\right),\quad (\text{see [1, 15–17]}), \end{aligned} \tag{1.6}$$

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where

$$K_\nu(z) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2z)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{\left(t^2+z^2\right)^{\nu+\frac{1}{2}}} dt.$$

We note that $y_n(x)$ are very similar to the modified spherical Bessel function of the second kind.

The first few are given as

$$\begin{aligned} &y_0(x) = 1, \quad y_1(x) = x+1, \quad y_2(x) = 3x^2+3x+1, \\ &y_3(x) = 15x^3+15x^2+6x+1, \\ &y_4(x) = 105x^4+105x^3+45x^2+10x+1, \quad \ldots. \end{aligned}$$

Carlitz reverse Bessel polynomials are defined by

$$p_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right), \quad (n \in \mathbb{N} \cup \{0\}), \quad (\text{see } [4, 15]). \tag{1.7}$$

These polynomials are also given by the generating function as

$$e^{x\left(1-\sqrt{1-2t}\right)} = \sum_{n=0}^\infty p_n(x) \frac{t^n}{n!}. \tag{1.8}$$

The explicit formulas for them are

$$\begin{aligned} p_n(x) &= \sum_{k=1}^n \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} x^k \\ &= (2n-3)!! x \, {}_1F_1(1-n; 2-2n; 2x), \quad (\text{see } [1, 15, 16]), \end{aligned} \tag{1.9}$$

where

$$n!! = \begin{cases} n(n-2)\cdots 5\cdot 3\cdot 1 & \text{if } n>0 \text{ odd}, \\ n(n-2)\cdots 6\cdot 4\cdot 2 & \text{if } n>0 \text{ even}, \\ 1 & \text{if } n=-1,0, \end{cases}$$

and

$$\begin{aligned} {}_1F_1(a;b;z) &= 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots \\ &= \sum_{k=0}^\infty \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)}\frac{z^k}{k!} \\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1}(1-t)^{b-a-1} dt. \end{aligned}$$

The first few polynomials are

$$\begin{aligned} &p_1(x) = x, \\ &p_2(x) = x^2+x, \\ &p_3(x) = x^3+3x^2+3x, \\ &p_4(x) = x^4+6x^3+15x^2+15x, \cdots. \end{aligned}$$

Recently, several authors have studied non-linear differential equations related to special polynomials (see [7–14]).

The reverse Bessel polynomials are used in the design of Bessel electronic filters.

In this paper, we consider linear differential equations arising from Carlitz reverse Bessel polynomials and give some new and explicit identities for Bessel polynomials.

2. Identities involving Bessel polynomials arising from linear differential equations

Let us put

$$F = F(t,x) = e^{x\left(1-\sqrt{1-2t}\right)}. \tag{2.1}$$

Thus, by (2.1), we get

$$F^{(1)} = \frac{d}{dt}F(t,x) = x(1-2t)^{-\frac{1}{2}}F, \tag{2.2}$$

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= \left(x(1-2t)^{-\frac{3}{2}} + x^2(1-2t)^{-1}\right)F, \end{aligned} \tag{2.3}$$

$$\begin{aligned} F^{(3)} &= \frac{d}{dt}F^{(2)} \\ &= \left(3x(1-2t)^{-\frac{5}{2}} + 3x^2(1-2t)^{-2} + x^3(1-2t)^{-\frac{3}{2}}\right)F, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} F^{(4)} &= \frac{dF^{(3)}}{dt} \\ &= \left(15x(1-2t)^{-\frac{7}{2}} + 15x^2(1-2t)^{-3} + 6x^3(1-2t)^{-\frac{5}{2}} + x^4(1-2t)^{-2}\right)F. \end{aligned} \tag{2.5}$$

Continuing this process, we set

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x) \\ &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right)F, \end{aligned} \tag{2.6}$$

where $N = 1,2,3,\ldots$.

From (2.6), we note that

$$\begin{aligned} &F^{(N+1)} \\ &= \frac{d}{dt}F^{(N)} \\ &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)\left(-\frac{i}{2}\right)(1-2t)^{-\frac{i}{2}-1}(-2)\right)F \\ &\quad + \sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}F^{(1)} \end{aligned} \tag{2.7}$$

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$$
\begin{aligned}
&= \left(\sum_{i=N}^{2N-1} i a_{i-N}(N,x)(1-2t)^{-\frac{i+2}{2}}\right) F \\
&\quad + \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right) x(1-2t)^{-\frac{1}{2}} F \\
&= \left(\sum_{i=N}^{2N-1} i a_{i-N}(N,x)(1-2t)^{-\frac{i+2}{2}}\right) F + \left(\sum_{i=N}^{2N-1} x a_{i-N}(N,x)(1-2t)^{-\frac{i+1}{2}}\right) F \\
&= \Big\{ x a_0(N,x)(1-2t)^{-\frac{N+1}{2}} + (2N-1) a_{N-1}(N,x)(1-2t)^{-\frac{2N+1}{2}} \\
&\quad + \sum_{i=N+1}^{2N-1} \left((i-1) a_{i-N-1}(N,x) + x a_{i-N}(N,x)\right)(1-2t)^{-\frac{i+1}{2}} \Big\} F.
\end{aligned}
$$

By replacing N by $N+1$ in (2.6), we get

$$
\begin{aligned}
F^{(N+1)} &= \left(\sum_{i=N+1}^{2N+1} a_{i-N-1}(N+1,x)(1-2t)^{-\frac{i}{2}}\right) F \\
&= \left(\sum_{i=N}^{2N} a_{i-N}(N+1,x)(1-2t)^{-\frac{i+1}{2}}\right) F.
\end{aligned} \tag{2.8}
$$

By comparing the coefficients on both sides (2.7) and (2.8), we have

$$
a_0(N+1,x) = x a_0(N,x), \tag{2.9}
$$

$$
a_N(N+1,x) = (2N-1) a_{N-1}(N,x), \tag{2.10}
$$

and

$$
a_{i-N}(N+1,x) = (i-1) a_{i-N-1}(N,x) + x a_{i-N}(N,x), \tag{2.11}
$$

where $N+1 \le i \le 2N-1$.

From (2.2) and (2.6), we can derive the following equation (2.11):

$$
x(1-2t)^{-\frac{1}{2}} F = F^{(1)} = a_0(1,x)(1-2t)^{-\frac{1}{2}} F. \tag{2.12}
$$

Thus, by (2.12), we have

$$
a_0(1,x) = x. \tag{2.13}
$$

From (2.9), we note that

$$
a_0(N+1,x) = x a_0(N,x) = x^2 a_0(N-1,x) = \cdots = x^N a_0(1,x) = x^{N+1}, \tag{2.14}
$$

and, by (2.10), we see

$$
\begin{aligned}
a_N(N+1,x) &= (2N-1) a_{N-1}(N,x) \\
&= (2N-1)(2N-3) a_{N-2}(N-1,x) \\
&\ \ \vdots \\
&= (2N-1)(2N-3)\cdots 3 \cdot 1 a_0(1,x) \\
&= (2N-1)!!x.
\end{aligned} \tag{2.15}
$$

The matrix $(a_i(j,x))_{0 \le i \le N-1, 1 \le j \le N}$ is given by

$$\begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&\cdots&N\end{array}\\ \begin{array}{c}0\\1\\2\\3\\\vdots\\N-1\end{array} & \begin{bmatrix} x&x^2&x^3&x^4&\cdots&x^N\\ &1!!x&&&&\\ &&3!!x&&&\\ &&&5!!x&&\\ &&&&\ddots&\\ &0&&&&(2N-3)!!x\end{bmatrix}\end{array}$$

From (2.11), we obtain

$$\begin{aligned}
(2.16)\qquad & a_1(N+1,x)\\
&= Na_0(N,x)+xa_1(N,x)\\
&= Na_0(N,x)+x(N-1)a_0(N-1,x)+x^2a_1(N-1,x)\\
&\vdots\\
&= \sum_{i=0}^{N-2} x^i(N-i)a_0(N-i,x)+x^{N-1}a_1(2,x)\\
&= \sum_{i=0}^{N-2} x^i(N-i)a_0(N-i,x)+x^{N-1}x\\
&= \sum_{i=0}^{N-1} x^i(N-i)a_0(N-i,x),
\end{aligned}$$

$$\begin{aligned}
(2.17)\qquad & a_2(N+1,x)\\
&= (N+1)a_1(N,x)+xa_2(N,x)\\
&= (N+1)a_1(N,x)+xNa_1(N-1,x)+x^2a_2(N-1,x)\\
&\vdots\\
&= \sum_{i=0}^{N-3} x^i(N+1-i)a_1(N-i,x)+x^{N-2}a_2(3,x)\\
&= \sum_{i=0}^{N-3} x^i(N+1-i)a_1(N-i,x)+3x^{N-2}a_1(2,x)\\
&= \sum_{i=0}^{N-2} x^i(N+1-i)a_1(N-i,x),
\end{aligned}$$

and

$$\begin{aligned}
(2.18)\qquad & a_3(N+1,x)\\
&= (N+2)a_2(N,x)+xa_3(N,x)\\
&= (N+2)a_2(N,x)+x(N+1)a_2(N-1,x)+x^2a_3(N-1,x)
\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&=\sum_{i=0}^{N-4}x^{i}\left(N-i+2\right)a_{2}\left(N-i,x\right)+5x^{N-3}a_{2}\left(3,x\right)\\
&=\sum_{i=0}^{N-3}x^{i}\left(N-i+2\right)a_{2}\left(N-i,x\right).
\end{aligned}$$

Continuing this process, we get

$$a_{j}\left(N+1,x\right)=\sum_{i=0}^{N-j}x^{i}\left(N-i+j-1\right)a_{j-1}\left(N-i,x\right), \tag{2.19}$$

where $j=1,2,\ldots,N-1$.

Now, we give explicit expressions for $a_{j}\left(N+1,x\right)$ $(j=1,2,\ldots,N-1)$. From (2.14) and (2.16), we can easily derive the following equation:

$$\begin{aligned}
a_{1}\left(N+1,x\right)&=\sum_{i_{1}=0}^{N-1}x^{i_{1}}\left(N-i_{1}\right)a_{0}\left(N-i_{1},x\right)\\
&=x^{N}\sum_{i_{1}=0}^{N-1}\left(N-i_{1}\right).
\end{aligned} \tag{2.20}$$

By (2.17), (2.18) and (2.19), we get

$$\begin{aligned}
a_{2}\left(N+1,x\right)&=\sum_{i_{2}=0}^{N-2}x^{i_{2}}\left(N-i_{2}+1\right)a_{1}\left(N-i_{2},x\right)\\
&=x^{N-1}\sum_{i_{2}=0}^{N-2}\sum_{i_{1}=0}^{N-2-i_{2}}\left(N-i_{2}+1\right)\left(N-i_{2}-i_{1}-1\right),
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
a_{3}\left(N+1,x\right)&=\sum_{i_{3}=0}^{N-3}x^{i_{3}}\left(N-i_{3}+2\right)a_{2}\left(N-i_{3},x\right)\\
&=x^{N-2}\sum_{i_{3}=0}^{N-3}\sum_{i_{2}=0}^{N-3-i_{3}}\sum_{i_{1}=0}^{N-3-i_{3}-i_{2}}\left(N-i_{3}+2\right)\left(N-i_{3}-i_{2}\right)\\
&\quad\times\left(N-i_{3}-i_{2}-i_{1}-2\right),
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
a_{4}\left(N+1,x\right)&=\sum_{i_{4}=0}^{N-4}x^{i_{4}}\left(N-i_{4}+3\right)a_{3}\left(N-i_{4},x\right)\\
&=x^{N-3}\sum_{i_{4}=0}^{N-4}\sum_{i_{3}=0}^{N-4-i_{4}}\\
&\quad\times\sum_{i_{2}=0}^{N-4-i_{4}-i_{3}}\sum_{i_{1}=0}^{N-4-i_{4}-i_{3}-i_{2}}\left(N-i_{4}+3\right)\left(N-i_{4}-i_{3}+1\right)\\
&\qquad\times\left(N-i_{4}-i_{3}-i_{2}-1\right)\left(N-i_{4}-i_{3}-i_{2}-i_{1}-3\right).
\end{aligned} \tag{2.23}$$

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Continuing this process, we get

(2.24)

$$\begin{aligned}
&a_j(N+1,x)\\
&= x^{N-j+1}\sum_{i_j=0}^{N-j}\sum_{i_{j-1}=0}^{N-j-i_j}\cdots\sum_{i_1=0}^{N-j-i_j-\cdots-i_2}\prod_{k=1}^{j}\left(N-i_j-\cdots-i_k-(j-(2k-1))\right).
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 1. *For $N\in\mathbb{N}$, the linear differential equations*

$$F^{(N)}=\left(\frac{d}{dt}\right)^N F(t,x)=\left(\sum_{i=N}^{2N-1}a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right)F$$

has a solution $F=F(t,x)=e^{x\left(1-\sqrt{1-2t}\right)}$, where

$$\begin{aligned}
&a_0(N,x)=x^N,\quad a_{N-1}(N,x)=(2n-3)!!x,\\
&a_j(N,x)=x^{N-j}\sum_{i_j=0}^{N-j-1}\sum_{i_{j-1}=0}^{N-j-1-i_j}\cdots\sum_{i_1=0}^{N-j-1-i_j-\cdots-i_2}\\
&\qquad\times\left(\prod_{k=1}^{j}\left(N-i_j-i_{j-1}-\cdots-i_k-(j-(2k-2))\right)\right).
\end{aligned}$$

Recall the the reverse Bessel polynomials $p_k(x)$ are given by the generating function as

$$\begin{aligned}
F=F(t,x)&=e^{x\left(1-\sqrt{1-2t}\right)}\\
&=\sum_{k=0}^{\infty}p_k(x)\frac{t^k}{k!}.
\end{aligned}\tag{2.25}$$

Thus, by (2.25), we get

$$\begin{aligned}
F^{(N)}&=\left(\frac{d}{dt}\right)^N F(t,x)\\
&=\sum_{k=N}^{\infty}p_k(x)(k)_N\frac{t^{k-N}}{k!}\\
&=\sum_{k=0}^{\infty}p_{k+N}(x)(k+N)_N\frac{t^k}{(k+N)!}\\
&=\sum_{k=0}^{\infty}p_{k+N}(x)\frac{t^k}{k!}.
\end{aligned}\tag{2.26}$$

On the other hand, by Theorem 1, we get

$$\begin{aligned}
F^{(N)}&=\left(\sum_{i=N}^{2N-1}a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right)F\\
&=\sum_{i=N}^{2N-1}a_{i-N}(N,x)\left(\sum_{l=0}^{\infty}\left(-\frac{i}{2}\right)_l\frac{(-2t)^l}{l!}\right)\left(\sum_{m=0}^{\infty}p_m(x)\frac{t^m}{m!}\right)
\end{aligned}\tag{2.27}$$

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$$= \sum_{k=0}^{\infty} \left\{ \sum_{i=N}^{2N-1} a_{i-N}(N,x) \sum_{l=0}^{k} \binom{k}{l} 2^l \left(\frac{i}{2} + l - 1 \right)_l p_{k-l}(x) \right\} \frac{t^k}{k!}.$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2. *For $k \in \mathbb{N} \cup \{0\}$, and $N \in \mathbb{N}$, we have*

$$p_{k+N}(x) = \sum_{i=N}^{2N-1} a_{i-N}(N,x) \sum_{l=0}^{k} \binom{k}{l} 2^l \left(\frac{i}{2} + l - 1 \right)_l p_{k-l}(x),$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$, $(n \geq 1)$, and $(x)_0 = 1$.

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On existence and comparison results for solutions to stochastic functional differential equations in the G-framework

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Abstract

With the advancement in stochastic calculus, stochastic differential equations have now become very common in different fields such as engineering, population dynamics, physics, system sciences, ecological sciences, medicine and financial mathematics. In several stochastic dynamic systems, one assumes that the future state of the system does not depend on its past states. However, under close analysis, it becomes evident that most realistic models would contain some of the past states of the system, and one would require stochastic functional differential equations in order to study such systems. This paper presents the existence theory for stochastic functional differential equations in the G-framework (in short G-SFDEs). The comparison theorem has been developed in a bid to obtain the required results. It is ascertained that the G-SFDEs, whose coefficients may be discontinuous functions, have more than one continuous and bounded solutions.

Key words: Existence, G-Brownian motion, Stochastic functional differential equations, discontinuous coefficients.

1 Introduction

In the last twenty years, the greater requirement for tools and procedure of stochastic calculus has been recorded in different scientific fields. In the study of financial markets, it has acquired the state of an essential element, projected in dynamic phenomena of routine changes in share and stock prices. Stochastic calculus has its applications in engineering, as well as in filtering and control theory, and even in physics, when it deals with the effect of random changes on different physical phenomena. In Biology, its main usage is in modeling the achievement of stochastic changes in reproduction on populations processes. The idea of G-Brownian motion, which is a new

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stochastic process, was given by a Chinese mathematician Shige Peng in 2006 [12]. This theory opened a new era in stochastic calculus and financial mathematics. This type of motion has a newer construction as it does not depend on a specific probability space. This motion explains the ancient Brownian motion in an extraordinary way. In the framework of a sublinear expectation (called as G-expectation), he established the associated Itô's calculus. During his research on stochastic calculus, Peng set up the existence and uniqueness of solutions for stochastic differential equations driven by G-Brownian motion in short (G-SDEs) with Lipschitz continuous coefficients [12, 13]. Then F. Gao generalized the associated Itô's calculus and the existence theory of G-SDEs with Lipschitz continuity condition using the concept of G-capacity and quasi-sure analysis [6]. Y. Ren and L. Hu proved the existence and uniqueness of solutions for G-SDEs under the Carathéodory conditions, while later on, X. Bai and Y. Lin extended the theory for G-SDEs to the integral Lipschitz conditions [1]. In the G-frame, stochastic functional differential equations were introduced by Ren, Bi and Sakthivel [14]. Then studied by Faizullah[4]. He used the Cauchy-Maruyama approximation scheme to establish the existence-and-uniqueness theorem for SFDEs in the G-frame with linear growth condition as well as Lipschitz continuity condition [4]. In a different manner, this paper explores the existence theory for SFDEs in the G-frame, whose coefficients may not be continuous. This is the generalization of the previous work by Faizullah, Mukhtar and Rana [5]. We consider stochastic functional differential equations in the G-framework of the following type

$$dY(t) = \kappa(t, Y_t)dt + \lambda(t, Y_t)d\langle B, B\rangle(t) + \lambda(t, Y_t)dB(t),\ 0 \leq t \leq T. \tag{1.1}$$

Recall that $Y_t = \{Y(t+\theta) : -\delta \leq \theta \leq 0, \delta > 0\}$ is a bounded continuous stochastic process from $[-\tau, 0]$ to $\mathbb{R}$ where at time t, the value of stochastic process is denoted by $Y(t)$ [4]. Also, Y_t indicates the collection of continuous bounded real-valued functions ψ defined on $[-\delta, 0]$ with norm $\|\psi\| = \sup_{-\delta\leq\theta\leq 0} \mid \psi(\theta) \mid$. Let κ, λ and μ are Borel measurable functions from $[0, T] \times BC([-\tau, 0]; \mathbb{R})$ to $\mathbb{R}$. We define the initial data of equation (1.1) as follows;

$$\begin{aligned} Y_{t_0} =& \zeta = \{\zeta(\theta) : -\tau < \theta \leq 0\}\ is\ \mathcal{F}_0 - measurable,\ BC([-\tau, 0]; \mathbb{R}) -\ valued \\ &random\ variable\ so\ that\ \zeta \in M_G^2\left([-\tau, 0]; \mathbb{R}\right). \end{aligned} \tag{1.2}$$

The integral form of problem (1.1) is given as the following

$$Y(t) = \zeta(0) + \int_0^t \kappa(s, Y_s)ds + \int_0^t \lambda(s, Y_s)d\langle B, B\rangle(s) + \int_0^t \mu(s, Y_s)dB(s).$$

The G-SFDE (1.1) admit at most solution $Y(t) \in M_G^2\left([-\tau, T]; \mathbb{R}\right)$ if all its coefficients gratify the linear growth condition as well as Lipschitz condition. [4, 14]. On the other hand, in this article we assume that the coefficients κ and λ may be discontinuous functions. The solution to problem 1.1 with initial data 1.2 is a real valued stochastic process $Y(t)$, $t \in [-\tau, T]$ if it holds the following characteristics

(**a**) For every $t \in [0, T]$, $Y(t)$ is $\mathcal{F}_t$-adapted as well as path-wise continuous.

(**b**) $\kappa(t, Y_t), \lambda(t, Y_t) \in \mathcal{L}^1([o, T]; \mathbb{R})$ and $\mu(t, Y_t) \in \mathcal{L}^2([o, T]; \mathbb{R})$;

(**c**) $Y_0 = \zeta$ and $dY(t) = \kappa(t, Y_t)dt + \lambda(t, Y_t)d\langle B, B\rangle(t) + \mu(t, Y_t)dB(t)$ q.s. for each $t \in [0, T]$.

The rest of the paper is organized as follows. Some basic definitions and notions are given in the subsequent section. Section 3 presents an important results known as the comparison theorem. The final section develops the existence theorem with possible discontinuous coefficients.

2 Preliminary Concerns

This section presents some basic notions and results, which are used in forthcoming research work of this paper [2, 3, 6, 13].

2.1 Sublinear Expectation

Suppose that Ω (sample space) is a grand set and H be a family of linear and real valued functions described on Ω. Suppose that H fulfil $k \in H$, for any constant k and $|Y| \in H$ if $Y \in H$. H containing the stochastic variables.

Definition 2.1. A functional E, where $E : H \to R$, is known as a G-expectation or sublinear expectation if

(**1**) E is monotonic, that is, if $Y \geq Z$ for all $Y, Z \in H \Rightarrow E[Y] \geq E[Z]$.

(**2**) E is constant conserving, that is, $E[k] = k \ \ k \in H$.

(**3**) E is sub-additive, that is, if $E[Y + Z] \leq E[Y] + E[Z]$, for each $Y, Z \in H$.

(**4**) E is positive homogeneous, that is, $E[bY] = b[y]$ for $b \geq 0$.

the space given by triple (Ω, H, E) is said to be sublinear expectation space. And E is nonlinear expectation if it satisfies the above two conditions. Sublinear expectation is also able to state the supremum of linear expectation

Definition 2.2. G-Brownian motion A d-dimensional process $(B_t)_{t\geq 0}$, define on $(\Omega, C_{l,lip}(H), E)$, is known as G-Browmain motion, if the following conditions are hold.

(**1**) $B_0(w) = 0$.

(**2**) The increment $B_{t+r} - B_t$ is G-normally distributed for any $t, r \geq 0$.

(**3**) $B_{t+r} - B_t$ is independent from $B_{t_1}, B_{t_2}, ...B_{t_n}$ for any $n \in N,\ t, r \geq 0$ and $0 \leq t_1 \leq t_2 \leq, ... \leq t_n \leq t$.

2.2 Ito's integral of G-Brownian motion

Definition 2.3. If $T \in R^+$, a partition π_T of the interval $[0, T]$ is

$$\pi_T = \{t_0, t_1, ..., t_N\},$$

since

$$\rho(\pi_T) = max\{|t_{\epsilon+1} - t_\epsilon| : \epsilon = 0, 1, ...N - 1\},$$

where

$$0 = t_0 \leq t_1 \leq, ... \leq t_N = T,$$

we customize $\pi_T^N = \{t_0^N, t_1^N, ...t_N^N\}$ to represent a sequence of partition of $[0,T]$ since

$$\lim_{N\to\infty} \rho(\pi_T^N) = 0.$$

Let $p \geq 1$. Suppose the following sort of processes of a partition

$$\pi_T = \{t_0, t\pi_1, ..., t_N\}.$$

We take,

$$\eta_t(\omega) = \sum_{m=0}^{N-1} \xi_m(\omega) I_{[t_m, t_{m+1})}(t)$$

where $\xi_m \in L_G^p(\omega_{tm})$, for all $m = 0, 1, 2, ...N-1$. The group of these process is represented by $M_G^{p,0}(0,T)$.

Definition 2.4. Let $\eta \in M_G^{1,0}(0,T)$ with

$$\eta_t(\omega) = \sum_{m=0}^{N-1} \xi_m(\omega) I_{[t_m, t_{m+1})}(t)$$

it can be written as,

$$\int_0^T \eta_t(\omega) dt = \sum_{m=0}^{N-1} \xi_m(\omega)(t_{m+1} - t_m)$$

Definition 2.5. For every $p \geq 1$, we represent by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} = \{E[\int_0^T |\eta_t|^p dt]\}^{1/p},$$

where for $1 \leq p \leq q$, $M_G^p(0,T) \supset M_G^q(0,T)$.

Definition 2.6. For every $\eta \in M_G^p(0,T)$ of the arrangement

$$\eta_t(w) = \sum_{\epsilon=0}^{N-1} \xi_\epsilon(w) I[t_\epsilon, t_{\epsilon+1})(t),$$

it can be written as,

$$I(\eta) = \int_0^T \eta_t dB_t = \sum_{\epsilon=0}^{N-1} \xi_\epsilon (B_{t_{\epsilon+1}} - B_{t_\epsilon}).$$

Lemma 2.7. *Let a function $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$, then it can be continuously extended to $I : M_G^2(0,T) \to L_G^2(\Omega_T)$. Moreover,*

$$E[\int_0^T \eta_t dB_t] = 0,$$

$$E[(\int_0^T \eta_t dB_t)^2] \leq \sigma^2 E[\int_0^T \eta_t^2 dt].$$

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2.3 (Peng's quadratic variation process $\langle B\rangle_t$)

Definition 2.8. A 1-dimensional G-quadratic variation process is introduced as follows. Let $\pi_t^N, N=1,2,....$, be a sequence of the partition $[0,T]$ then

$$\begin{aligned} B_t^2 &= \sum_{\epsilon=0}^{N-1}(B^2_{tN_{\epsilon+1}} - B^2_{tN_\epsilon}) \\ &= \sum_{\epsilon=0}^{N-1} 2B_{tN_\epsilon}(B_{tN_{i+\epsilon}} - B_{tN_\epsilon}) + \sum_{\epsilon=0}^{N-1}(B_{tN_{\epsilon+1}} - B_{tN_\epsilon})^2. \end{aligned}$$

Taking limit $\mu(\pi_t^N) \to 0$

$$\sum_{\epsilon=0}^{N-1} 2B_{tN_\epsilon}(B_{tN_{\epsilon+1}} - B_{tN_\epsilon}) \quad \textit{converges to} \quad 2\int_0^t B_s dB_s,$$

and we have

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_s dB_s.$$

Definition 2.9. Let $\mathcal{P}$ be a (weakly compact) collection of probability measures P defined on $(\Omega, \mathcal{B}(\Omega))$ then the capacity $\hat{c}(.)$ associated to $\mathcal{P}$ is defined by

$$\hat{c}(B) = \sup_{P\in\mathcal{P}} P(B), \quad B \in \mathcal{B}(\Omega),$$

where $\mathcal{B}(\Omega)$ is the Borel σ-algebra of Ω. A set B is said to be polar if its capacity is zero, that is, $\hat{c}(B) = 0$ and a statement holds quasi-surely in short (q.s.) if it holds except on a polar set.

3 An important result

In this section, we establish an important result known as comparison theorem. First, we assume two stochastic functional integral equations given as follows.

$$Y(t) = \zeta_1(0) + \int_{t_0}^t \kappa_1(s, Y_s)ds + \int_{t_0}^t \lambda_1(s, Y_s)d\langle B, B\rangle(s) + \int_{t_0}^t \mu(s, Y_s)dB(s), \qquad t \in [0,T], \quad (3.1)$$

$$Y(t) = \zeta_2(0) + \int_{t_0}^t \kappa_2(s, Y_s)ds + \int_{t_0}^t \lambda_2(s, Y_s)d\langle B, B\rangle(s) + \int_{t_0}^t \mu(s, Y_s)dB(s), \qquad t \in [0,T]. \quad (3.2)$$

Theorem 3.1. *Let Y^1 and Y^2 are the respective unique solutions of equations* (3.1) *and* (3.2). *Suppose that $\kappa_1(s, Y_s) \le \kappa_2(s, Y_s)$ and $\lambda_1(s, Y_s) \le \lambda_2(s, Y_s)$ are componentwise for every $t \in [t_0, T]$, $y \in BC([-\tau, 0]; \mathbb{R}^d)$ and $\zeta^1 \le \zeta^2$. Also, let the coefficients κ_1, λ_1 or κ_2, λ_2 are increasing functions. Then for every $t > 0$, $Y^1 \le Y^2$ q.s.*

Proof. Suppose that κ_2 and λ_2 are increasing and consider the problem

$$\begin{aligned} Z(t) = \zeta_2(0) + \int_{t_0}^{t} \kappa_2(s, \max\{{Y_s}^1, Z_s\})ds + \int_{t_0}^{t} \lambda_2(s, \max\{{Y_s}^1, Y_s\})d\langle B, B\rangle(s) \\ + \int_{t_0}^{t} \mu(s, \max\{{Y_s}^1, Z_s\})dB(s), \ \ t_0 \leq t \leq T, \end{aligned} \tag{3.3}$$

where the function $x \to \max\{y, z\}$ satisfies the growth condition $|\max\{y, z\}| \leq |y| + |z|$ and the Lipschitz condition with constant one. It follows that all coefficients of the above equation 3.3 gratify the growth condition as well as Lipschitz condition. Thus problem 3.3 admit the only one solution say $Z(t)$. Now one has to show that $Z(t) \geq Y_s^1$ q.s. First define stopping times δ_1 and δ_2 as follows. More details on stopping times can be found in [9, 10, 11].

$$\begin{aligned} \delta_1 &= \inf\{t \in [t_0, T] : Y_s^1 - Z(t) > 0\} \ \ \textit{where} \ \ \delta_1 < T, \\ \delta_2 &= \inf\{t \in [\tau_1, T] : Y_s^1 - Z(t) < 0\}. \end{aligned}$$

Contrary assume that $(\delta_1, \delta_2) \subset [t_0, T]$ be an arbitrary interval, such that $Z(\delta_1) = Y^1(\delta_1) = \zeta^*(0)$ and $Z(t) \leq Y^1(t)$ for every $t \in (\delta_1, \delta_2)$. Then,

$$\begin{aligned} Z(t) - Y^1(t) = \zeta^*(0) &+ \int_{\delta_1}^{t} \kappa_2(s, \max\{{Y_s}^1, Z_s\})ds + \int_{\delta_1}^{t} \lambda_2(s, \max\{{Y_s}^1, Z_s\})d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} \mu(s, \max\{{Y_s}^1, Z_s\})dB(s) - \zeta^*(0) - \int_{\delta_1}^{t} \kappa_1(s, {Y_s}^1)ds \\ &- \int_{\delta_1}^{t} \lambda_1(s, {Y_s}^1)d\langle B, B\rangle(s) - \int_{\delta_1}^{t} \mu(s, {Y_s}^1)dB(s), \ \ t \in (\delta_1, \delta_2). \end{aligned}$$

$$\begin{aligned} Z(t) - Y^1(t) = &\int_{\delta_1}^{t} [\kappa_2(s, \max\{{Y_s}^1, Z_s\}) - \kappa_1(s, {Y_s}^1)]ds \\ &+ \int_{\delta_1}^{t} [\lambda_2(s, \max\{{Y_s}^1, Z_s\}) - \lambda_1(s, {Y_s}^1)]d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} [\mu(s, \max\{{Y_s}^1, Z_s\}) - \mu(s, {Y_s}^1)]dB(s), \ \ t \in (\delta_1, \delta_2). \end{aligned}$$

But the assumption $Z(t) \leq Y^1(t)$ gives $\max[Y^1, Z] = Y^1$. So, we have

$$\begin{aligned} Z(t) - Y^1(t) = &\int_{\delta_1}^{t} [\kappa_2(s, {Y_s}^1) - \kappa_1(s, {Y_s}^1)]ds \\ &+ \int_{\delta_1}^{t} [\lambda_2(s, {Y_s}^1) - \lambda_1(s, {Y_s}^1)]d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} [\mu(s, {Y_s}^1) - \mu(s, {Y_s}^1)]dB(s), \end{aligned}$$

which gives $Z(t) \geq Y^1(t)$ because $\kappa_2(t, y) \geq \kappa_1(t, y)$ and $\lambda_2(t, y) \geq \lambda_1(t, y)$. This gives contradiction. So, the supposition $Z(t) \leq Y^1(t)$ for every $t \in (\delta_1, \delta_2)$ is not true. Thus $Z(t) \geq Y^1(t)$ q.s. and hence $\max\{Y^1, Z\} = Z$. It follows that $Z = Y^2 \geq Y^1$ because problem (3.3) admit a single solution Y^2. The proof is complete. □

4 Existence of solutions to SFDEs in the G-framework

Next, we assume that the coefficients κ and λ are not continuous. However, they are increasing, left continuous and $\kappa(t,y) \geq 0$, $\lambda(t,y) \geq 0$ for every $(t,y) \in [0,T] \times BC([-\delta,0];\mathbb{R})$. Assume a sequence of problems given as follows.

$$Y^l(t) = \zeta(0) + \int_0^t \kappa(s, Y_s^{l-1})ds + \int_0^t \lambda(s, Y_s^l)d\langle B, B\rangle(s) + \int_0^t \mu(s, Y_s^l)dB(s), \ t \in [0,T], \tag{4.1}$$

where $Y^0 = L_t$, L_t is the unique solution of the equation given by

$$L_t = \zeta + \int_0^t \mu(s, L_s)dB(s), \tag{4.2}$$

where $t \in [0,T]$. By our supposition $\kappa(t,y) \geq 0$, $\lambda(t,y) \geq 0$ and comparison result we obtain $Y^1 \geq L_t$. Thus, one can see that the sequence $\{Y^l : l \geq 1\}$ is increasing. In the following lemma we show that Y^l is bounded.

Lemma 4.1. *Let $Y^l(t)$ denotes a solution of equation* (4.1)*. Then*

$$E\left(\sup_{-\delta \leq s \leq T} |Y^l(s)|^2\right) \leq K,$$

where $K = C_6 e^{C_5 T}$, $C_6 = E[\|\zeta\|] + C_4$, $C_5 = 4(C_1 + C_2 + C_3)$, $C_4 = 4[E|\zeta|^2 + C_1 T + C_2 T + C_3 T]$, C_1, C_2 and C_3 are positive constants.

Proof. Define the following stopping time, for any $l \geq 1$

$$\delta_m = T \wedge inf\{t \in [t_0, T] : \|Y_t^l\| \geq m\}.$$

We get $\delta_m \uparrow T$ and define $Y^{l,m}(t) = Y^l(t \wedge \delta_m)$ for $t \in (-\tau, T)$. Next we proceed as follows.

$$Y^{l,m}(t) = \zeta(0) + \int_0^t \kappa(s, Y_s^{l-1,m})I_{[o,\delta_m]}ds + \int_0^t \lambda(s, Y_s^{l,m})I_{[o,\delta_m]}d\langle B, B\rangle_s + \int_0^t \mu(s, Y_s^{l,m})I_{[o,\delta_m]}dB_s.$$

$$\begin{aligned}
|Y^{l,m}(t)|^2 &= |\zeta(0) + \int_0^t \kappa(s, Y_s^{l-1,m})I_{[0,\delta_m]}ds + \int_0^t \lambda(s, Y_s^{l,m})I_{[0,\delta_m]}d\langle B, B\rangle_s \\
&\quad + \int_0^t \mu(s, Y_s^{l,m})I_{[0,\delta_m]}dB_s|^2 \\
&\leq 4|\zeta(0)|^2 + 4|\int_0^t \kappa(s, Y_s^{l-1,m})I_{[0,\delta_m]}ds|^2 + 4|\int_0^t \lambda(s, Y_s^{l,m})I_{[0,\delta_m]}d\langle B, B\rangle_s|^2 \\
&\quad + 4|\int_0^t \mu(s, Y_s^{l,m})I_{[0,\delta_m]}dB_s|^2
\end{aligned}$$

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By taking G-expectation on both sides, using the linear growth condition and Burkholder-Davis-Gundy inequalities [6, 13] we proceed as follows

$$
\begin{aligned}
E[|Y^{l,m}(t)|^2] \leq\ & 4E|\zeta(0)|^2 + 4C_1\int_0^t [1+E|Y_s^{l-1,m}|^2]ds + 4C_2\int_0^t [1+E|Y_s^{l,m}|^2]ds| \\
& + 4C_3\int_0^t [1+E|Y_s^{l,m}|^2]ds \\
& \leq 4E|\zeta(0)|^2 + 4C_1\int_0^t ds + 4C_1\int_0^t E|Y_s^{l-1,m}|^2 ds + 4C_2\int_0^t dt + 4C_2\int_0^t E|Y_s^{l,m}|^2 ds \\
& + 4C_3\int_0^t ds + 4C_3\int_0^t E|Y_s^{l,m}|^2 ds \\
& = 4E|\xi(0)|^2 + 4C_1T + 4C_1\int_0^t E|Y_s^{l-1,m}|^2 ds + 4C_2T + 4C_2\int_0^t E|Y_s^{l,m}|^2 ds \\
& + 4C_3T + 4C_3\int_0^t E|Y_s^{l,m}|^2 ds.
\end{aligned}
$$

For any $j \in \mathbb{N}$ we get,

$$
\max_{1\leq l\leq j} E[|Y^{l,m}(t)|^2] \leq C_4 + 4C_1\int_0^t \max_{1\leq l\leq j} E|Y_s^{l-1,m}|^2 ds + 4C_2\int_0^t \max_{1\leq l\leq j} E|Y_s^{l,m}|^2 ds + 4C_3\int_0^t \max_{1\leq l\leq j} E|Y_s^{l,m}|^2 ds,
$$

where $C_4 = 4[E|\zeta|^2 + C_1T + C_2T + C_3T]$. Hence by Doob's martingale inequality we get for any $l, m \in \mathbb{N}$

$$
E[\sup_{0\leq s\leq t} |Y^{l,m}(s)|^2] \leq C_4 + C_5\int_0^t E|Y_s^{l,m}|^2 ds, \tag{4.3}
$$

where $C_5 = 4(C_1 + C_2 + C_3)$. One can observe the fact [11],

$$
\sup_{-\delta\leq s\leq t} |Y^{l,m}(v)|^2 \leq \|\zeta\| + \sup_{0\leq s\leq t} |Y^{l,m}(s)|^2,
$$

and hence 4.3 gives

$$
\begin{aligned}
E[\sup_{-\delta\leq s\leq t} |Y^{l,m}(s)|^2] &\leq E[\|\zeta\|] + C_4 + C_5\int_0^t E|Y_s^{l,m}|^2 ds \\
&\leq C_6 + C_5\int_0^t E[\sup_{-\delta\leq q\leq s} |Y^{l,m}(q)|^2] ds,
\end{aligned}
$$

where $C_6 = E[\|\zeta\|] + C_4$. Finally, taking $m \to \infty$ and by the Gronwall's inequality we get,

$$
E[\sup_{-\delta\leq s\leq t} |Y^{l}(s)|^2] \leq C_6 e^{C_5 t}.
$$

Letting $t = T$ we have

$$
E[\sup_{-\delta\leq s\leq T} |Y^{l}(s)|^2] \leq K,
$$

where $K = C_6 e^{C_5 T}$. Hence, the proof stands completed. □

Theorem 4.2. *Let the coefficients $\kappa(t,y)$ and $\lambda(t,y)$ are increasing in the second variable y and left continuous. For all $(t,y) \in [0,T] \times BC([-\tau,0];\mathbb{R})$, $\kappa(t,y) \geq 0$ and $\lambda(t,y) \geq 0$. Then there exists at least one solution $Y(t) \in M_G^2([-\tau,T];\mathbb{R})$ to problem* (1.1).

Proof. Theorem 3.1 follows that the sequence $\{Y^l\}$ is increasing. On the other hand, Lemma 4.1 shows that $\{Y^l\}$ is a bounded sequence in the norm $\mathbb{L}^2$. Thus dominated convergence theorem yields that Y^n converges in $\mathbb{L}^2$. Let Y be the limit of Y^l. Then for almost all w, we have

$$\kappa(t,Y^l(t)) \to \kappa(t,Y(t)) \ as\ l \to \infty,$$
$$\lambda(t,Y^l(t)) \to \lambda(t,Y(t)) \ as\ l \to \infty.$$

Also

$$|\kappa(t,Y^l(t))| \leq K(1+\sup_l |Y^l(t)|) \in L^1([t_0,T]),$$
$$|\lambda(t,Y^l(t))| \leq K(1+\sup_l |Y^l(t)|) \in L^1([t_0,T]).$$

Since $\langle B\rangle$ is continuous, so, for uniformly in t and almost all w

$$\int_0^t \kappa(s,Y^l(s))ds \to \int_0^t \kappa(s,Y(s))ds,\ l \to \infty,$$
$$\int_0^t \lambda(s,Y^l(s))\langle B,B\rangle(s) \to \int_0^t \lambda(s,Y(s))\langle B,B\rangle(s),\ l \to \infty.$$

Since G-integral is continuous we get,

$$\sup_{0\leq t\leq T}\left|\int_0^t \mu(s,Y^l(s))dB(s) - \int_0^t \mu(s,Y(s))dB(s)\right| \to 0\ (q.s),\ l \to \infty.$$

Obviously, the sequence Y^l converges uniformly to Y in t, hence Y is continuous. Taking limits $l \to \infty$ on both sides of equation (4.1), we obtain that Y is the solution to G-SFDE (1.1) with initial condition (1.2). □

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Interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and their calculations†

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Abstract: It is necessary to assume additivity and independent among decision making criteria for traditional multiple decision making (MDM) in which the weights given by decision makers based on a additive measure. However, most criteria have inter-dependent or interactive characteristics in the real decision making problems. Furthermore, with respect to multiple attribute group decision making (MAGDM) problems in which the attribute weights and the expert weights take the form of real numbers and the attribute values take the form of interval-valued intuitionistic sets, we propose interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and discuss their calculations in this paper. First, we introduce some concepts of fuzzy measure, interval-valued intuitionistic sets and Archimedean t-norm. Then, the representations and transformations of Archimedean t-norm and Archimedean t-conorm are obtained, and the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are presented under intuitionistic fuzzy environment. Finally, as fuzzy Choquet integral operators, some aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are given.
Keywords: Intuitionistic sets; Fuzzy Choquet integral operators; Archimedean t-norm.

1. Introduction

Multiple attribute decision making (MADM) problem is an important research topic in decision theory. Because the objects are fuzzy and uncertain, the attributes involved in decision problems are not always expressed as real numbers, and some better suited to be denoted by fuzzy numbers, such as interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic numbers on uncertain linguistic variables, and intuitionistic fuzzy numbers. Because Zadeh initially proposed the basic model of fuzzy decision making based on the theory of fuzzy mathematics, fuzzy MADM has been receiving more and more attention. We also notice that the main technologies in multiple attribute decision making, whether the situation is certain or vague, are how to define and calculate the aggregation operators proposed in the practice.

The fuzzy set (FS) theory proposed by Zadeh [1] was a very good tool to research the fuzzy MADM problems, the fuzzy set is used to character the fuzziness just by membership degree. Different from fuzzy set, there is another parameter: non-membership degree in intuitionistic fuzzy set (IFS) which is proposed by Atanassov [2, 3]. Clearly, the IFS can describe and character the fuzzy essence of the objective world more accurately [2] than the fuzzy set, and has received more and more attention since its appearance. Later, Atanassov and Gargov [4, 5] further introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are interval numbers rather real numbers.

Base on Archimedean t-conorm and t-norm [6, 7], and the aggregation functions for the classical fuzzy sets (FSs), Beliakov et al. gave some operations about intuitionistic fuzzy sets, proposed two general concepts for constructing other types of aggregation operators for intuitionistic fuzzy sets (IFSs)

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extending the existing methods and showed that the operators obtained by using the Lukasiewicz t-norm are consistent with the ones on ordinary FSs. We can find above aggregation operators are all based on different relationships of the aggregated arguments, which can provide more choices for the decision makers.

As an aggregation function, it is well-known that Choquet integral [8] based on non-additive fuzzy measure, is a kind of non-additive and non-linear integral, and has been successfully used for handling information fusion and decision making problems (MCDM). The main characteristic of this aggregation function is that it is able to flexibly describe the relative importance of decision criteria as well as their interactions. There are many works on the Choquet integral of single-valued functions, set-valued functions and studied their mathematical properties. It is of interest to combine the Choquet integral and the IFS theory or MCDM under intuitionistic fuzzy environment, because, by doing this, we cannot only deals with the imprecise and uncertain decision information but also efficiently take into account the various interactions among the decision criteria. The intuitionistic fuzzy-valued Choquet integral, the combination of the Choquet integral and the IFS theory, can also act an aggregation tool employed in MCDM as well as other multicriteria analysis field. In this paper, we propose the interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and discuss their calculations. First, we introduced some concepts of fuzzy measure and interval-valued intuitionistic sets based on Archimedean t-norm. Then, interval-valued intuitionistic weighted average(geometric) operator based on Archimedean t-norm, interval-valued intuitionistic ordered weighted average operator based on Archimedean t-norm are developed.

The rest of this study is organized as follows. In section 2, we recall the definitions of intuitionistic fuzzy set、Archimedean t-norm and Choquet integral. In section 3, the representations and transformations of Archimedean t-norm and Archimedean t-conorm are proposed and inveastigated, and some of its properties are investigated in detail by means of the representation theorem. In section 4, the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm is presented under intuitionistic fuzzy environment. In section 5, an aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are defined and discussed.

2. Definitions and preliminaries

A fuzzy measure on X is a set function $\mu : P(X) \rightarrow [0,1]$ such that
(i) $\mu(\emptyset) = 0, \mu(X) = 1$;
(ii) $A, B \subseteq X$, $A \subseteq B$ implies $\mu(A) \leqslant \mu(B)$.

Definition 2.1. Let $A, B \in P(X), A \cap B = \varnothing$. If fuzzy measure g satisfies the following conditions:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B)$$

and $\lambda \in (-1, \infty)$.

Especially if $\lambda = 0$, then g is an additive measure, which means there is no interaction between coalitions A and B .

Let $X = \{x_1, x_2, ...x_n\}$ be a attribute index set, if $i, j = 1, 2, ..., n$ and $i \neq j, x_i \cap x_j = \varnothing, \bigcup_{i=1}^n x_i = X$, then

$$g(X) = \begin{cases} \frac{1}{\lambda}(\prod_{i=1}^n [1 + \lambda g(x_i)] - 1) & \lambda \neq 0, \\ \sum_{i=1}^n g(x_i) & \lambda = 0, \end{cases} \tag{1}$$

From Eq. (1), for the $A \in P(X)$, g can be expressed by

$$g(X) = \begin{cases} \frac{1}{\lambda}(\prod_{i \in A} [1 + \lambda g(x_i)] - 1) & \lambda \neq 0, \\ \sum_{i \in A} g(x_i) & \lambda = 0, \end{cases} \tag{2}$$

For x_i, $g(x_i)$ is called a fuzzy measure function, and it indicates the importance degree of x_i.

From $g(X) = 1$, we know λ is determined by $\lambda + 1 = \prod_{i=1}^n (1 + \lambda g(x_i))$.

Definition 2.2. Let f be a positive real-valued function on X, the discrete Choquet integral of f with respect to a fuzzy measure μ on X is defined as

$$C_\mu(f(x_{(1)}), ..., f(x_{(n)})) = \sum_{i=1}^n f(x_{(i)})[\mu(A_{(i)}) - \mu(A_{(i+1)})]$$

where $(\cdot)$ indicates a permutation on X such that $f(x_{(1)}) \leqslant \cdots \leqslant f(x_{(n)})$. $A_{(i)} = (i, \ldots, n)$, and $A_{(n+1)} = \emptyset$.

A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions [8,9]:
1) $T(1, x) = x$, for all x.
2) $T(x, y) = T(y, x)$, for all x and y.
3) $T(x, T(y, z)) = T(T(x, y), z)$, for all x, y and z.
4) $x \leqslant x', y \leqslant y'$ implies $T(x, y) \leqslant T(x', y'),\ x, y, x', y' \in [0,1]$.

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions[8,9]:
1) $S(0, x) = x$, for all x.
2) $S(x, y) = S(y, x)$, for all x and y.
3) $S(x, S(y, z)) = S(S(x, y), z)$, for all x, y and z.
4) $x \leqslant x', y \leqslant y'$ implies $S(x, y) \leqslant S(x', y'),\ x, y, x', y' \in [0,1]$.

Definition 2.3 [8,9]. A t-norm function $T(x, y)$ is called Archimedean t-norm if it is continuous and $T(x, x) < x$ for all $x \in [0,1]$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x, y \in (0,1)$.

A t-conorm function $S(x, y)$ is called Archimedean t-conorm if it is continuous and $S(x, x) > x$ for all $x \in [0,1]$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x, y \in (0,1)$.

Definition 2.4. Let X be in a given domain. Then,

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

is called an interval-valued intuitionistic fuzzy set ($IVIFS$), where $\mu_A : X \to I \subset [0,1],\ \ \nu_A : X \to J \subset [0,1]$ and I, J are closed intervals in $[0,1]$, the following condition is met: $\sup \mu_A(x) + \sup \nu_A(x) \leqslant 1\ ,\ x \in X$. The intervals $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x on X.

Thus for each x, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals and their lower and upper end points are, respectively, denoted by $\mu_A^L(x)$, $\mu_A^U(x)$,$\nu_A^L(x)$,$\nu_A^U(x)$. We can denote by

$$A = \{\langle x, [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]]\rangle | x \in X\},$$

where $0 \leqslant \mu_A^U(x) + \nu_A^U(x) \leqslant 1,\ x \in X,\ \mu_A^L(x) \geqslant 0$ and $\nu_A^L(x) \geqslant 0$.

Simply, we write $A = \langle[\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]$.

For each element x, we can compute its hesitation interval of x as:

$$\pi_A(x) = [\pi_A^L(x), \pi_A^U(x)] = [1 - \nu_A^U(x) - \mu_A^U(x), 1 - \nu_A^L(x) - \mu_A^L(x)].$$

3. The representations and transformations of Archimedean t-norm and Archimedean t-conorm

Definition 3.1. A mapping $N : [0,1] \to [0,1]$ is called negation operator, if N is decreasing and $N(0) = 1$, $N(1) = 0$. Especially, we have

(i) If $N(x) = 1 - x$, it is called standard negation operator.

(ii) $\forall x \in [0,1]$, if $N(N(x)) = x$, then it is called cyclotron negation operator. Obviously, cyclotron negation operator is continuous and strictly increasing.

(iii) For each negation operator, T and S are dual with respect to $N(x)$ if and only if $T(N(x), N(y)) = N(S(x, y))$.

It is well known [9] that a strict Archimedean t-norm is expressed via its additive generator g as $T(x, y) = g^{-1}(g(x) + g(y))$, and similarly, applied to its dual t-conorm $S(x, y) = h^{-1}(h(x) + h(y))$ with $h(t) = g(N(t))$. We notice that an additive generator of a continuous Archimedean t-norm is a strictly decreasing function $g : [0,1] \to [0, +\infty)$ such that $g(1) = 0$. If we assign specific forms to the function g, then some well-known t-conorms and t-norms can be obtained. Let me emphasize that the results (1-4) were shown in [11], however, considering that the representation of the negation operator is always restricted by the policy mak- ers' historical knowledge, perceptual judgement and other factors in the game playing, benefit groups' voting or decision making process, we could define the negation operator by means of the fuzzy logic non-portal operators in this paper and calculate Archimedean t-norm and Archimedean t-conorm as results (5-8) as follows.

Theorem 3.1 Let $T(x,y)$ be Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm. Then we have the following statements:

If $N(x)=1-x$, i.e. $h(t)=g(1-t)$), then the following are valid:

(1) Let $g(t)=-\log t$, then $h(t)=-\log(1-t), g^{-1}(t)=\exp^{-t}, h^{-1}(t)=1-\exp^{-t}$, and Algebraic t-conorm and t-norm [10] are obtained as follows:

$$T^A(x,y)=x\cdot y,\quad S^A(x,y)=x+y-xy.$$

(2) Let $g(t)=\log(\frac{2-t}{t})$, then $h(t)=\log(\frac{2-(1-t)}{1-t}), g^{-1}(t)=\frac{2}{\exp^t+1}, h^{-1}(t)=1-\frac{2}{\exp^t+1}$, and we get Einstein t-conorm and t-norm [10]:

$$T^E(x,y)=\frac{xy}{1+(1-x)(1-y)}\ ,\ S^E(x,y)=\frac{x+y}{1+xy}.$$

(3) Let $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then we have $h(t)=\log(\frac{\gamma+(1-\gamma)(1-t)}{1-t}), g^{-1}(t)=\frac{\gamma}{\exp^t+\gamma-1}$, $h^{-1}(t)=1-\frac{\gamma}{\exp^t+\gamma-1}$, and Hamacher t-conorm and t-norm [10] are obtained as follows:

$$T^H_\gamma(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}\ ,\ \gamma>0,$$

$$S^H_\gamma(x,y)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}\ ,\ \gamma>0.$$

Especially, if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Algebraic t-conorm and t-norm respectively; if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Einstein t-conorm and t-norm respectively.

(4) Let $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then $h(t)=\log(\frac{\gamma-1}{\gamma^{1-t}-1}), g^{-1}(t)=\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}, h^{-1}(t)=1-\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}$, and we have Frank t-conorm and t-norm [10] as follows:

$$T^F_\gamma(x,y)=\log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})\ ,\ \gamma>1,$$

$$S^F_\gamma(x,y)=1-\log_\gamma(1+\frac{(\gamma^{1-x}-1)(\gamma^{1-y}-1)}{\gamma-1})\ ,\ \gamma>1.$$

Especially, if $\gamma\to1$, then we have

$$\lim_{\gamma\to1}g(t)=\lim_{\gamma\to1}\log(\frac{\gamma-1}{\gamma^t-1})=\lim_{\gamma\to1}\log(\frac{1}{t\gamma^{t-1}-1})=-\log t.$$

which indicates that $\lim_{\gamma\to1}S^F_\gamma(x,y)=S^A_\gamma(x,y)$ and $\lim_{\gamma\to1}T^F_\gamma(x,y)=T^A_\gamma(x,y)$.

If $N(x)=1-x^2$, i.e. $h(t)=g(1-t^2)$ then the following are also valid:

(5) Let $g(t)=-\log t$, then $h(t)=-\log(1-t^2), g^{-1}(t)=\exp^{-t}, h^{-1}(t)=\sqrt{1-\exp^{-t}}$, and Algebraic t-conorm and t-norm [10] are obtained as follows:

$$T^A_2(x,y)=xy\ ,\ S^A_2(x,y)=\sqrt{1-(1-x^2)(1-y^2)}.$$

(6) Let $g(t)=\log(\frac{2-t}{t})$, then we have $h(t)=\log\frac{1+t^2}{1-t^2}, g^{-1}(t)=\frac{2}{\exp^t+1}, h^{-1}(t)=\sqrt{\frac{\exp^t-1}{\exp^t+1}}$, and we get Einstein t-conorm and t-norm [10] are obtained as follows:

$$T^E_2(x,y)=\frac{xy}{1+(1-x)(1-y)}\ ,\ S^E_2(x,y)=\sqrt{\frac{x^2+y^2}{1+x^2y^2}}/$$

(7) Let $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then we have $h(t)=\log(\frac{\gamma+(1-\gamma)(1-t^2)}{1-t^2}), g^{-1}(t)=\frac{\gamma}{\exp^t+\gamma-1}$, $h^{-1}(t)=\sqrt{1-\frac{\gamma}{\exp^t+\gamma-1}}$, and Hamacher t-conorm and t-norm [10] are obtained as follows:

$$T^H_{2\gamma}(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}\ ,\ \gamma>0,$$

$$S^H_{2\gamma}(x,y)=\sqrt{\frac{x^2+y^2-x^2y^2-(1-\gamma)x^2y^2}{1-(1-\gamma)x^2y^2}}\ ,\ \gamma>0.$$

Especially, if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Algebraic t-conorm and t-norm respectively; if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Einstein t-conorm and t-norm respectively.

(8) Let $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then $h(t)=\log(\frac{\gamma-1}{\gamma^{1-t^2}-1}), g^{-1}(t)=\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}$, $h^{-1}(t)=\sqrt{1-\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}}$,
and we have Frank t-conorm and t-norm [10] as follows:

$$T^F_{2\gamma}(x,y)=\log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})\ ,\ \gamma>1,$$

$$S^F_{2\gamma}(x,y)=\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-x^2}-1)(\gamma^{1-y^2}-1)}{\gamma-1})}\ ,\ \gamma>1.$$

Especially, if $\gamma\to1$, then we have

$$\lim_{\gamma\to1}g(t)=\lim_{\gamma\to1}\log(\frac{\gamma-1}{\gamma^t-1})=\lim_{\gamma\to1}\log(\frac{1}{t\gamma^{t-1}-1})=-\log t.$$

which indicates that $\lim_{\gamma\to1}S^F_\gamma(x,y)=S^A_\gamma(x,y)$ and $\lim_{\gamma\to1}T^F_\gamma(x,y)=T^A_\gamma(x,y)$.

4. The operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm

Definition 4.1. Let $\widetilde{\alpha}_i=\langle[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2)$ be two interval-valued intuitionistic fuzzy sets, $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, and $\lambda\geqslant0$. We can define the operational rules about $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ based on Archimedean t-norm as follows

(1) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[S(\mu^L(\alpha_1),\mu^L(\alpha_2)),S(\mu^U(\alpha_1),\mu^U(\alpha_2))],[T(\nu^L(\alpha_1),\nu^L(\alpha_2)),T(\nu^U(\alpha_1),\nu^U(\alpha_2))]\rangle$
$=\langle[h^{-1}(h(\mu^L(\alpha_1))+h(\mu^L(\alpha_2))),h^{-1}(h(\mu^U(\alpha_1))+h(\mu^U(\alpha_2)))],$
$[g^{-1}(g(\nu^L(\alpha_1))+g(\nu^L(\alpha_2))),g^{-1}(g(\nu^U(\alpha_1))+g(\nu^U(\alpha_2)))]\rangle$;

(2) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[T(\mu^L(\alpha_1),\mu^L(\alpha_2)),T(\mu^U(\alpha_1),\mu^U(\alpha_2))],[S(\nu^L(\alpha_1),\nu^L(\alpha_2)),S(\nu^U(\alpha_1),\nu^U(\alpha_2))]\rangle$
$=\langle[g^{-1}(g(\mu^L(\alpha_1))+g(\mu^L(\alpha_2))),g^{-1}(g(\mu^U(\alpha_1))+g(\mu^U(\alpha_2)))],$
$[h^{-1}(h(\nu^L(\alpha_1))+h(\nu^L(\alpha_2))),h^{-1}(h(\nu^U(\alpha_1))+h(\nu^U(\alpha_2)))]\rangle$;

(3) $\lambda\widetilde{\alpha}_1=\langle[h^{-1}(\lambda h(\mu^L(\alpha_1))),h^{-1}(\lambda h(\mu^U(\alpha_1)))],[g^{-1}(\lambda g(\nu^L(\alpha_1))),g^{-1}(\lambda g(\nu^U(\alpha_1)))]\rangle$;

(4) $\widetilde{\alpha}_1^\lambda=\langle[g^{-1}(\lambda g(\mu^L(\alpha_1))),g^{-1}(\lambda g(\mu^U(\alpha_1)))],[h^{-1}(\lambda h(\nu^L(\alpha_1))),h^{-1}(\lambda h(\nu^U(\alpha_1)))]\rangle$.

Obviously, the above operational result is still an the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm. According Theorem 3.1 and Definition 4.1, we have Theorem 4.1 and Theorem 4.2, the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are obtained as follows.

Theorem 4.1. Let $\widetilde{\alpha}_i=\langle[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2)$ be two interval-valued fuzzy intuitionistic sets, $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, $N(x)=1-x$. Then the following operational rules based on Archimedean t-norm are hold:

(1) If $g(t)=-\log t$, then [9]

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2),\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\nu^L(\alpha_1)\nu^L(\alpha_2),\nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle$;

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\mu^L(\alpha_1)\mu^L(\alpha_2),\mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2),\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1=\langle[s_{\lambda\times\theta(\alpha_1)},s_{\lambda\times\tau(\alpha_1)}],[1-(1-\mu^L(\alpha_1))^\lambda,1-(1-\mu^U(\alpha_1))^\lambda],[(\nu^L(\alpha_1))^\lambda,(\nu^U(\alpha_1))^\lambda]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda=\langle[s_{(\theta(\alpha_1))^\lambda},s_{(\tau(\alpha_1))^\lambda}],[(\mu^L(\alpha_1))^\lambda,(\mu^U(\alpha_1))^\lambda],[1-(1-\nu^L(\alpha_1))^\lambda,1-(1-\nu^U(\alpha_1))^\lambda]\rangle$.

(2) If $g(t)=\log(\frac{2-t}{t})$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\frac{\mu^L(\alpha_1)+\mu^L(\alpha_2)}{1+\mu^L(\alpha_1)\mu^L(\alpha_2)},\frac{\mu^U(\alpha_1)+\mu^U(\alpha_2)}{1+\mu^U(\alpha_1)\mu^U(\alpha_2)}],[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{1+(1-\nu^L(\alpha_1))(1-\nu^L(\alpha_2))},\frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{1+(1-\nu^U(\alpha_1))(1-\nu^U(\alpha_2))}]\rangle$;

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{1+(1-\mu^L(\alpha_1))(1-\mu^L(\alpha_2))},\frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{1+(1-\mu^U(\alpha_1))(1-\mu^U(\alpha_2))}],[\frac{\nu^L(\alpha_1)+\nu^L(\alpha_2)}{1+\nu^L(\alpha_1)\nu^L(\alpha_2)},\frac{\nu^U(\alpha_1)+\nu^U(\alpha_2)}{1+\nu^U(\alpha_1)\nu^U(\alpha_2)}]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1=\langle[\frac{(1+\mu^L(\alpha_1))^\lambda-(1-\mu^L(\alpha_1))^\lambda}{(1+\mu^L(\alpha_1))^\lambda+(1-\mu^L(\alpha_1))^\lambda},\frac{(1+\mu^U(\alpha_1))^\lambda-(1-\mu^U(\alpha_1))^\lambda}{(1+\mu^U(\alpha_1))^\lambda+(1-\mu^U(\alpha_1))^\lambda}],[\frac{2(\nu^L(\alpha_1))^\lambda}{(2-\nu^L(\alpha_1))^\lambda+\nu^L(\alpha_1)^\lambda},\frac{2(\nu^U(\alpha_1))^\lambda}{(2-\nu^U(\alpha_1))^\lambda+\nu^U(\alpha_1)^\lambda}]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda=\langle[\frac{2(\mu^L(\alpha_1))^\lambda}{(2-\mu^L(\alpha_1))^\lambda+\mu^L(\alpha_1)^\lambda},\frac{2(\mu^U(\alpha_1))^\lambda}{(2-\mu^U(\alpha_1))^\lambda+\mu^U(\alpha_1)^\lambda}],[\frac{(1+\nu^L(\alpha_1))^\lambda-(1-\nu^L(\alpha_1))^\lambda}{(1+\nu^L(\alpha_1))^\lambda+(1-\nu^L(\alpha_1))^\lambda},\frac{(1+\nu^U(\alpha_1))^\lambda-(1-\nu^U(\alpha_1))^\lambda}{(1+\nu^U(\alpha_1))^\lambda+(1-\nu^U(\alpha_1))^\lambda}]\rangle$.

(3) If $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\frac{\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2)-(1-\gamma)\mu^L(\alpha_1)\mu^L(\alpha_2)}{1-(1-\gamma)\mu^L(\alpha_1)\mu^L(\alpha_2)},\frac{\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2)-(1-\gamma)\mu^U(\alpha_1)\mu^U(\alpha_2)}{1-(1-\gamma)\mu^U(\alpha_1)\mu^U(\alpha_2)}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{\gamma+(1-\gamma)(\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2))},\frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{\gamma+(1-\gamma)(\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2))}]\rangle$;

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{\gamma+(1-\gamma)(\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2))},\frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{\gamma+(1-\gamma)(\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2))}],$
$[\frac{\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2)-(1-\gamma)\nu^L(\alpha_1)\nu^L(\alpha_2)}{1-(1-\gamma)\nu^L(\alpha_1)\nu^L(\alpha_2)},\frac{\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2)-(1-\gamma)\nu^U(\alpha_1)\nu^U(\alpha_2)}{1-(1-\gamma)\nu^U(\alpha_1)\nu^U(\alpha_2)}]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1=\langle[\frac{(1+(\gamma-1)\mu^L(\alpha_1))^\lambda-(1-\mu^L(\alpha_1))^\lambda}{(1+(\gamma-1)\mu^L(\alpha_1))^\lambda+(\gamma-1)(1-\mu^L(\alpha_1))^\lambda},\frac{(1+(\gamma-1)\mu^U(\alpha_1))^\lambda-(1-\mu^U(\alpha_1))^\lambda}{(1+(\gamma-1)\mu^U(\alpha_1))^\lambda+(\gamma-1)(1-\mu^U(\alpha_1))^\lambda}],$
$[\frac{\gamma(\nu^L(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\nu^L(\alpha_1)))^\lambda+(\gamma-1)(\nu^L(\alpha_1))^\lambda},\frac{\gamma(\nu^U(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\nu^U(\alpha_1)))^\lambda+(\gamma-1)(\nu^U(\alpha_1))^\lambda}]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda=\langle[\frac{\gamma(\mu^L(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\mu^L(\alpha_1)))^\lambda+(\gamma-1)(\mu^L(\alpha_1))^\lambda},\frac{\gamma(\mu^U(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\mu^U(\alpha_1)))^\lambda+(\gamma-1)(\mu^U(\alpha_1))^\lambda}],$
$[\frac{(1+(\gamma-1)\nu^L(\alpha_1))^\lambda-(1-\nu^L(\alpha_1))^\lambda}{(1+(\gamma-1)\nu^L(\alpha_1))^\lambda+(\gamma-1)(1-\nu^L(\alpha_1))^\lambda},\frac{(1+(\gamma-1)\nu^U(\alpha_1))^\lambda-(1-\nu^U(\alpha_1))^\lambda}{(1+(\gamma-1)\nu^U(\alpha_1))^\lambda+(\gamma-1)(1-\nu^U(\alpha_1))^\lambda}]\rangle$.

(4) If $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[1-\log_\gamma(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)(\gamma^{1-\mu^L(\alpha_2)}-1)}{\gamma-1}),1-\log_\gamma(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)(\gamma^{1-\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[\log_\gamma(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)(\gamma^{\nu^L(\alpha_2)}-1)}{\gamma-1}),\log_\gamma(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)(\gamma^{\nu^U(\alpha_2)}-1)}{\gamma-1})]\rangle$;

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\log_\gamma(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)(\gamma^{\mu^L(\alpha_2)}-1)}{\gamma-1}),\log_\gamma(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)(\gamma^{\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[1-\log_\gamma(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)(\gamma^{1-\nu^L(\alpha_2)}-1)}{\gamma-1}),1-\log_\gamma(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)(\gamma^{1-\nu^U(\alpha_2)}-1)}{\gamma-1})]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1=\langle[1-\log_\gamma(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}),1-\log_\gamma(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})],$
$[\log_\gamma(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}),\log_\gamma(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda=\langle[\log_\gamma(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}),\log_\gamma(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})],$
$[1-\log_\gamma(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}),1-\log_\gamma(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})]\rangle$.

Theorem 4.2. Let $\widetilde{\alpha}_i=\langle[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2)$ be be two interval-valued intuitionistic fuzzy sets , $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, $N(x)=1-x$. Then the following operational rules based on Archimedean t-norm valid:

(1) If $g(t)=-\log t$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=$
$\langle[\sqrt{(\mu^L(\alpha_1))^2+(\mu^L(\alpha_2))^2-(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2},\sqrt{(\mu^U(\alpha_1))^2+(\mu^U(\alpha_2))^2-(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}],$
$[\nu^L(\alpha_1)\nu^L(\alpha_2),\nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle$;

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\mu^L(\alpha_1)\mu^L(\alpha_2),\mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\sqrt{(\nu^L(\alpha_1))^2+(\nu^L(\alpha_2))^2-(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2},\sqrt{(\nu^U(\alpha_1))^2+(\nu^U(\alpha_2))^2-(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1=\langle[\sqrt{1-(1-(\mu^L(\alpha_1))^2)^\lambda},\sqrt{1-(1-(\mu^U(\alpha_1))^2)^\lambda}],[(\nu^L(\alpha_1))^\lambda,(\nu^U(\alpha_1))^\lambda]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda=\langle[(\mu^L(\alpha_1))^\lambda,(\mu^U(\alpha_1))^\lambda],[\sqrt{1-(1-(\nu^L(\alpha_1))^2)^\lambda},\sqrt{1-(1-(\nu^U(\alpha_1))^2)^\lambda}]\rangle$.

(2) If $g(t)=\log(\frac{2-t}{t})$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\sqrt{\frac{(\mu^L(\alpha_1))^2+(\mu^L(\alpha_2))^2}{1+(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}},\sqrt{\frac{(\mu^U(\alpha_1))^2+(\mu^U(\alpha_2))^2}{1+(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{1+(1-\nu^L(\alpha_1))(1-\nu^L(\alpha_2))},\frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{1+(1-\nu^U(\alpha_1))(1-\nu^U(\alpha_2))}]\rangle$;

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{1+(1-\mu^L(\alpha_1))(1-\mu^L(\alpha_2))},\frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{1+(1-\mu^U(\alpha_1))(1-\mu^U(\alpha_2))}],$
$[\sqrt{\frac{(\nu^L(\alpha_1))^2+(\nu^L(\alpha_2))^2}{1+(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}},\sqrt{\frac{(\nu^U(\alpha_1))^2+(\nu^U(\alpha_2))^2}{1+(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}}]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1=\langle[\sqrt{\frac{(1+(\mu^L(\alpha_1))^2)^\lambda-(1-(\mu^L(\alpha_1))^2)^\lambda}{(1+(\mu^L(\alpha_1))^2)^\lambda+(1-(\mu^L(\alpha_1))^2)^\lambda}},\sqrt{\frac{(1+(\mu^U(\alpha_1))^2)^\lambda-(1-(\mu^U(\alpha_1))^2)^\lambda}{(1+(\mu^U(\alpha_1))^2)^\lambda+(1-(\mu^U(\alpha_1))^2)^\lambda}}],$
$[\frac{2(\nu^L(\alpha_1))^\lambda}{(2-\nu^L(\alpha_1))^\lambda+(\nu^L(\alpha_1))^\lambda},\frac{2(\nu^U(\alpha_1))^\lambda}{(2-\nu^U(\alpha_1))^\lambda+(\nu^U(\alpha_1))^\lambda}]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda = \langle[\frac{2(\mu^L(\alpha_1))^\lambda}{(2-\mu^L(\alpha_1))^\lambda+(\mu^L(\alpha_1))^\lambda}, \frac{2(\mu^U(\alpha_1))^\lambda}{(2-\mu^U(\alpha_1))^\lambda+(\mu^U(\alpha_1))^\lambda}],$
$[\sqrt{\frac{(1+(\nu^L(\alpha_1))^2)^\lambda-(1-(\nu^L(\alpha_1))^2)^\lambda}{(1+(\nu^L(\alpha_1))^2)^\lambda+(1-(\nu^L(\alpha_1))^2)^\lambda}}, \sqrt{\frac{(1+(\nu^U(\alpha_1))^2)^\lambda-(1-(\nu^U(\alpha_1))^2)^\lambda}{(1+(\nu^U(\alpha_1))^2)^\lambda+(1-(\nu^U(\alpha_1))^2)^\lambda}}]\rangle.$

(3) If $g(t) = \log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma > 0$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle[\sqrt{\frac{(\mu^L(\alpha_1))^2+(\mu^L(\alpha_2))^2-(\mu^L(\alpha_1))^2(\mu^L(\alpha_1))^2-(1-\gamma)(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}{1-(1-\gamma)(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}},$
$\sqrt{\frac{(\mu^U(\alpha_1))^2+(\mu^U(\alpha_2))^2-(\mu^U(\alpha_1))^2(\mu^U(\alpha_1))^2-(1-\gamma)(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}{1-(1-\gamma)(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{\gamma+(1-\gamma)(\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2))}, \frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{\gamma+(1-\gamma)(\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2))}]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle[\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{\gamma+(1-\gamma)(\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2))}, \frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{\gamma+(1-\gamma)(\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2))}],$
$[\sqrt{\frac{(\nu^L(\alpha_1))^2+(\nu^L(\alpha_2))^2-(\nu^L(\alpha_1))^2(\nu^L(\alpha_1))^2-(1-\gamma)(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}{1-(1-\gamma)(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}},$
$\sqrt{\frac{(\nu^U(\alpha_1))^2+(\nu^U(\alpha_2))^2-(\nu^U(\alpha_1))^2(\nu^U(\alpha_1))^2-(1-\gamma)(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}{1-(1-\gamma)(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}}]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle[\sqrt{\frac{(1+(\gamma-1)(\mu^L(\alpha_1))^2)^\lambda-(1-(\mu^L(\alpha_1))^2)^\lambda}{(1+(\gamma-1)(\mu^L(\alpha_1))^2)^\lambda+(\gamma-1)(1-(\mu^L(\alpha_1))^2)^\lambda}}, \sqrt{\frac{(1+(\gamma-1)(\mu^U(\alpha_1))^2)^\lambda-(1-(\mu^U(\alpha_1))^2)^\lambda}{(1+(\gamma-1)(\mu^U(\alpha_1))^2)^\lambda+(\gamma-1)(1-(\mu^U(\alpha_1))^2)^\lambda}}],$
$[\frac{\gamma(\nu^L(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\nu^L(\alpha_1)))^\lambda+(\gamma-1)(\nu^L(\alpha_1))^\lambda}, \frac{\gamma(\nu^U(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\nu^U(\alpha_1)))^\lambda+(\gamma-1)(\nu^U(\alpha_1))^\lambda}]\rangle;$

(iv) $\widetilde{\alpha}_1^\lambda = \langle[\frac{\gamma(\mu^L(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\mu^L(\alpha_1)))^\lambda+(\gamma-1)(\mu^L(\alpha_1))^\lambda}, \frac{\gamma(\mu^U(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\mu^U(\alpha_1)))^\lambda+(\gamma-1)(\mu^U(\alpha_1))^\lambda}],$
$[\sqrt{\frac{(1+(\gamma-1)(\nu^L(\alpha_1))^2)^\lambda-(1-(\nu^L(\alpha_1))^2)^\lambda}{(1+(\gamma-1)(\nu^L(\alpha_1))^2)^\lambda+(\gamma-1)(1-(\nu^L(\alpha_1))^2)^\lambda}}, \sqrt{\frac{(1+(\gamma-1)(\nu^U(\alpha_1))^2)^\lambda-(1-(\nu^U(\alpha_1))^2)^\lambda}{(1+(\gamma-1)(\nu^U(\alpha_1))^2)^\lambda+(\gamma-1)(1-(\nu^U(\alpha_1))^2)^\lambda}}]\rangle.$

(4) If $g(t) = \log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma > 1$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 =$
$\langle[\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-(\mu^L(\alpha_1))^2}-1)(\gamma^{1-(\mu^L(\alpha_2))^2}-1)}{\gamma-1})}, \sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-(\mu^U(\alpha_1))^2}-1)(\gamma^{1-(\mu^U(\alpha_2))^2}-1)}{\gamma-1})}],$
$[\log_\gamma(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)(\gamma^{1-\nu^L(\alpha_2)}-1)}{\gamma-1}), \log_\gamma(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)(\gamma^{1-\nu^U(\alpha_2)}-1)}{\gamma-1})]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle[\log_\gamma(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)(\gamma^{1-\mu^L(\alpha_2)}-1)}{\gamma-1}), \log_\gamma(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)(\gamma^{1-\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-(\nu^L(\alpha_1))^2}-1)(\gamma^{1-(\nu^L(\alpha_2))^2}-1)}{\gamma-1})}, \sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-(\nu^U(\alpha_1))^2}-1)(\gamma^{1-(\nu^U(\alpha_2))^2}-1)}{\gamma-1})}]]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle[\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-(\mu^L(\alpha_1))^2}-1)^\lambda}{(\gamma-1)^{\lambda-1}})}, \sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-(\mu^U(\alpha_1))^2}-1)^\lambda}{(\gamma-1)^{\lambda-1}})}],$
$[\log_\gamma(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \log_\gamma(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})]\rangle;$

(iv) $\widetilde{\alpha}_1^\lambda = \langle[\log_\gamma(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \log_\gamma(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})],$
$[\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-(\nu^L(\alpha_1))^2}-1)^\lambda}{(\gamma-1)^{\lambda-1}})}, \sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-(\nu^U(\alpha_1))^2}-1)^\lambda}{(\gamma-1)^{\lambda-1}})}]\rangle.$

Theorem 4.3. Let $\widetilde{\alpha}_i$ $(i = 1, 2)$ be be two interval-valued intuitionistic fuzzy sets, $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. We can easily prove the the following statements:

(1) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \widetilde{\alpha}_2 \oplus \widetilde{\alpha}_1$;
(2) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \widetilde{\alpha}_2 \otimes \widetilde{\alpha}_1$;
(3) $\lambda(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2) = \lambda\widetilde{\alpha}_1 \oplus \lambda\widetilde{\alpha}_2,\ \lambda > 0$;
(4) $\lambda_1\widetilde{\alpha}_1 \oplus \lambda_2\widetilde{\alpha}_1 = (\lambda_1 + \lambda_2)\widetilde{\alpha}_1,\ \lambda_1, \lambda_2 > 0$;
(5) $\widetilde{\alpha}_1^{\lambda_1} \otimes \widetilde{\alpha}_1^{\lambda_2} = (\widetilde{\alpha}_1)^{\lambda_1+\lambda_2},\ \lambda_1, \lambda_2 > 0$;
(6) $\widetilde{\alpha}_1^\lambda \otimes \widetilde{\alpha}_2^\lambda = (\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2)^\lambda,\ \lambda > 0$.

According Theorem 3.1 and Definition 4.1, Theorem 4.3 is easy to prove.

5. Aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm

Definition 5.1. Let $\widetilde{\alpha}_1 = \langle[\mu^L(\alpha_1), \mu^U(\alpha_1)], [\nu^L(\alpha_1), \nu^U(\alpha_1)]\rangle$ be an interval-valued fuzzy intuitionistic sets. An expected value $E(\widetilde{\alpha}_1)$ of $\widetilde{\alpha}_1$ can be represented as follows

$$E(\widetilde{\alpha}_1) = \frac{1}{2} \times (\frac{\mu^L(\alpha_1) + \mu^U(\alpha_1)}{2} + 1 - \frac{\nu^L(\alpha_1) + \nu^U(\alpha_1)}{2})$$

$$= (\mu^L(\alpha_1) + \mu^U(\alpha_1) + 2 - \nu^L(\alpha_1) - \nu^U(\alpha_1))/4.$$

An accuracy function $H(\widetilde{\alpha}_1)$ can be represented as follows

$$H(\widetilde{\alpha}_1) = (\frac{\mu^L(\alpha_1) + \mu^U(\alpha_1)}{2} + \frac{\nu^L(\alpha_1) + \nu^U(\alpha_1)}{2})$$
$$= (\mu^L(\alpha_1) + \mu^U(\alpha_1) + \nu^L(\alpha_1) + \nu^U(\alpha_1))/4.$$

Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2)$ be two interval-valued fuzzy intuitionistic sets. Then

(1) If $E(\widetilde{\alpha}_1) > E(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$.

(2) If $E(\widetilde{\alpha}_1) = E(\widetilde{\alpha}_2)$, then:

If $H(\widetilde{\alpha}_1) > H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$.

If $H(\widetilde{\alpha}_1) = H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$.

Based on the the above operational rules, we propose weighted average (geometric) operator, ordered weighted average (geometric) operator and hybrid average (geometric) operator for interval-valued intuitionistic fuzzy sets based on Archimedean t-norm in this part.

Definition 5.2. Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic fuzzy sets, $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. We define interval-valued intuitionistic fuzzy weighted average operator based on Archimedean t-norm as follows: $ATS - IVIFWA : \Omega^n \to \Omega$,

$$ATS - IVIFWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \sum_{j=1}^{n} \mu_j \widetilde{\alpha}_j,$$

Specifically, if $\mu = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $ATS - IVIFWA$ operator degenerates interval-valued intuitionistic fuzzy arithmetic average operator based on Archimedean t-norm $(ATS - IVIFAA)$:

$$ATS - IVIFAA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \frac{1}{n}(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 \oplus \ldots \oplus \widetilde{\alpha}_n).$$

Similarly, we could define interval-valued intuitionistic fuzzy weighted geometric average operator based on Archimedean t-norm, $ATS - IVIFWGA : \Omega^n \to \Omega$, as follows

$$ATS - IVIFWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \prod_{j=1}^{n} (\widetilde{\alpha}_j)^{\mu_j},$$

Specifically, if $\mu = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $ATS - IVIFWGA$ operator degenerates interval-valued intuitionistic fuzzy arithmetic geometric average operator based on Archimedean t-norm $(ATS - IVIFGA)$:

$$ATS - IVIFGA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = (\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 \otimes \ldots \otimes \widetilde{\alpha}_n)^{\frac{1}{n}}.$$

where Ω is the set of all interval-valued fuzzy intuitionistic sets, and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j (j = 1, 2, \ldots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^{n} \mu_j = 1$, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$.

Theorem 5.1. Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic fuzzy sets, $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. Then, the result aggregated by Definition 5.1 is still an intuitionistic fuzzy set, and

$(i) ATS - IVIFWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) =$
$\langle[h^{-1}(\sum_{j=1}^{n} \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{n} \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^{n} \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{n} \mu_j g(\nu^U(\alpha_j)))]\rangle$.

$(ii) ATS - IVIFWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) =$
$\langle[g^{-1}(\sum_{j=1}^{n} \mu_j g(\mu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{n} \mu_j g(\mu^U(\alpha_j)))], [h^{-1}(\sum_{j=1}^{n} \mu_j h(\nu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{n} \mu_j h(\nu^U(\alpha_j)))]\rangle$,
where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^{n} \mu_j = 1$, the parentheses used for indices represent a permutation on X such that $\widetilde{\alpha}_1 \leqslant \widetilde{\alpha}_2 \leqslant \cdots \leqslant \widetilde{\alpha}_n$, $A_{(j)} = (j, ..., n)$, $A_{(n+1)} = \varnothing$.

Theorem 5.1 can be proven by mathematical induction. The steps in the proof are as follows:

Proof. We only prove that (i) holds. the proof of (ii) is similar.

(1) When $n = 1$, obviously, it is right.

(2) When $n = 2$,

$\mu_1\widetilde{\alpha}_1 = \langle[h^{-1}(\mu_1 h(\mu^L(\alpha_1))), h^{-1}(\mu_1 h(\mu^U(\alpha_1)))], [g^{-1}(\mu_1 g(\nu^L(\alpha_1))), g^{-1}(\nu_1 g(\nu^U(\alpha_1)))]\rangle$.

$\mu_2\widetilde{\alpha}_2 = \langle[h^{-1}(\mu_2 h(\mu^L(\alpha_2))), h^{-1}(\mu_2 h(\mu^U(\alpha_2)))], [g^{-1}(\mu_2 g(\nu^L(\alpha_2))), g^{-1}(\nu_2 g(\nu^U(\alpha_2)))]\rangle$.

$ATS - ATS - IVIFWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2) = \mu_1\widetilde{\alpha}_1 \oplus \mu_2\widetilde{\alpha}_2$
$= [h^{-1}(\mu_1 h(\mu^L(\alpha_1))), h^{-1}(\mu_1 h(\mu^U(\alpha_1)))], [g^{-1}(\mu_1 g(\nu^L(\alpha_1))), g^{-1}(\nu_1 g(\nu^U(\alpha_1)))]\rangle$
$\oplus\langle[h^{-1}(\mu_2 h(\mu^L(\alpha_2))), h^{-1}(\mu_2 h(\mu^U(\alpha_2)))], [g^{-1}(\mu_2 g(\nu^L(\alpha_2))), g^{-1}(\nu_2 g(\nu^U(\alpha_2)))]\rangle =$
$\langle[h^{-1}(h(h^{-1}(\mu_1 h(\mu^L(\alpha_1)))) + h(h^{-1}(\mu_2 h(\mu^L(\alpha_2))))), h^{-1}(h(h^{-1}(\mu_1 h(\mu^U(\alpha_1)))) + h(h^{-1}(\mu_2 h(\mu^U(\alpha_2)))))],$
$[g^{-1}(g(g^{-1}(\mu_1 g(\nu^L(\alpha_1)))) + g(g^{-1}(\mu_2 g(\nu^L(\alpha_2))))), g^{-1}(g(g^{-1}(\mu_1 g(\nu^U(\alpha_1)))) + g(g^{-1}(\mu_2 g(\nu^U(\alpha_2)))))]\rangle =$
$\langle[h^{-1}(\sum_{j=1}^2 \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^2 \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^2 \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^2 \mu_j g(\nu^U(\alpha_j)))]\rangle$.

Therefore, when $n = 2$, the conclusion is right.

(3) Suppose when $n = k$, the conclusion is right, i.e.

$ATS - IVIFWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_k) =$
$\langle[h^{-1}(\sum_{j=1}^k \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^k \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^k \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^k \mu_j g(\nu^U(\alpha_j)))]\rangle$.

Then, when $n = k + 1$,

$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_k, \widetilde{\alpha}_{k+1}) =$
$\langle[h^{-1}(\sum_{j=1}^k \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^k \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^k \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^k \mu_j g(\nu^U(\alpha_j)))]\rangle$
$\oplus\langle[h^{-1}(\mu_{k+1} h(\mu^L(\alpha_{k+1}))), h^{-1}(\mu_{k+1} h(\mu^U(\alpha_{k+1})))], [g^{-1}(\mu_{k+1} g(\nu^L(\alpha_{k+1}))), g^{-1}(\nu_{k+1} g(\nu^U(\alpha_{k+1})))]\rangle =$
$\langle[h^{-1}(h(h^{-1}(\sum_{j=1}^k \mu_j h(\mu^L(\alpha_j)))) + h(h^{-1}(\mu_{k+1} h(\mu^L(\alpha_{k+1}))))),$
$h^{-1}(h(h^{-1}(\sum_{j=1}^k \mu_j h(\mu^U(\alpha_j)))) + h(h^{-1}(\mu_{k+1} h(\mu^U(\alpha_{k+1})))))],$
$[g^{-1}(g(g^{-1}(\sum_{j=1}^k \mu_j g(\nu^L(\alpha_j)))) + g(g^{-1}(\mu_{k+1} g(\nu^L(\alpha_{k+1}))))),$
$g^{-1}(g(g^{-1}(\sum_{j=1}^k \mu_j g(\nu^U(\alpha_j)))) + g(g^{-1}(\mu_{k+1} g(\nu^U(\alpha_{k+1})))))]\rangle =$
$\langle[h^{-1}(\sum_{j=1}^{k+1} \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{k+1} \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^{k+1} \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{k+1} \mu_j g(\nu^U(\alpha_j)))]\rangle$.

So, when $n = k + 1$, the conclusion is right, too.

According to steps (1), (2) and (3), we can conclude the conclusion is right for all n.

6. Conclusions

The main technologies in multiple attribute decision making, whether the situation is certain or vague, are how to define and calculate aggregation operators proposed in the practice. In this study we only discussed and investigated the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm, and the aggregating of interval-valued intuitionistic fuzzy sets based on Archimed -ean t-norm. In order to do this we also obtained the representations and transformations of Archimedean t-norm and Archimedean t-conorm. Based on these operators proposed in this note, we could make multiple attribute group decision making problems easily. Limited to the length of this paper it can not be discussed. However, it will be our main work in the future.

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Approximate bi-homomorphisms and bi-derivations in intuitionistic fuzzy ternary normed algebras

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Abstract. In this paper, we generalize the concept of homomorphisms and derivations in intuitionistic fuzzy normed algebras for 2-dimensional functional equations. Furthermore, we investigate the Hyers-Ulam stability bi-homomorphisms and bi-derivations in intuitionistic fuzzy ternary normed algebras concerning a 2-dimensional bi-additive functional equation.

1. Introduction and preliminaries

We say a functional equation (ζ) is stable if any function g satisfying the equation (ζ) approximately is near to true solution of (ζ). Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it. The stability problem of functional equations originated from a question of Ulam [37] in 1940, concerning the stability of group homomorphisms. We are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive mappings was solved by Hyers [11] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Rassias [28]. In 1991, Gajda [8] answered the question for the case $p > 1$, which was raised by Rassias. For more information on functional equations, see [18, 25, 26, 27, 32, 34, 35].

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. This new theory was introduced by Zadeh [38], in 1965 and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories has also been made. It has also very useful application in various fields, e.g. population dynamics [5], chaos control [7], computer programming [9], nonlinear dynamical systems [10], fuzzy physics [12], fuzzy topology [31], fuzzy stability [13, 14, 15, 16, 24], nonlinear operators [20], statistical convergence [21, 23], etc. The concept of intuitionistic fuzzy normed spaces, initially has been introduced by Saadati and Park [29]. In [30], by modifying the separation condition and strengthening some conditions in the definition of Saadati and Park, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces. Many authors have considered the intuitionistic fuzzy normed linear spaces, and intuitionistic fuzzy 2-normed spaces(see [3, 4, 6, 19]).

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Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(N1)$ $N(x, c) = 0$ for $c \leqslant 0$;
$(N2)$ $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
$(N3)$ $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
$(N4)$ $N(x + y, s + t) \geqslant \min\{N(x, s), N(y, t)\}$;
$(N5)$ $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
$(N6)$ For $x \neq 0, N(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement the norm of x is less than or equal to the real number t.

The stability problem for a 2-dimensional bi-additive functional equation was proved by Bae and Park [1] for mappings $f : X \times X \to Y$, where X is a real normed space and Y is a Banach space.

In this paper, we determine some stability results of bi-homomorphism and bi-derivation concerning the 2-dimensional bi-additive functional equation

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w) \tag{1.1}$$

in intuitionistic fuzzy ternary normed algebras. It has been discussed that $f(x, y) = ax^2 + by^2$ is a solution of (1.1) (see [2]).

We recall some notations and basic definitions used in this paper.

We use the definition of intuitionistic fuzzy normed spaces given in [17, 22, 29] to investigate some stability results for the functional equation (1.1) in the intuitionistic fuzzy normed vector space setting.

Definition 1.1. ([33]) *A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is said to be a continuous* t*-norm if it satisfies the following conditions:*
(a) *is commutative and associative;*
(b) *is continuous;*
(c) $a * 1 = a$ *for all* $a \in [0, 1]$*;*
(d) $a * b \leqslant c * d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$ *for all* $a, b, c, d \in [0, 1]$.

Definition 1.2. ([33]) *A binary operation* $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ *is said to be a continuous* t*-conorm if it satisfies the following conditions:*
(a) *is commutative and associative;*
(b) *is continuous;*
(c) $a \diamond 0 = a$ *for all* $a \in [0, 1]$*;*
(d) $a \diamond b \leqslant c \diamond d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$ *for all* $a, b, c, d \in [0, 1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [29] have introduced the concept of intuitionistic fuzzy normed space.

Definition 1.3. ([22, 29]) *The five-tuple* $(X, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space (for short, IFNS) if* X *is a vector space,* $*$ *is a continuous* t*-norm,* $\diamond$ *is a continuous* t*-conorm, and* μ, ν *are fuzzy sets on* $X \times (0, \infty)$ *satisfying the following conditions: for every* $x, y \in X$ *and* $s, t > 0$,

(i) $\mu(x, t)+\nu(x, t) \leqslant 1$, (ii) $\mu(x, t) > 0$, (iii) $\mu(x, t) = 1$ *if and only if* $x = 0$, (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$, (v) $\mu(x, t) * \mu(y, s) \leqslant \mu(x + y, t + s)$, (vi) $\mu(x, .) : (0, \infty) \to [0, 1]$ *is continuous,* (vii) $\lim_{t\to\infty} \mu(x, t) = 1$ *and* $\lim_{t\to 0} \mu(x, t) = 0$, $(viii)$ $\nu(x, t) < 1$, (ix) $\nu(x, t) = 0$ *if and only if* $x = 0$, (x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$, (xi) $\nu(x, t)\diamond\nu(y, s) \geqslant \nu(x+y, t+s)$, (xii) $\nu(x, .) : (0, 1) \to [0, 1]$ *is continuous,* $(xiii)$ $\lim_{t\to\infty} \nu(x, t) = 0$ *and* $\lim_{t\to 0} \nu(x, t) = 1$.

Definition 1.4. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $\{x_n\}$ is said to be* intuitionistic fuzzy convergent *to $L \in X$ if $\lim_{k\to\infty} \mu(x_k - L, t) = 1$ and $\lim_{k\to\infty} \nu(x_k - L, t) = 0$ for all $t > 0$. In this case we write $x_k \to L$ as $k \to \infty$. A sequence $\{x_n\}$ is said to be* intuitionistic fuzzy Cauchy sequence *if $\lim_{k\to\infty} \mu(x_{k+p} - x_k, t) = 1$ and $\lim_{k\to\infty} \nu(x_{k+p} - x_k, t) = 0$ for all $p \in \mathbb{N}$ and all $t > 0$. Then IFNS $(X, \mu, \nu, *, \diamond)$ is said to be* complete *if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$ and $(X, \mu, \nu, *, \diamond)$ is also called an* intuitionistic fuzzy Banach space.

The concepts of convergent sequence and Cauchy sequence in an intuitionistic fuzzy normed space are studied in [29].

Definition 1.5. *Let X be a ternary algebra with $[\cdot,\cdot,\cdot]$ and $(X, \mu, \nu, *, \diamond)$ be an IFNS.*
(1) *The intuitionistic fuzzy normed space $(X, \mu, \nu, *, \diamond)$ is called an intuitionistic fuzzy ternary normed algebra if*

$$\mu([x, y, z], stu) \geqslant \mu(x, s) * \mu(y, t) * \mu(z, u)$$
$$\nu([x, y, z], stu) \geqslant \nu(x, s) * \nu(y, t) * \nu(z, u)$$

for all $x, y, z \in X$ and $s, t, u > 0$.

(2) *A complete intuitionistic fuzzy ternary normed algebra is called an intuitionistic fuzzy ternary Banach algebra.*

Definition 1.6. *Let X be a ternary normed (Banach) algebra and (Y, μ, ν) an intuitionistic fuzzy ternary Banach algebra.*

(1) *A bi-additive mapping $H : X \times X \to Y$ is called a ternary bi-homomorphism if*

$$H([x, y, z], [w, w, w]) = [H(x, w), H(y, w), H(z, w)],$$
$$H([x, x, x], [y, z, w]) = [H(x, y), H(x, z), H(x, w)]$$

for all $x, y, z, w \in X$.

(2) *A bi-additive mapping $\delta : X \times X \to X$ is called a ternary bi-derivation if*

$$\delta([x, y, z], w) = [\delta(x, w), y, z] + [x, \delta(y, w), z] + [x, y, \delta(z, w)],$$
$$\delta(x, [y, z, w]) = [\delta(x, y), z, w] + [y, \delta(x, z), w] + [y, z, \delta(x, w)]$$

for all $x, y, z, w \in X$.

2. Bi-homomorphisms in intuitionistic fuzzy ternary normed algebras

We begin with a Hyers-Ulam type theorem in intuitionistic fuzzy ternary normed algebras to approximate bi-homomorphism associated to the functional equation (1.1). For notational convenience, given a function $f : X \times X \to Y$, we define the difference operator

$$D_q f(x, y, z, w) = f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) + 2f(y, w)$$

Lemma 2.1. ([36, Theorem 3.1]) *Let X be a linear space and let (Z, μ', ν') be an IFNS. Let $\varphi : X^4 \to Z$ be a mapping such that, for some $0 < \alpha < 4$.*

$$\begin{cases} \mu'(\varphi(2x, 2y, 2z, 2w), t) \geqslant \mu'(\alpha\varphi(x, y, z, w), t), \\ \nu'(\varphi(2x, 2y, 2z, 2w), t) \leqslant \nu'(\alpha\varphi(x, y, z, w), t), \end{cases} \tag{2.1}$$

for all $x, y, z, w \in X$ and all $t > 0$. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and let $f : X \times X \to Y$ be a mapping satisfying $f(0, 0) = 0$ and

$$\begin{cases} \mu(D_q f(x, y, z, w), t) \geqslant \mu'(\varphi(x, y, z, w), t), \\ \nu(D_q f(x, y, z, w), t) \leqslant \nu'(\varphi(x, y, z, w), t) \end{cases} \tag{2.2}$$

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for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-additive mapping $H : X \times X \to Y$ satisfying (1.1) *such that*

$$\begin{cases} \mu\Big(H(x,y) - f(x,y), t\Big) \\ \quad \geqslant *^{\infty}\mu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) *^{\infty} \mu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) *^{\infty} \mu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big), \\ \nu\Big(H(x,y) - f(x,y), t\Big) \\ \quad \leqslant \diamond^{\infty}\nu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) \diamond^{\infty} \nu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) \diamond^{\infty} \nu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big) \end{cases} \tag{2.3}$$

for all $x, y, z, w \in X$ and all $t > 0$, where $*^{\infty}a := a * a * \cdots$ and $\diamond^{\infty}a := a \diamond a \diamond \cdots$ for all $a \in [0,1]$.

Theorem 2.2. *Let X be a ternary algebra and let (Z, μ', ν) be an IFNS. Let $\varphi : X^4 \to Z$ be a mapping satisfying* (2.1). *Let (Y, μ, ν) be an intuitionistic fuzzy ternary Banach algebra and let $f : X \times X \to Y$ be a mapping satisfying $f(0,0) = 0$,* (2.2) *and*

$$\begin{cases} \mu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \geqslant \mu'(\varphi(x,y,z,w), t), \\ \nu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\nu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \leqslant \nu'(\varphi(x,y,z,w), t) \end{cases} \tag{2.4}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-homomorphism $H : X \times X \to Y$ satisfying (1.1) *and* (2.3).

Proof. In Lemma 2.1, the mapping $H : X \times X \to Y$ was defined by $H(x,y) = \lim_{n\to\infty} \frac{f(2^n x, 2^n y)}{4^n}$ for all $x, z \in X$.

From (2.4) and definition of H, it follows that

$$\begin{aligned} &\mu(H([x,y,z],[w,w,w]) - [H(x,y), H(y,w), H(z,w)], t) \\ &\quad + \mu(H([x,x,x],[y,z,w]) - [H(x,y), H(x,z), H(x,w)], t) \\ &\quad = \mu\Big(\frac{f([2^n x, 2^n y, 2^n z],[2^n w, 2^n w, 2^n w])}{64^n} - \Big[\frac{f(2^n x, 2^n w)}{4^n}, \frac{f(2^n y, 2^n w)}{4^n}, \frac{f(2^n z, 2^n w)}{4^n}\Big], t\Big) \\ &\quad + \mu\Big(\frac{f([2^n x, 2^n x, 2^n x],[2^n x, 2^n y, 2^n z])}{64^n} - \Big[\frac{f(2^n x, 2^n y)}{4^n}, \frac{f(2^n x, 2^n z)}{4^n}, \frac{f(2^n x, 2^n w)}{4^n}\Big], t\Big) \\ &\quad \geqslant \mu'(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^{3n} t) \geqslant \mu'(\varphi(x,y,z,w), \frac{4^{3n}}{\alpha^n} t) \to 1 \end{aligned}$$

as $n \to \infty$ for all $x, y, z, w \in X$ and all $t > 0$, and similarly

$$\begin{aligned} &\nu(H([x,y,z],[w,w,w]) - [H(x,y), H(y,w), H(z,w)], t) \\ &\quad + \nu(H([x,x,x],[y,z,w]) - [H(x,y), H(x,z), H(x,w)], t) \leqslant 0 \end{aligned}$$

for all $x, y, z, w \in X$ and all $t > 0$. So we conclude that

$$\begin{aligned} H([x,y,z],[w,w,w]) &= [H(x,w), H(y,w), H(z,w)], \\ H([x,x,x],[y,z,w]) &= [H(x,y), H(x,z), H(x,w)] \end{aligned}$$

for all $x, y, z, w \in X$. □

Corollary 2.3. *Let p be a nonnegative real number with $p < 2$, X be a ternary normed algebra with norm $\|.\|$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (Y, μ, ν) be a complete intuitionistic fuzzy ternary normed algebra, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0, 0) = 0$ and*

$$\left\{\begin{array}{l} \mu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad + \mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \\ \quad \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \\ \nu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad + \mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \\ \quad \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \end{array}\right. \tag{2.5}$$

and

$$\left\{\begin{array}{l} \mu(D_q f(x,y), t) \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \\ \nu(D_q f(x,y), t) \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \end{array}\right. \tag{2.6}$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-homomorphism $H : X \times X \to Y$ such that

$$\left\{\begin{array}{l} \mu(H(x,y) - f(x,y), t) \geqslant *^2 \mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\right) \\ \nu(H(x,y) - f(x,y), t) \leqslant *^2 \nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\right) \end{array}\right.$$

for all $x, y \in X$ and $t > 0$.

Lemma 2.4. ([36, Theorem 3.3]) *Let X be a linear space and let (Z, μ', ν') be an IFNS. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping such that, for some $\alpha > 4$,*

$$\left\{\begin{array}{l} \mu'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \geqslant \mu'(\varphi(x,y,z,w), \alpha t), \\ \nu'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \leqslant \nu'(\varphi(x,y,z,w), \alpha t), \end{array}\right. \tag{2.7}$$

for all $x, y, z, w \in X$ and all $t > 0$. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and let $f : X \times X \to Y$ be a φ-approximately bi-additive mapping in the sense of (2.2) and (2.4) with $f(0, 0) = 0$. Then there exists a unique bi-additive mapping $H : X \times X \to Y$ such that

$$\begin{aligned} \mu(H(x,y) - f(x,y), t) \geqslant *^\infty\, \mu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) *^\infty\, \mu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big) \\ *^\infty\, \mu'\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \mu(H(x,y) - f(x,y), t) \leqslant \diamond^\infty\, \nu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) \diamond^\infty\, \nu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big) \\ \diamond^\infty\, \nu'\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big) \end{aligned} \tag{2.9}$$

for all $x, y \in X$ and all $t > 0$.

Theorem 2.5. *Let X be a ternary algebra and let (Z, μ', ν') be an IFNS. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping satisfying (2.7). Let (Y, μ, ν) be an intuitionistic fuzzy ternary Banach algebra and let $f : X \times X \to Y$ be a φ-approximately bi-additive mapping in the sense of (2.2) and (2.4) with $f(0, 0) = 0$. Then there exists a unique bi-homomorphism $H : X \times X \to Y$ satisfying (2.8) and (2.9).*

Proof. The proof is similar to the proof of Theorem 2.2. □

Corollary 2.6. *Let p be a nonnegative real number with $p > 2$, X be a ternary normed algebra with norm $\|.\|$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (Y, μ, ν) be a complete intuitionistic fuzzy ternary normed algebra, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0,0) = 0$, (2.5) and (2.6). then there exists a unique bi-homomorphism $H : X \times X \to Y$ such that*

$$\begin{cases} \mu(H(x,y) - f(x,y), t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \mu'\Big((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\Big) \\ \nu(H(x,y) - f(x,y), t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \nu'\Big((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\Big) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

3. Bi-derivations on intuitionistic fuzzy ternary normed algebras

In this section, we investigate generalized Hyers-Ulam stability of bi-derivations on intuitionistic fuzzy ternary normed algebrasfor the functional equation (1.1).

Theorem 3.1. *Let X be an intuitionistic fuzzy ternary Banach algebra and let (Z, μ', ν') be an IFNS. Let $f : X \times X \to X$ be a mapping with $f(0,0) = 0$ for which there exists a mapping $\varphi : X \times X \times X \times X \to Z$ such that, for some $0 < \alpha < 4$ satisfying (2.1), (2.2) and*

$$\begin{cases} \mu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\ \qquad +\mu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\ \quad \geqslant \mu'(\varphi(x,y,z,w),t), \\ \nu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\ \qquad +\nu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\ \quad \leqslant \nu'(\varphi(x,y,z,w),t) \end{cases} \tag{3.1}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-derivation $\delta : X \times X \to X$ satisfying (1.1) such that

$$\begin{cases} \mu\Big(\delta(x,y) - f(x,y), t\Big) \\ \geqslant *^\infty\mu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) *^\infty \mu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) *^\infty \mu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big), \\ \nu\Big(\delta(x,y) - f(x,y), t\Big) \\ \leqslant \diamond^\infty\nu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) \diamond^\infty \nu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) \diamond^\infty \nu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big) \end{cases} \tag{3.2}$$

for all $x, y, z, w \in X$ and all $t > 0$, where $*^\infty a := a * a * \cdots$ and $\diamond^\infty a := a \diamond a \diamond \cdots$ for all $a \in [0,1]$.

Proof. By the same argument as in the proof of Theorem 2.2, there exists a unique bi-additive mapping $\delta : X \times X \to X$ satisfying (3.2). The mapping δ is given by

$$\delta(x,y) = \lim_{n\to\infty} \frac{1}{4} f(2^n x, 2^n y)$$

for all $x, y \in X$. It follows from (3.1) that

$$\begin{aligned}
&\mu(\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w),z] - [x,y,\delta(z,w)],t) \\
&\quad + \mu(\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x,z),w] - [y,z,\delta(x,w)],t) \\
&\quad = \mu\Big(\frac{1}{4^{3n}} f(2^{3n}[x,y,z],2^{3n}w) - \Big[\frac{1}{4^n} f(2^n x, 2^n w), y, z\Big] \\
&\qquad - \Big[x, \frac{1}{4^n} f(2^n x, 2^n w), z\Big] - \Big[x, y, \frac{1}{4^n} f(2^n z, 2^n w)\Big], t\Big) \\
&\qquad + \mu\Big(\frac{1}{4^{3n}} f(2^{3n}x, 2^{3n}[y,z,w]) - \Big[\frac{1}{4^n} f(2^n x, 2^n y), z, w\Big] \\
&\qquad - \Big[y, \frac{1}{4^n} f(2^n x, 2^n z), w\Big] - \Big[y, z, \frac{1}{4^n} f(2^n x, 2^n w)\Big], t\Big) \\
&\quad = \mu\Big(\frac{1}{4^{3n}} f([2^n x, 2^n y, 2^n z], 2^{3n} w) - \frac{1}{4^{3n}}[f(2^n x, 2^{3n} w), 2^n y, 2^n z] \\
&\qquad - \frac{1}{4^{3n}}[2^n x, f(2^n y, 2^{3n} w), 2^n z] - \frac{1}{4^{3n}}[2^n x, 2^n y, f(2^n z, 2^{3n} w)], t\Big) \\
&\qquad + \mu\Big(\frac{1}{4^{3n}} f(2^{3n} x, [2^n y, 2^n z, 2^n w]) - \frac{1}{4^{3n}}[f(2^{3n} x, 2^n y), 2^n z, 2^n w] \\
&\qquad - \frac{1}{4^{3n}}[2^n y, f(2^{3n} x, 2^n z), 2^n w] - \frac{1}{4^{3n}}[2^n y, 2^n z, f(2^{3n} x, 2^n w)], t\Big) \\
&\qquad \leqslant \mu'(\varphi(2^n x, 2^n y, 2^n z, 2^{3n} w), 4^{3n} t) + \mu'(\varphi(2^{3n} x, 2^n y, 2^n z, 2^n w), 4^{3n} t)) \\
&\qquad \leqslant 2\mu'\Big(\varphi(x,y,z,w), \frac{4^{3n} t}{\alpha^{3n}}\Big) \longrightarrow 1
\end{aligned}$$

as $n \to \infty$ for all $x, y, z, w \in \mathcal{A}$. Similarly, we obtain

$$\begin{aligned}
&\nu(\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w),z] - [x,y,\delta(z,w)],t) \\
&\quad + \nu(\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x,z),w] - [y,z,\delta(x,w)],t) = 0
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. Thus

$$\begin{aligned}
\delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w),z] + [x,y,\delta(z,w)], \\
\delta(x,[y,z,w]) &= [\delta(x,y),z,w] + [y,\delta(x,z),w] + [y,z,\delta(x,w)]
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. So we conclude that δ is a unique bi-derivation satisfying (3.2). □

Corollary 3.2. *Let p be a nonnegative real number with $p < 2$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (X, μ, ν) be a complete intuitionistic fuzzy ternary Banach algebra, and let $z_0 \in Z$. If $f : X \to X$ is a mapping with $f(0,0) = 0$ such that*

$$\begin{cases}
\mu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\
\qquad + \mu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\
\qquad \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \\
\nu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\
\qquad + \nu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\
\qquad \geqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t)
\end{cases} \tag{3.3}$$

and

$$\begin{cases} \mu(D_q f(x,y),t) \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \\ \nu(D_q f(x,y),t) \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \end{cases} \tag{3.4}$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-derivation $\delta : X \times X \to X$ such that

$$\begin{cases} \mu(\delta(x,y) - f(x,y),t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \mu'\Big((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\Big) \\ \nu(\delta(x,y) - f(x,y),t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \nu'\Big((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\Big) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

Theorem 3.3. *Let X be an intuitionistic fuzzy ternary Banach algebra and let $(Z < \mu', \nu,)$ be an IFNS. Let $f : X \times X \to Y$ be a mapping with $f(0,0) = 0$ for which there exists a mapping $\varphi : X \times X \times X \times X \to Z$ satisfying (2.1), (2.7) and (3.1) for some $\alpha > 4$. Then there exists a unique bi-derivation $\delta : X \times X \to X$ such that*

$$\mu(\delta(x,y) - f(x,y),t) \geqslant *^\infty\, \mu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) *^\infty\, \mu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big)$$
$$*^\infty\, \mu\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big)$$

and

$$\mu(\delta(x,y) - f(x,y),t) \leqslant \diamond^\infty\, \nu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) \diamond^\infty\, \nu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big)$$
$$\diamond^\infty\, \nu'\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big)$$

for all $x, y \in X$.

Proof. The proof is similar to the proof of Theorems 2.5 and 3.1. □

Corollary 3.4. *Let p be a nonnegative real number with $p > 2$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (X, μ, ν) be a complete intuitionistic fuzzy ternary Banach algebra and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0,0) = 0$, (3.3) and (3.4), then there exists a unique bi-derivation $\delta : X \times X \to X$ such that*

$$\begin{cases} \mu(\delta(x,y) - f(x,y),t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \mu'\Big((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\Big) \\ \nu(\delta(x,y) - f(x,y),t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \nu'\Big((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\Big) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

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ON NEW REFINEMENTS AND APPLICATIONS OF EFFICIENT QUADRATURE RULES USING N-TIMES DIFFERENTIABLE MAPPINGS

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Abstract. In this paper, new efficient quadrature rules are established using a newly developed special type of kernel for n-times differentiable mappings, having five steps. Some previous inequalities are also recaptured as special cases of our main inequalities. At the end, efficiency of the newly developed quadrature rules are discussed.

1. Introduction

In 1938, Ostrowski [13] first announced his inequality for different differentiable mappings, which is given below:

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (I° is the interior of I) and let $a, b \in I^{\circ}$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) i.e. $\|f'\|_{\infty} = \sup\limits_{t\in[a,b]} |f''(t)| < \infty$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_{\infty}, \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In 1976, Milovanovic et. al [11], proved a generalization of Ostrowski's inequality for n-time differentiable mappings. Up till now, a large number of research papers and books have been written on inequalities and their applications (see for instance [2]-[5], [8] and [14]-[16]). In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities like Ostrowski are very helpful for this purpose. Ostrowski type inequalities have immediate applications in numerical integration, optimization theory, statistics, and integral operator theory.

We indicate another inequality called Grüss inequality [11] which is stated as the integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals, which is given below.

Theorem 2. *Let $f,\ g : [a,b] \to \mathbb{R}$ be integrable functions such that $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for some constants $\varphi,\ \Phi,\ \gamma,\ \Gamma$ and $x \in [a,b]$. Then*

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx . \frac{1}{b-a}\int_a^b g(x)dx \right| \tag{1.2}$$
$$\leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).$$

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Dragomir et. al [4] combined Ostrowski and Grüss inequality to give a new inequality which they named Ostrowski-Grüss type inequality. Dragomir [3], Liu [6], Alomari [1] and Liu et. al [8] established some companions of ostrowski type integral inequalities.

Recently, Liu [7] proved the following companions of ostrowski type inequalities for 3-step kernels.

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be differentiable in* (a,b). *If* $f' \in L^1[a,b]$, *and* $\gamma \leq f'(x) \leq \Gamma$, *for all* $x \in [a,b]$, *then for all* $x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\left| \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{b-a}{4} + \left|x - \frac{3a+b}{4}\right|\right](S-\gamma) \tag{1.3}$$

and

$$\begin{aligned} &\left| \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \left[\frac{b-a}{4} + \left|x - \frac{3a+b}{4}\right|\right](\Gamma - S). \end{aligned} \tag{1.4}$$

More recently, Qayyum et. al [9]-[10] proved companions of Ostrowski inequality for 5-step linear and quadratic kernels but in this paper, we establish our results for 5-step kernel for n-times differentiable mappings. In this paper, new ontrowski inequalities are extended. Using these inequalities, some efficient quadrature rules are established. Some previous inequalities are also recaptured as special cases of our main inequalities. At the end, efficiency of the newly developed quadrature rules are discussed.

2. Derivation of Ostrowski inequalities using 5-step kernel

We will start our work by introducing a new Peano kernel defined by $P(x,.) : [a,b] \to \mathbb{R}$

$$P_n(x,t) = \begin{cases} \frac{1}{n!}(t-a)^n, & t \in \left(a, \frac{a+x}{2}\right], \\ \frac{1}{n!}\left(t - \frac{3a+b}{4}\right)^n, & t \in \left(\frac{a+x}{2}, x\right], \\ \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n, & t \in (x, a+b-x], \\ \frac{1}{n!}\left(t - \frac{a+3b}{4}\right)^n, & t \in \left(a+b-x, \frac{a+2b-x}{2}\right], \\ \frac{1}{n!}(t-b)^n, & t \in \left(\frac{a+2b-x}{2}, b\right], \end{cases} \tag{2.1}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

The following lemma is the main tool to prove the main results.

Lemma 1. *Let* $f : [a,b] \to \mathbb{R}$ *be an n-times differentiable function such that* $f^{(n-1)}(x)$ *for* $n \in N$ *is absolutely continuous on* $[a,b]$ *then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b P_n(x,t) f^{(n)}(t)dt \\ &= \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right. \end{aligned} \tag{2.2}$$

$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\Bigg]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt,$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. The proof of (2.2) is established using mathematical induction.

Take $n=1$,

$$L.H.S \text{ of } (2.2)=\int_a^b P_1(x,t)f'(t)dt. \tag{2.3}$$

After integrating by parts, we get

$$\frac{1}{b-a}\int_a^b P_1(x,t)f'(t)dt \tag{2.4}$$
$$=\frac{1}{4}\left[f\left(\frac{a+x}{2}\right)+f(x)+f(a+b-x)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt.$$

We have

$$L.H.S=\int_a^b P_1(x,t)f'(t)dt.$$

Equation (2.3), is identical to the $R.H.S$ of (2.2).

Assume that (2.2) is true for n.

$$\int_a^b P_{n+1}(x,t)f^{(n+1)}(t)dt$$
$$=\sum_{k=0}^{n}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+(-1)^{n+1}\int_a^b f(t)dt,$$

where

$$P_{n+1}(x,t)=\begin{cases}\frac{1}{(n+1)!}\left(t-a\right)^{n+1}, & t\in\left(a,\frac{a+x}{2}\right],\\ \frac{1}{(n+1)!}\left(t-\frac{3a+b}{4}\right)^{n+1}, & t\in\left(\frac{a+x}{2},x\right],\\ \frac{1}{(n+1)!}\left(t-\frac{a+b}{2}\right)^{n+1}, & t\in\left(x,a+b-x\right],\\ \frac{1}{(n+1)!}\left(t-\frac{a+3b}{4}\right)^{n+1}, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right],\\ \frac{1}{(n+1)!}\left(t-b\right)^{n+1}, & t\in\left(\frac{a+2b-x}{2},b\right].\end{cases}$$

After integeration by parts, we get

$$\begin{aligned}&\int_a^b P_{n+1}(x,t)f^{(n+1)}(t)dt\\ &=\frac{1}{(n+1)!}\Bigg[\frac{1}{2^{n+1}}\left(x-a\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(x)\\ &-\frac{1}{2^{n+1}}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+(-1)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(a+b-x)\\ &-\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(x)+\left(\frac{-1}{2}\right)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\\ &+(-1)^n\left\{\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(a+b-x)+\left(\frac{x-a}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right\}\Bigg]\end{aligned}$$

$$\begin{aligned}&-\frac{1}{n!}\Bigg[\int_a^{\frac{a+x}{2}}\left(t-a\right)^n f^{(n)}(t)dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^n f^{(n)}(t)dt\\ &+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^n f^{(n)}(t)dt+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{3a+b}{4}\right)^n f^{(n)}(t)dt\\ &+\int_{\frac{a+2b-x}{2}}^{b}\left(t-b\right)^n f^{(n)}(t)dt\Bigg]\end{aligned}$$

$$\begin{aligned}&=\frac{1}{(n+1)!}\Bigg[\frac{1}{2^{n+1}}\left(x-a\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(x)\\ &-\frac{1}{2^{n+1}}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+(-1)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(a+b-x)\\ &-\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(x)+\left(\frac{-1}{2}\right)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\\ &-(-1)^{n+1}\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(a+b-x)\\ &-\left(\frac{-1}{2}\right)^{n+1}\left(x-a\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\Bigg]-\int_a^b P_n(x,t)f^{(n)}(t)dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(n+1)!}\left[\left\{(x-a)^{n+1}-\left(x-\frac{a+b}{2}\right)^{n+1}\right\}\frac{1}{2^{n+1}}f^{(n)}\left(\frac{a+x}{2}\right)\right.\\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{n+1}-\left(x-\frac{a+b}{2}\right)^{n+1}\right\}f^{(n)}(x)\\
&+\left\{\left(x-\frac{a+b}{2}\right)^{n+1}-\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}(-1)^{n+1}f^{(n)}(a+b-x)\\
&\left.+\left\{\left(x-\frac{a+b}{2}\right)^{n+1}-(x-a)^{n+1}\right\}\left(\frac{-1}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{n-1}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.\\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)\\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)\\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]\\
&+(-1)^{n+1}\int_a^b f(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{k=0}^{n}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.\\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)\\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)\\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]\\
&+(-1)^{n+1}\int_a^b f(t)dt.
\end{aligned}
$$

This completes the proof of lemma 1. □

Now we will present our results by imposing three different conditions on $f^{(n)}$ and $f^{(n+1)}$.

3. Case A: When $f^{(n)} \in L^1[a,b]$

Theorem 4. *Let $f\colon [a,b] \to \mathbb{R}$ be an n-times differentiable function on (a,b), $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $\gamma \leq f^{(n)}(t) \leq \Gamma$, $\forall\ t \in [a,b]$, then for*

all $x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \left[\frac{1}{2^{k+1}} \left\{(x-a)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \tag{3.1}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\left[\times\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + (1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq \delta(x)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \left[\frac{1}{2^{k+1}} \left\{(x-a)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \tag{3.2}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\times\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + (1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq \delta(x)(b-a)(\Gamma - S),$$

where

$$S = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}, \tag{3.3}$$

$$\delta(x) = \max\left\{\left|\frac{1}{n!}\left(\frac{x-a}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{n!}\left(x-\frac{3a+b}{4}\right)^2 - \frac{\lambda(x)}{b-a}\right|,\right.$$
$$\left.\left|\frac{1}{n!}\left(x-\frac{a+b}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{4n!}\left(x-\frac{a+b}{2}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \frac{\lambda(x)}{b-a}\right\}$$

and

$$\lambda(x) = \frac{1}{(n+1)!}\left[(1+(-1)^n)\left\{\frac{1}{2^{n+1}}(x-a)^{n+1} + \left(x-\frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left.+\left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

Proof. Let

$$\frac{1}{b-a}\int_a^b P_n(x,t)dt \tag{3.4}$$
$$= \frac{1}{(b-a)(n+1)!}\left[(1+(-1)^n)\left\{\frac{1}{2^{n+1}}(x-a)^{n+1} + \left(x-\frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left.+\left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

Using (3.4), we get

$$\frac{1}{b-a}\int_a^b P_n(x,t)f^{(n)}(t)dt - \frac{1}{(b-a)^2}\int_a^b P_n(x,t)dt\int_a^b f^{(n)}(t)dt \tag{3.5}$$
$$= \sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\left[\times\frac{1}{2^{n+1}}(1+(-1)^n)(x-a)^{n+1} + (1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.+\left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

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Denote the L.H.S of (3.5) by $R_n(x)$. If $C \in R$ is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a}\int_a^b \left(f^{(n)}(t) - C\right)\left[P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right]dt. \tag{3.6}$$

Furthermore, we have

$$|R_n(x)| \le \frac{1}{b-a}\max_{t\in[a,b]}\left|P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right| \int_a^b \left|f^{(n)}(t) - C\right| dt. \tag{3.7}$$

Now

$$\left|P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right| \tag{3.8}$$
$$= \max\left\{\begin{array}{c} \left|\frac{1}{n!}\left(\frac{x-a}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{n!}\left(x-\frac{3a+b}{4}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \\ \left|\frac{1}{n!}\left(x-\frac{a+b}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{4n!}\left(x-\frac{a+b}{2}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \frac{\lambda(x)}{b-a} \end{array}\right\} = \delta(x),$$

where

$$\lambda(x) = \frac{1}{(n+1)!}\left[(1+(-1)^n)\left\{\frac{1}{2^{n+1}}(x-a)^{n+1} + \left(x - \frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left. + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x - \frac{a+b}{2}\right)^{n+1}\right].$$

We also have

$$\int_a^b \left|f^{(n)}(t) - \gamma\right| dt = (S-\gamma)(b-a), \tag{3.9}$$

$$\int_a^b \left|f^{(n)}(t) - \Gamma\right| dt = (\Gamma - S)(b-a). \tag{3.10}$$

Using (3.4) to (3.10) and using $C = \gamma$ and $C = \Gamma$ in (3.7), we can obtain (3.1) and (3.2). □

Remark 1. *If we substitute $n = 2$ in (3.1) and (3.2), we get Qayyum et. al result proved in* [9].

Corollary 1. *Substitution of $x = a$ in (3.1) and (3.2) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\}\right. \tag{3.11}$$
$$\left. + \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1+(-1)^n)\right|$$
$$\le \delta(a)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left\{(-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right) f^{(k)}(b)\right\}\right. \tag{3.12}$$

$$\left. + \frac{(-1)^n}{b-a} \int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!} \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1+(-1)^n)\right|$$

$$\leq \delta(a)(b-a)(\Gamma - S).$$

Corollary 2. *Substitution of* $x = \frac{a+b}{2}$ *in (3.1) and (3.2) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left\{f^{(k)}\left(\frac{3a+b}{4}\right) + \left(1+(-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right)\right.\right. \tag{3.13}$$

$$\left. + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\} + \frac{(-1)^n}{b-a} \int_a^b f(t)dt$$

$$\left. - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \times \frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right|$$

$$\leq \delta\left(\frac{a+b}{2}\right)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left\{f^{(k)}\left(\frac{3a+b}{4}\right) + \left(1+(-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right)\right.\right. \tag{3.14}$$

$$\left. + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\} + \frac{(-1)^n}{b-a} \int_a^b f(t)dt$$

$$\left. - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \times \frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right|$$

$$\leq \delta\left(\frac{a+b}{2}\right)(b-a)(\Gamma - S).$$

Corollary 3. *Substitution of* $x = \frac{3a+b}{4}$ *in (3.1) and (3.2) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{\left(1+(-1)^k\right)}{2^{k+1}} f^{(k)}\left(\frac{7a+b}{8}\right) + (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)\right.\right. \tag{3.15}$$

$$\left. + f^{(k)}\left(\frac{a+3b}{4}\right) + \frac{1}{2^{k+1}}\left(1+(-1)^k\right) f^{(k)}\left(\frac{a+7b}{8}\right)\right] + \frac{(-1)^n}{b-a} \int_a^b f(t)dt$$

$$\left. - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \frac{1}{(n+1)!} \frac{(b-a)^{n-1}}{4^{n+1}} \left\{1 + (-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n}\right\}\right|$$

$$\leq \delta\left(\frac{3a+b}{4}\right)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left[\frac{1}{2^{k+1}}\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{7a+b}{8}\right)\right.\right. \tag{3.16}$$
$$+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)+f^{(k)}\left(\frac{a+3b}{4}\right)$$
$$\left.+\frac{1}{2^{k+1}}\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{a+7b}{8}\right)\right]+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$
$$\left.-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}\frac{(b-a)^{n+1}}{4^{n+1}}\left\{1+(-1)^n+\frac{1}{2^n}+\frac{(-1)^n}{2^n}\right\}\right|$$
$$\leq \delta\left(\frac{3a+b}{4}\right)(b-a)(\Gamma-S).$$

4. **Case B:** When $f^{(n+1)}\in L^2[a,b]$

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be an n-times differentiable function on* (a,b), $f^{(n+1)}\in L^2[a,b]$, *then for all* $x\in\left[a,\frac{a+b}{2}\right]$, *we have*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{f^{(k)}\left(\frac{a+x}{2}\right)}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}\right.\right. \tag{4.1}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\times\left[(1+(-1)^n)\left\{\frac{1}{2^{n+1}}(x-a)^{n+1}+\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq\frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{(x-a)^{2n+1}}{2^{2n}}+2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$
$$\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}$$
$$-\frac{1}{(b-a)(n+1)!}\left\{(1+(-1)^n)\left(\frac{(x-a)^{n+1}}{2^{n+1}}+\left(x-\frac{3a+b}{4}\right)^{n+1}\right)\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}-\frac{(-1)^n}{2^{n+1}}-(-1)^n-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}}.$$

Proof. Substitute $C = f^{(n)}\left(\frac{a+b}{2}\right)$, in $R_n(x)$ given in (3.5) and use the Cauchy Inequality, then we get

$$
\begin{aligned}
&|R_n(x)| \qquad (4.2)\\
&\leq \frac{1}{b-a}\int_a^b \left|f^{(n)}(t) - f^{(n)}\left(\frac{a+b}{2}\right)\right| \left|P^{(n)}(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right| dt\\
&\leq \frac{1}{b-a}\left[\int_a^b \left(f^{(n)}(t) - f^{(n)}\left(\frac{a+b}{2}\right)\right)^2 dt\right]^{\frac{1}{2}}\\
&\times \left[\int_a^b \left(P^{(n)}(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right)^2 dt\right]^{\frac{1}{2}}.
\end{aligned}
$$

Use the Diaz-Metcalf inequality [12] or [17], to get

$$
\int_a^b \left(f^{(n)}(t) - f^{(n)}\left(\frac{a+b}{2}\right)\right)^2 dt \leq \frac{(b-a)^2}{\pi^2}\left\|f^{(n+1)}\right\|_2^2.
$$

Therefore, using the above relations, we obtain (4.1). $\square$

Corollary 4. *Substitution of $x = a$ in (4.1) gives*

$$
\begin{aligned}
&\Bigg|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right) f^{(k)}(b)\right\}\\
&+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1+(-1)^n)\Bigg|\\
&\qquad\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{(b-a)^{2n+1}}{2^{2n}(n!)^2(2n+1)} - \frac{(1+(-1)^n)^2(b-a)^{2n+1}}{2^{2n+2}(n+1)!}\right]^{\frac{1}{2}}.
\end{aligned}
$$

Corollary 5. *Substitution of $x = \frac{a+b}{2}$ in (4.1) gives*

$$
\begin{aligned}
&\Bigg|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)\right. \qquad (4.3)\\
&\left.+\left(1+(-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right) + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}\\
&+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\Bigg|\\
&\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{1}{(n!)^2(2n+1)}\frac{4}{4^{2n+1}}(b-a)^{2n+1}\right.\\
&\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{2(b-a)^{n+1}}{4^{n+1}}(1+(-1)^n)\right\}^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

Corollary 6. *Substitution of* $x = \frac{3a+b}{4}$ *in (4.1) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{\left\{1+(-1)^k\right\}}{2^{k+1}} f^{(k)}\left(\frac{7a+b}{8}\right) \right.\right. \tag{4.4}$$
$$+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right)$$
$$\left. + \frac{1}{2^{k+1}}\left(1+(-1)^k\right) f^{(k)}\left(\frac{a+7b}{8}\right)\right] + \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)$$
$$\left. \times \frac{1}{(n+1)!}\frac{(b-a)^{n-1}}{4^{n+1}}\left\{1+(-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n}\right\}\right|$$
$$\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2 \left[\frac{1}{(n!)^2(2n+1)}\frac{(b-a)^{2n+1}}{4^{2n+1}}\left\{\frac{4}{2^{2n+1}} - 2\right\}\right.$$
$$\left. - \frac{1}{b-a}\frac{1}{(n+1)!}\frac{(b-a)^{n+1}}{4^{n+1}}\left\{(1+(-1)^n)\left(2+\frac{1}{2^{n+1}}\right)\right\}^2\right]^{\frac{1}{2}}.$$

5. Case C: When $f^{(n)} \in L^2[a,b]$.

Theorem 6. *Let* $f : [a;b] \to R$ *be an n-times differentiable function on* (a,b)*, with* $f^{(n)} \in L^2[a,b]$*. Then, we have*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \tag{5.1}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left. + \left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\times\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + \left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left. + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{(x-a)^{2n+1}}{2^{2n}} + 2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$
$$\left. - \left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}$$

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$$-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(x-a)^{n+1}}{2^{n+1}}(1+(-1)^n)+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}-\frac{(-1)^n}{2^{n+1}}-(-1)^n-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}},$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, *where*

$$\sigma\left(f^{(n)}\right) = \left\|f^{(n)}\right\|_2^2-\frac{\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)^2}{b-a}=\left\|f^{(n)}\right\|_2^2-k^2(b-a), \tag{5.2}$$

where S *is as defined in Theorem 4.*

Proof. Let $R_n(x)$ is defined as in (3.5). If we choose $C=\frac{1}{b-a}\int_a^b f^{(n)}(s)ds$ in (3.6) and use the Cauchy inequality and (3.5), then we get

$$\begin{aligned}
&|R_n(x)|\\
&\le\frac{1}{b-a}\int_a^b\left|f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right|\left|P_n(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right|dt\\
&\le\frac{1}{b-a}\left[\int_a^b\left(f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right)^2dt\right]^{\frac{1}{2}}\\
&\times\left[\int_a^b\left(P_n(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right)^2dt\right]^{\frac{1}{2}}\\
&=\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\int_a^b(P_n(x,t))^2-\frac{1}{b-a}\left(\int_a^b P^{(n)}(x,t)dt\right)^2\right]^{\frac{1}{2}}\\
&\le\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{1}{2^{2n}}(x-a)^{2n+1}+2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.\\
&\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}\\
&-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{1}{2^{n+1}}(1+(-1)^n)(x-a)^{n+1}\right.\\
&+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\\
&\left.\left.\times\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Hence theorem is completed. □

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Corollary 7. *Substitution of $x=a$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a)+\left(1+\frac{1}{2^{k+1}}+\frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\}\right. \tag{5.3}$$

$$\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)(1+(-1)^n)\right|$$

$$\leq\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{-1}{(n!)^2(2n+1)}\frac{(b-a)^{2n+1}}{4^{3n+1}}\right.$$

$$\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(b-a)^{n+1}}{2^{n+1}}(1+(-1)^n)\right\}^2\right]^{\frac{1}{2}}.$$

Corollary 8. *Substitution of $x=\frac{a+b}{2}$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)\right.\right. \tag{5.4}$$

$$\left.+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$

$$\left.-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right|$$

$$\leq\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{(b-a)^{2n+1}}{2^{2n}(n!)^2(2n+1)}\left(1+\frac{1}{2^{2n+1}}\right)-\frac{(b-a)^{2n+1}(1+(-1)^n)^2}{4^{2n+1}(n+1)!}\right]^{\frac{1}{2}}.$$

Corollary 9. *Substitution of $x=\frac{3a+b}{4}$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left[\frac{\left(1+(-1)^k\right)}{2^{k+1}}f^{(k)}\left(\frac{7a+b}{8}\right)\right.\right. \tag{5.5}$$

$$+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)+f^{(k)}\left(\frac{a+3b}{4}\right)$$

$$\left.+\frac{1}{2^{k+1}}\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+7b}{8}\right)\right]+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$

$$-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right).\frac{1}{(n+1)!}\frac{(b-a)^{n-1}}{4^{n+1}}$$

$$\left.\times\left(1+(-1)^n+\frac{1}{2^n}+\frac{(-1)^n}{2^n}\right)\right|$$

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$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\frac{2(b-a)^{2n+1}}{4^{2n+1}}\left(\frac{1}{2^{2n}}+1\right)\right.$$

$$\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(b-a)^{n+1}}{4^{n+1}}\left(1+(-1)^n\right)\left(1+\frac{1}{2^n}\right)\right\}^2\right]^{\frac{1}{2}}.$$

Remark 2. *By choosing $n = 1$ in case A, B and C, we get all results obtained in* [10]*.*

Remark 3. *By choosing $n = 2$ in case A, B and C, we get all results obtained in* [9]*.*

6. Derivation of Numerical Quadrature Rules

We propose some new quadrature rules involving higher order derivatives of the function f. In fact, the following new quadrature rules can be obtained while investigating error bounds using theorem 5.

$$Q_{n,1}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(b-a)^{k+2}}{2^{k+1}(k+1)!}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]$$
$$+\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\frac{(b-a)^n}{2^{n+1}(n+1)!}\left(1+(-1)^n\right),$$

$$Q_{n,2}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(b-a)^{k+2}(-1)^k}{4^{k+1}(k+1)!}\left[f^{(k)}\left(\frac{3a+b}{4}\right)\right.$$
$$\left.+\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{a+b}{2}\right)+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right]$$
$$+\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2}{4^{n+1}}\frac{(b-a)^n}{(n+1)!}\left((-1)^n+1\right),$$

$$Q_{n,3}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}\frac{(b-a)^{k+2}}{4^{k+1}}\left[\frac{1}{2^{k+1}}\left(1+(-1)^k\right)\left(f^{(k)}\left(\frac{7a+b}{8}\right)+f^{(k)}\left(\frac{a+7b}{8}\right)\right)\right.$$
$$\left.+\left\{(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)+f^{(k)}\left(\frac{a+3b}{4}\right)\right\}\right]$$
$$+\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\times\frac{(b-a)^n}{4^{n+1}(n+1)!}\left((-1)^n+1\right)\left(\frac{1}{2^n}+1\right).$$

Performance of the efficient quadrature rules

	Method	$n:\quad Q_{n,1}(f)$	$n:\quad Q_{n,2}(f)$	$n:\quad Q_{n,3}(f)$	Exact Value
1.	$\int_0^1 f_1(x)dx$	**2:** 2.83333	**2:** 2.83333	**2:** 2.83333	2.83333
	Error:	0	0	0	
2.	$\int_0^1 f_2(x)dx$	**6:** 0.30117	**4:** 0.301172	**4:** 0.30117	0.301169
	Error:	1.1381×10^{-6}	3.08726×10^{-6}	1.38925×10^{-6}	
3.	$\int_0^1 f_3(x)dx$	**6:** 0.909328	**4:** 0.909324	**4:** 0.909327	0.909331
	Error:	2.33999×10^{-6}	7.13925×10^{-6}	3.21638×10^{-6}	
4.	$\int_0^1 f_4(x)dx$	**5:** 0.793022	**4:** 0.793031	**4:** 0.793031	0.793031
	Error:	8.63182×10^{-6}	2.9641×10^{-7}	1.33626×10^{-7}	
5.	$\int_0^1 f_5(x)dx$	**11:** 1.46266	**7:** 1.46265	**6:** 1.46266	1.46265
	Error:	5.8789×10^{-6}	2.29707×10^{-6}	5.20247×10^{-6}	
6.	$\int_0^1 f_6(x)dx$	**11:** 1.31384	**6:** 1.31383	**6:** 1.31383	1.31383
	Error	7.37624×10^{-6}	2.13363×10^{-6}	1.73918×10^{-6}	
7.	$\int_0^1 f_7(x)dx$	**6:** 1.34146	**4:** 1.34137	**4:** 1.34147	1.34147
	Error:	1.4808×10^{-7}	5.42574×10^{-7}	2.44601×10^{-7}	
8.	$\int_0^1 f_8(x)dx$	**9:** 0.62977	**5:** 0.629762	**4:** 0.629774	0.629769
	Error:	1.18074×10^{-6}	6.3567×10^{-6}	5.647×10^{-6}	

Table:

$$
\begin{aligned}
&f_1(x) = x^2 + x + 2, && f_2(x) = x \sin x, \\
&f_3(x) = e^x \sin x \ , && f_4(x) = x^2 + \sin x, \\
&f_5(x) = e^{x^2}, && f_6(x) = e^x \cos\left(e^x - 2x\right), \\
&f_7(x) = x + \cos x, && f_8(x) = \log\left(x^2+2\right) \sin\left[\log\left(x^2+2\right)\right].
\end{aligned} \tag{6.1}
$$

From the above table, we observe that all three quadrature rules show exact value of the integral of f_1 for $n = 2$. For any polynomial of degree k, $n = k + 1$ will give exact value of the integral f_1. Acceptable error estimates can be obtained for smaller values of n to save computational time.

The integral of f_5 , $Q_{n,3}(f)$ report an error of the order of 10^{-6} for $n = 6$ while the other two quadrature rules give a similar error for $n = 7$ and $n = 11$. Similarly for all other functions $Q_{n,3}(f)$ report errors of the order of 10^{-6} or 10^{-7} for relatively smaller values of n as compared to the other two quadrature rules. Specifically, $Q_{n,3}(f)$ give an excellent estimate for the integrals of f_5 and f_8 at $n = 6$ and $n = 4$ respectively. In general $Q_{n,3}(f)$ gave better results as compared to the rest of the quadrature rules for much smaller values of n. Therefore we can conclude that overall $Q_{n,3}(f)$ is computationally more efficient both in terms of error approximation, simplicity, and time. As a rough estimate we integrated $\log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right]$ using the built in algorithms of Mathematica 10.0 which took 26.30 seconds to give its approximate answer. To obtain similar approximation for the integral of f_8, $Q_{n,3}(f)$ took less than a second.

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Based on this analysis, we can conjecture that $Q_{n,3}(f)$ is the most efficient quadrature rule, while $Q_{n,2}(f)$ comes second in terms of performance. It should be noted that if desired the value of n can be adjusted to improve the error bounds or decrease computational time.

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Duality in multiobjective nonlinear programming under generalized second order $(F, b, \phi, \rho, \theta)-$ univex functions

Falleh R. Al-Solamy, Meraj Ali Khan

Abstract

In the present paper, second order duality for multiobjective nonlinear programming are investigated under the second order generalized $(F, b, \phi, \rho, \theta)-$ univex functions. The weak, strong and converse duality theorems are proved. Further, we also illustrated an example of $(F, b, \phi, \rho, \theta)-$ univex functions. Results obtained in this paper extend some previously known results of multiobjective nonlinear programming in the literature.

Keywords: Duality, Multiobjective programming, Univex functions

Mathematics Subject Classification (2000): 90C32, 49K35, 49N15

1 Introduction

In recent years, the concept of convexity and generalized convexity is well known in optimization theory and plays a central role in mathematical economics, management science and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important aspects in mathematical programming. In particular, the concept of generalized $(F, \rho)-$ convexity introduced by Preda [8]. In [9, 13], the concept of $V - \rho$-invexity and $(F, \alpha, \rho, d)-$ convexity were introduced respectively. Zhang and Mond [12] extended the class of $(F, \rho)-$ convex functions to second oder $(F, \rho)-$ convex functions and obtained the duality results for Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective dual problems. Motivated by Liang et al. [13] and Aghezzaf [2], I. Ahmad and Z. Husain [5] introduced second order $(F, \alpha, \rho, d)-$ convex functions and their generalization and they investigate weak, strong and strict converse duality theorems for second order Mond Weir type Multiobjective dual. Bector et al. [15] generalized the notion of convex function to univex functions. Rueda et al. [16] obtained optimality and duality results for several mathematical programs by combining the concepts of type I and univex functions. Mishra [8] obtained optimality results and saddle point results for multiobjective programs under generalized type I univex functions. Recently, Zalmai [14] introduced the notion of second order $(F, b, \phi, \rho, \theta)-$ univex functions and he called these functions $(F, b, \phi, \rho, \theta)-$sounivex functions, these function generalize the second order $(F, \alpha, \rho, d)-$convex functions defined by Ahmad and Husain [5].

The concept of second order duality in nonlinear programming problems was first introduced by Mangasarian [11]. One significant practical application of second order dual over first order is that it may provide tighter bounds for value of objective function because there ae more parameters involved, several researchers [1, 4, 7, 21] considered second order dual models for multiobjective

programming. In this paper , we formulate second order dual model and investigate weak, strong and strict converse duality theorems under $(F, b, \phi, \rho, \theta)-$ sounivexity assumptions. Further, an example have been constructed, which shows the existence of $(F, b, \phi, \rho, \theta)-$ sounivex functions.

2 Notations and Preliminaries

We consider the following multiobjective nonlinear programming problem:

$$(\mathbf{P}) \qquad \text{Minimize} \qquad f(x),$$
$$\text{subject to} \qquad g(x) \leqq 0, \quad x \in X, \tag{1}$$

where $f = (f_1, f_2, \ldots, f_k) : X \to R^k$, $g = (g_1, g_2, \ldots g_m) : X \to R^m$ are assumed to be twice differentiable function over X, an open subset of R^n.

Definition 2.1. A function $\mathcal{F} : X \times X \times R^n \to R$, where $X \subseteq R^n$ is said to be sublinear in its third argument, if $\forall\ x, \bar{x} \in X$,
(i) $\mathcal{F}(x, \bar{x}; a_1 + a_2) \leq \mathcal{F}(x, \bar{x}; a_1) + \mathcal{F}(x, \bar{x}; a_2),\ \forall\ a_1, a_2 \in R^n$,

(ii) $\mathcal{F}(x, \bar{x}; \alpha a) = \alpha \mathcal{F}(x, \bar{x}; a),\ \forall\ \alpha \in R_+, a \in R^n$.

Definition 2.2. A point $\bar{x} \in S$ is said to efficient solution of (P), if there exists no other feasible point x such that $f(x) \leq f(\bar{x})$ for each $x, \bar{x} \in X$.

Let $u \in R^n$ and assume that the function $f : X \to R$ is twice differentiable at u.

Definition 2.3. [14] The function f is said to be (strictly) $(F, b, \phi, \rho, \theta)-$ sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X (x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2} p^t \nabla^2 f(u) p)(>) \geqq F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u) p])$$
$$+\rho(x, u) \|\theta(x, u)\|^2,$$

where $\|.\|^2$ is a norm on R^n.

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$sounivex at u. Now we define generalized $(F, b, \phi, \rho, \theta)-$ sounivex functions

Definition 2.4. A twice differentiable function f, over X is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X (x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2} p^t \nabla^2 f(u) p) < 0$$
$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u) p]) < -\rho(x, u) \|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$pseudo sounivex at u.

Definition 2.5. A twice differentiable function f, over X is said to be strictly $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \geqq -\rho(x, u)\|\theta(x, u)\|^2$$

$$\Rightarrow \quad \phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) > 0,$$

or equivalently

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leqq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) < -\rho(x, u)\|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be strictly $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components is strictly f_i is $(F, b, \phi, \rho, \theta)-$pseudo sounivex at u.

Definition 2.6. A twice differentiable function f, over X is said to be $(F, b, \phi, \rho, \theta)-$ quasi sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leqq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \leqq -\rho(x, u)\|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$quasi sounivex at u.

Definition 2.7. A twice differentiable function f, over X is said to be strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \leq -\rho(x, u)\|\theta(x, u)\|^2$$

A twice differentiable vector function $f : X \to R^k$ is said to be strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u.

Every $(F, b, \phi, \rho, \theta)-$ sounivex function need not to be second order $(F, \alpha, \rho, d)-$ convex, definded in [5]. To show this, consider the following example.

Example 2.1. Let $f : X = (0, \infty) \to R$ be defined as $f(x) = -x^2 - x$. Let $\phi(t) = -t$, $b(x, u) = x - u$, $\rho = -10$, $\theta(x, u) = \frac{u+2}{2}$ and sublinear function is defined as $F(x, u, a) = a(x - u) + x$

$$F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) = -(x^2 - u^2)(2u + 1 + 2p) + x - 10\|\frac{u+2}{2}\|^2,$$

at $u = 0$,

$$F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p]) = -x^2(1 + 2p) + x - 10$$

and

$$f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p = -x^2 - x + u^2 + u - p^2,$$

at $u = 0$

$$\phi(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) = x^2 + x + p^2,$$

and it is easy to see that

$$\phi(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) - F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p])$$

$$= x^2 + p^2 + x^2(1 + 2p) + 10 \geq 0$$

for all $x \in R$ and $-1 \leq p < \infty$, so the function is $(F, b, \rho, \phi, \theta)-$ sounivex at $x = 0$, but at $p = -1, x = 10$

$$(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) - F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p]) < 0$$

Hence, the function is not $(F, \alpha, \rho, d)-$convex at $x = 0$.

Now we have following Kuhn-Tucker type necessary conditions, which will be useful to prove the strong duality theorem.

Theorem 2.1. (*Kuhn-Tucker type necessary conditions*) Assume that x^* is an efficient solution for (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\lambda^* \in R^k$ and $y^* \in R^m$, such that

$$\lambda^{*t}\nabla f(x^*) + y^{*t}\nabla g(x^*) = 0,$$

$$y^{*t}\nabla g(x^*) = 0,$$

$$y^* \geqq 0,$$

$$\lambda^* \geq 0.$$

3 Second order Mond-Weir type duality

In this section, we consider the following Mond-Weir second order dual associated with multiobjective problem (P) and establish weak, strong and strict converse duality theorems under generalized $(F, b, \rho, \phi, \theta)-$ sounivexity

(**MD**) Maximize

$$f(u) - \frac{1}{2}p^t\nabla^2 f(u)p$$

Subject to

$$\nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p + \nabla\lambda^t g(u) + \nabla^2\lambda^t g(u)p = 0, \tag{2}$$

$$y^t g(u) - \frac{1}{2}p^t\nabla^2 y^t g(u)p \geqq 0, \tag{3}$$

$$y \geqq 0, \tag{4}$$

$$\lambda \geq 0, \tag{5}$$

where λ is a $k-$dimensional vector, and y is an $m-$dimensional vector.

Theorem 2 (*weak duality*) Suppose that for all feasible solutions x in (P) and all feasible solutions (u, y, λ, p) in MD

(i) $y^t g(0)$ is $(F, b, \phi, \rho, \theta)-$quasi sounivex at u,

(ii) $\lambda > 0$, and $f(.)$ is strong $(F, b_1, \phi, \rho_1, \theta)-$ pseudo sounivex at u with $b^{-1}\rho + b_1^{-1}\rho_1\lambda \geqq 0$,

(iii) $u \leq 0 \Rightarrow \phi(u) \leq 0$ and $v \leqq 0 \Rightarrow \phi(v) \leqq 0$, for all $u, v \in R^n$.

Then the following can not hold

$$f(x) \leq f(u) - \frac{1}{2}p^t\nabla^2 f(u)p. \tag{6}$$

Proof. Now suppose contrary to the result that (6) holds, i.e.,

$$f(x) \leq f(u) - \frac{1}{2}p^t\nabla^2 f(u)p,$$

or

$$f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p \leq 0,$$

then by assumption (iii)

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leq 0, \tag{7}$$

which by virtue of assumption (ii) leads

$$F(x, u, b_1(x, u)\{\nabla f(u) + \nabla^2 f(u)p\}) \leq -\rho_1\|\theta(x, u)\|^2. \tag{8}$$

On multiplying (8) by $\lambda > 0$ and using sublinearity of F with $b_1(x, u) > 0$, we have

$$F(x, u, \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p) < -b_1^{-1}(x, u)\rho_1\lambda\|\theta(x, u)\|^2. \tag{9}$$

The first dual constraint and sublinearity of F give

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u)p) \geqq -F(x, u; \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p).$$

Applying (9) in above inequality, we have

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u)) > b_1^{-1}(x, u)\rho_1\lambda\|\theta(x, u)\|^2. \tag{10}$$

Let x be any feasible solution in (P) and (u, y, λ, p) be any feasible solution in (MD). Then we have

$$y^t g(x) \leqq 0 \leqq y^t g(u) - \frac{1}{2} p^t \nabla^2 y^t g(u) p), \tag{11}$$

by assumption (iii), (11) yields

$$\phi(y^t g(x) - y^t g(u) + \frac{1}{2} p^t \nabla^2 y^t g(u) p) \leqq 0. \tag{12}$$

Using $(F, b, \phi, \rho, \theta)$-quasi sounivexity of $y^t g(.)$, we have

$$F(x, u; b(x, u)\{\nabla y^t g(u) + \nabla^2 y^t g(u) p)\}) \leqq -\rho \|\theta(x, u)\|^2. \tag{13}$$

Since $b(x, u) > 0$, the above inequality with the sublinearity of F give

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u) p) \leqq -b^{-1} \rho \|\theta(x, u)\|^2. \tag{14}$$

Now using the assumption $b^{-1}\rho + b_1^{-1}\rho_1\lambda \geqq 0$, the above inequality yields

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u) p) \leqq b_1^{-1} \rho_1 \lambda \|\theta(x, u)\|^2. \tag{15}$$

Which contradict (10), hence (6) can not hold.

Theorem 3 (*Strong duality*). Let $\bar{x}$ be an efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{y} \in R^m$ and $\bar{\lambda} \in R^k$, such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is a feasible for (MD) and the corresponding values of (P) and (MD) are equal. If in addition, the assumptions of weak duality (Theorem 2) hold for all feasible solutions of (P) and (MD), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution of (MD).

Proof. Since $\bar{x}$ is an efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied, then by Theorem 1, there exist $\bar{y} \in R^m$ and $\bar{\lambda} \in R^k$, such that

$$\bar{\lambda}^t \nabla f(\bar{x}) + \bar{y}^t \nabla g(\bar{x}) = 0,$$

$$\bar{y}^t \nabla g(\bar{x}) = 0,$$

$$\bar{y} \geqq 0,$$

$$\bar{\lambda} \geq 0.$$

Therefore $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is feasible for (MD) and the corresponding values of (P) and (MD) are equal. The efficiency of this feasible solution for (MD) thus follows from weak duality (Theorem 2).

Theorem 4 (*Strict converse duality*) Let $\bar{x}$ and $(\bar{u}, \bar{y}, \bar{\lambda}, \bar{p})$ be the efficient solution of (P) and (MD), respectively such that

$$\bar{\lambda}^t f(\bar{x}) = \bar{\lambda}^t f(\bar{u}) - \frac{1}{2} \bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u}) \bar{p}. \tag{16}$$

Suppose

(i) $y^t g(.)$ is $(F, b, \phi, \rho, \theta)-$quasi sounivex at $\bar{u}$,

(ii) $\bar{\lambda}^t f(.)$ be $(F, b_1, \phi, \rho_1, \theta)-$ pseudo sounivex at $\bar{u}$ with $b^{-1}\rho + b_1^{-1}\rho_1\lambda \geqq 0$,

(iii) $u \leqq 0 \Rightarrow \phi(u) \leqq 0$ and $v < 0 \Rightarrow \phi(v) < 0$, for all $u, v \in R^n$.

Then $\bar{x} = \bar{u}$, that is $\bar{u}$ is an efficient solution.

Proof. We assume that $\bar{x} \neq \bar{u}$ and reach a contradiction, since $\bar{x}$ and $(\bar{u}, \bar{y}, \bar{\lambda}, p)$ are respectively the feasible solution of (P) and (MD), we have

$$\bar{y}^t g(\bar{x}) - \bar{y}^t g(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{y}^t g(\bar{u})\bar{p} \leqq 0. \tag{17}$$

Using the assumption (iii), we have

$$\phi(\bar{y}^t g(\bar{x}) - \bar{y}^t g(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{y}^t g(\bar{u})\bar{p}) \leqq 0. \tag{18}$$

By $(F, b, \phi, \rho, \theta)-$quasi sounivexity of $\bar{y}^t g(.)$ at $\bar{u}$, we get

$$F(\bar{x}, \bar{u}; b(\bar{x}, \bar{u})\{\nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p})\}) \leqq -\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{19}$$

Since $b(\bar{x}, \bar{u}) > 0$, the inequality (19) along with the sublinearity of F, imply

$$F(\bar{x}, \bar{u}; \nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p}) \leqq -b^{-1}(x, u)\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{20}$$

The first dual constraint and sublinearity of F imply

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) \geqq -F(\bar{x}, \bar{u}, \nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p}).$$

Applying (20) and $b^{-1}\rho + b_1^{-1}\rho \geqq 0$ in above inequality, we get

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) \geqq -b_1^{-1}(\bar{x}, \bar{u})\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{21}$$

Suppose (16) does not holds, then we have

$$\bar{\lambda}^t f(\bar{x}) < \bar{\lambda}^t f(\bar{u}) - \frac{1}{2}\bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u})\bar{p},$$

now using assumption (iii)

$$\phi(\bar{\lambda}^t f(\bar{x}) - \bar{\lambda}^t f(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u})\bar{p}) < 0.$$

Now by the assumption (ii), the above inequality gives

$$F(\bar{x}, \bar{u}; b_1(\bar{x}, \bar{u})(\nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p})) < -\rho\|\theta(\bar{x}, \bar{u})\|^2,$$

or

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) < -b_1^{-1}(\bar{x}, \bar{u})\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{22}$$

Which contradict (21). Hence result.

4 Conclusion

In this paper anew concept of generalized invex functions is introduced. Under this generalized invexity we establish weak, strong and converse duality theorems. These duality relations lead to duality in nonlinear fractional programming problems.

5 Authors contributions

Both the authors contributed equally to writing of this paper and the final manuscript is read and approved by the authors.

6 Competing interests

The author declare that they have no competing interests.

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STABILITY OF FRACTIONAL DIFFERENTIAL EQUATION WITH BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we prove the Hyers-Ulam stability of a fractional differential equation of order $\alpha \in (1, 2)$ with certain boundary conditions.

1. INTRODUCTION

The recent concentric area in the research world of mathematics is fractional differential equations. The concept of fractional derivative is not new and is very much as old as classical differential equations. In recent years, many authors disscussed and proved the existence results of fractional differential equations using various methods. For example, one can refer the monographs of Kilbas et al. [10], Miller and Ross [14], Podulbny [20], Diethelm et al. [4, 5], Benchora [2] and so on. Obviously, the differential equations of fractional order has been proved to be a valuable tool in the modeling of many phenomena in various fields of science and engineering. Indeed, one can find many applications in electromagnetic, control, electrochemistry, etc. (see [6, 7]).

At the same instance, the stability concept is more devoloped in the research world of mathematics, particularly in functional equations. But the analysis of stability concepts of fractional differential equations has been very slow and there are only countable number of works. In 2009, Li [12], first proposed the Mittag-Leffler stability and in 2010 [13], the fractional Lyapunovs second method. In the next year, Li and Zhang [11] have been given a brief overview on the stability of the fractional differential equations. However, there are only few works available on the local stability and Mittag-Leffler stability for fractional differential equations and very rare works on the Ulam stability of fractional differential equations.

In 2011, Wang [24] carried out a pioneering work on the Hyers-Ulam stability and data dependence for fractional differential equations with Caputo derivative. Wang [25] proved the Hyers-Ulam stability of fractional differential equation of order $0 < \alpha < 1$ via a generalized fixed point approach, by adopting some part idea of Wang et al. [24], Cadariu and Radu [3] and Jung [9] in the next year. Particularly, there are very rare works on the Hyers-Ulam stability of fractional differential equations with boundary conditions. Recently, Rabha [8], Muniyappan and Rajan [16] had given Ulam stabilities with boundary conditions in the interval $(0, 1)$. For more information on functional equations and their stability problems, see [15, 17, 18, 19, 21, 22, 23].

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In this paper, the Hyers-Ulam stability of the following fractional boundary value problem is proved.

$$^{C}D^{\alpha}y(t) = F(t, y(t)), \quad 1 < \alpha < 2 \tag{1.1}$$

$$y(0) = y_0 \quad y(T) = y_T \tag{1.2}$$

This paper is organized as follows: In Section 2, basic definitions and notations are given. In Section 3, the Hyers-Ulam stability of the above fractional boundary value problem is proved.

2. Preliminaries

Throughout this paper, we assume that Y is a normed space and $I = [0, T]$ is a given interval.

Definition 2.1. ([2]) The fractional order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function.

Definition 2.2. ([2]) For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h, is defined by

$$(^{c}D_{a+}^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. ([1]) A function $y \in C^2(I, \mathbb{R})$ is said to be a solution of (1.1)-(1.2) if y satisfies the equation $^{c}D^{\alpha}y(t) = F(t, y(t))$ on I, and the condition $y(0) = y_0$ and $y(T) = y_T$

Lemma 2.4. ([1]) *Let $1 < \alpha < 2$ and let $F : I \to \mathbb{R}$ be continuous. A function $y \in C^2(I, \mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T$$

if and only if y is a solution of the fractional boundary value problem

$$^{c}D^{\alpha}y(t) = F(t, y(t)), t \in [0, T]$$

$$y(0) = y_0 \quad y(T) = y_T$$

Definition 2.5. ([25]) The fractional differential equation (1.1) is Hyers-Ulam stable if there exists a continuously differentiable function $f : I \to Y$ satisfying the inequality

$$\|^{c}D^{\alpha}y(t) - F(t, y(t))\| \le \epsilon$$

for all $t \in I$ and for some $\epsilon > 0$, there exists a solution $f_0 : I \to Y$ of the fractional differential equation 1.1 such that

$$\|f(t) - f_0(t)\| \le K\epsilon$$

for all $t \in I$.

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Definition 2.6. ([25]) The fractional differential equation (1.1) is Hyers-Ulam-Rassias stable if there exists a continuously differentiable function $f : I \to Y$ satisfying the inequality

$$\|^c D^\alpha y(t) - F(t, y(t))\| \leq \varphi(t)$$

for all $t \in I$, there exists a solution $f_0 : I \to Y$ of the fractional differential equation (1.1) such that

$$\|f(t) - f_0(t)\| \leq \Phi(t)$$

for all $t \in I$. where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly.

Definition 2.7. ([25]) For a nonempty set X, a function $d : X \times X \to [0, \infty]$ is called generalized metric on X if and only if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Such a space is called a generalized complete metric space.

Theorem 2.8. ([3]) *Let (X,d) be a generalised complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$, If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k) < \infty$ for some $x \in X$, then the following are true:*

(1) *The sequence $\{\Lambda^n x\}$ converges to a fixed end point x^* of Λ*
(2) *x^* is the unique fixed point of Λ in $X^* = \left\{y \in X | d(\Lambda^k x, y) < \infty\right\}$;*
(3) *If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$*

3. Hyers-Ulam stability

In this section, we first investigate the Hyers-Ulam stability of the fractional differential equation (1.1) with boundary condition (1.2) via Theorem 2.8.

Theorem 3.1. *Let $I = [0, T]$ be a closed interval. Let K, P, and L be positive constants with $0 < KPL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the standard Lipschitz condition*

$$|F(t, y) - F(t, z)| \leq L\,|y - z| \tag{3.1}$$

for all $t \in I$ and $y, z \in \mathbb{R}$. If a continuously differential function $y : I \to \mathbb{R}$ satisfies

$$|^c D^\alpha y(t) - F(t, y(t))| \leq \varphi(t) \tag{3.2}$$

for all $t \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function with

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(\tau) d\tau \right| \leq K\varphi(t) \tag{3.3}$$

for all $t \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T \tag{3.4}$$

and

$$|y(t) - y_0(t)| \leq \frac{K}{1 - KPL} \varphi(t) \tag{3.5}$$

for all $t \in I$.

Proof. Let us define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} | f \text{ is continuous}\}. \tag{3.6}$$

Similar to [9, Theorem 3.1], one can introduce a generalised complete metric on X as follows

$$d(f,g) = \inf \{C \in [0,\infty] | \, |f(t) - g(t)| \le C\varphi(t) \quad \text{for all} \quad t \in I\}. \tag{3.7}$$

Define an operator $\Lambda : X \to X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T \tag{3.8}$$

for all $f \in X$.
It is easy to see that Λ is well defined, since F and f are continuous functions.
To acheive our aim, we need to prove that Λ is strictly contractive on X.
For any $f, g \in X$, let $C_{fg} \in [0,\infty]$ be an arbitrary constant with $d(f,g) \le C_{fg}$, that is, by (3.7), we have

$$|f(t) - g(t)| \le C_{fg}\varphi(t) \tag{3.9}$$

for all $t \in I$. It then follows from (3.1), (3.3), (3.7), (3.8) and (3.9) that

$$\begin{aligned}
|(\Lambda f)t - (\Lambda g)t| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
&\quad + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
&\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - g(s)| \, ds + \frac{tL}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s) - g(s)| \, ds \\
&\le \frac{L}{\Gamma(\alpha)} C_{fg} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds + \frac{tL}{T\Gamma(\alpha)} C_{fg} \int_0^T (T-s)^{\alpha-1} \varphi(s) ds \\
&\le KPLC_{fg}\varphi(t)
\end{aligned}$$

for all $t \in I$, where $P = \left(1 + \frac{t}{T}\right)$. That is,

$$d(\Lambda f, \Lambda g) \le KPLC_{fg}.$$

Hence we can conclude that

$$d(\Lambda f, \Lambda g) \le KPLd(f,g)$$

for all $f, g \in X$, where we note that $0 < KPL < 1$.
It follows from (3.6) and (3.8) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$\begin{aligned}
|(\Lambda g_0)(t) - g_0(t)| &= \Big| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds \\
&\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T - g_0(t) \Big| \\
&\le C\varphi(t)
\end{aligned}$$

for all $t \in I$, since $f(t, g_0(t))$ and $g_0(t)$ are bounded on I and $min_{t\in I}\varphi(t) > 0$.
Thus (3.7) implies that

$$d(\Lambda g_0, g_0) < \infty.$$

Therefore, according to Theorem 2.8, there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) and $\Lambda y_0 = y_0$, that is, y_0 satisfies (3.4) for every $t \in I$.
we will now verify that $\{g \in X/d(g_0, g) < \infty\} = X$.
For any $g \in X$, since g and g_0 are bounded on I and $min_{t\in I}\varphi(t) > 0$, there exists a constant $0 < C_g < \infty$ such that $|g_0(t) - g(t)| \leq C_g\varphi(t)$
Hence, we have $d(g_0, g) < \infty$ for all $g \in X$, that is $\{g \in X/d(g_0, g) < \infty\} = X$.
Hence in view of Theorem 2.8, we conclude that y_0 is the unique continuous function with the property (3.4). On the other hand, it follows from (3.2) that

$$-\varphi(t) \leq^c D^{\alpha}_{a+}y(t) - F(t, y(t)) \leq \varphi(t)$$

for all $t \in I$.
If we integrate each term in the above inequality and substitute the boundary conditions, then we obtain

$$|y(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, y(s))ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s, y(s))ds - \left(\frac{t}{T} - 1\right)y_0 + \frac{t}{T}y_T|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\varphi(t)ds$$

for all $t \in I$.
Thus by (3.3) and (3.8) we get

$$|y(t) - (\Lambda y)(t)| \leq K\varphi(t)$$

for each $t \in I$, which implies that

$$d(y, \Lambda y) \leq K. \tag{3.10}$$

Finally, Theorem 2.8 and (3.10) imply that

$$d(y, y_0) \leq \frac{1}{1 - KPL}d(y, \Lambda y) \leq \frac{K}{1 - KPL}.$$

□

Now, we will prove the Hyers-Ulam stability of the (1.1) with boundary condition (1.2)

Theorem 3.2. *Let $I = [0, T]$ be a closed interval. Let $r > 0$ be a positive constant with $0 \leq t, T \leq r$ and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies a Lipschitz condition (3.1) for all $t \in I$ and $y, z \in \mathbb{R}$, where L is a constant with $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfying the differential inequality*

$$\left|{}^cD^{\alpha}_{a+}y(t) - F(t, y(t))\right| \leq \epsilon \tag{3.11}$$

for all $t \in I$ and for some $\epsilon \geq 0$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying (3.4) and

$$|y(t) - y_0(t)| \leq \frac{r^\alpha}{\Gamma(\alpha+1) - LPr^\alpha}\epsilon \tag{3.12}$$

for all $t \in I$.

Proof. First, we define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} | f \text{ is continuous}\}$$

and introduce a generalized complete metric on X as follows

$$d(f, g) = \inf\{C \in [0, \infty] | \, |f(t) - g(t)| \le C \quad \text{for all} \quad t \in I\}$$

Define an operator $\Lambda : X \to X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T$$

for all $f \in X$.

We now assert that Λ is strictly contractive on X.

For all $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \le C_{fg}$, that is, let us assume that

$$|f(t) - g(t)| \le C_{fg} \tag{3.13}$$

for any $t \in I$. Moreover, it follows from (3.1), (3.8) and (3.13) that

$$\begin{aligned}
|(\Lambda f)t - (\Lambda g)t| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
&\quad + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
&\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - g(s)| \, ds \\
&\quad + \frac{tL}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s) - g(s)| \, ds \\
&\le LC_{fg} \left[\frac{r^\alpha}{\alpha\Gamma(\alpha)} + \frac{tr^\alpha}{T\alpha\Gamma(\alpha)}\right] \\
&\le \frac{LC_{fg} r^\alpha}{\Gamma(\alpha+1)} \left[\frac{t+T}{T}\right] \\
&\le \frac{LPC_{fg} r^\alpha}{\Gamma(\alpha+1)}
\end{aligned}$$

for all $t \in I$, where $P = \left(1 + \frac{t}{T}\right)$, that is

$$d(\Lambda f, \Lambda g) \le \frac{LPr^\alpha}{\Gamma(\alpha+1)} C_{fg}.$$

Thus it follows that

$$d(\Lambda f, \Lambda g) \le \frac{LPr^\alpha}{\Gamma(\alpha+1)} d(f, g)$$

for all $f, g \in X$, and we note that $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$.

Analogously to the proof of Theorem 3.1, we can show that each $g_0 \in X$ satisfies the property $d(\Lambda g_0, g_0) < \infty$.

Therefore, Theorem 2.8 implies that there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) as $n \to \infty$, and such that $y_0 = \Lambda y_0$, that is, y_0 satisfies the equation (3.4) for all $t \in I$.

If $g \in X$, then g_0 and g are continuous functions defined on a compact interval I. Hence, there exists a constant $C > 0$ with $|g_0(t) - g(t)| \leq C$ for all $t \in I$. This implies that $d(g_0, g) < \infty$ for every $g \in X$, or equivalently, $\{g \in X | d(g_0, g) < \infty\} = X$. Therefore, according to Theorem 2.8, y_0 is a unique continuous function with property (3.4). Furtheremore, it follows from (3.11) that

$$-\epsilon \leq^c D^\alpha_{a+} y(t) - F(t, y(t)) \leq \epsilon$$

for all $t \in I$. If we integrate each term of the above inequality and appling the boundary conditions, then we have

$$|(\Lambda y)(t) - y(t)| \leq \frac{r^\alpha}{\Gamma(\alpha+1)}\epsilon$$

for all $t \in I$, that is, it holds that $d(\Lambda y, y) \leq \frac{r^\alpha}{\Gamma(\alpha+1)}\epsilon$.
It now follows from Theorem 2.8 that

$$d(y, y_0) \leq \frac{1}{1 - \frac{LPr^\alpha}{\Gamma(\alpha+1)}} d(\Lambda y, y) \leq \frac{r^\alpha}{\Gamma(\alpha+1) - LPr^\alpha}\epsilon, \tag{3.14}$$

which implies the validity of (3.12) for each $t \in I$ □

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Bernstein-Stancu type operators which preserve polynomials

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Abstract

In the last years there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions. This idea motivated us to modify the sequence of linear Bernstein Stancu type operators. Using numerical examples we show that these operators present a better degree of approximation than the original ones. In this note the modified Bernstein Stancu operators are studied in regard to uniform convergence and global smoothness preservation.

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1 Introduction

In 1912 in Bernstein's constructive proof of the Weierstrass approximation theorem [3] were introduced the classical Bernstein operators $B_n : C[0,1] \to C[0,1]$, defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad \text{where } p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Lemma 1.1. *The Bernstein operators verify the following identities*

(i) $B_n(e_0;x) = 1$,
(ii) $B_n(e_1;x) = x$,
(iii) $B_n(e_2;x) = \frac{x}{n}(1 + xn - x)$, *where* $e_i(t) = t^i$, $i = 0, 1, \ldots$.

In the last years there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions. King [12] consider for the first time this kind of modification for the Bernstein operators and proved that the modified operators reproduce the functions $e_i(x) = x^i$ for $i = 0, 2$ and approximate each continuous function on $[0,1]$ with an order of approximation at least as good as that of the classic Bernstein whenever $0 \le x < \frac{1}{3}$. Using the same type of technique introduced by King or new methods many authors published new results in regard with this subject. Cárdenas-Morales et al. [4] extended this result considering a family of sequences of operators $B_{n,\alpha}$ that preserve e_0 and $e_2 + \alpha e_1$ with $\alpha \in [0,\infty)$. Gonska et al. [11] studied the sequence $V_n^\tau : C[0,1] \to C[0,1]$ defined by

$$V_n^\tau f := (B_n f) \circ (B_n \tau)^{-1} \circ \tau,$$

where τ is a continuous strictly increasing function defined on $[0,1]$ with $\tau(0) = 0$ and $\tau(1) = 1$. Note that if $\tau = \frac{e_2 + \alpha e_1}{1+\alpha}$, then $V_n^\tau = B_{n,\alpha}$ and the operators V_n^τ preserve e_0 and τ. In [5], the authors inspired by the above ideas consider the sequence of linear Bernstein-type operators defined for $f \in C[0,1]$ by $B_n(f \circ \tau^{-1}) \circ \tau$, τ being any function that is continuously differentiable ∞ times on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$. Note that the Korovkin set $\{1, e_1, e_2\}$ is generalized to $\{1, \tau, \tau^2\}$ and these operators present a better degree of approximation than B_n.

Since the modified operators present a better degree of approximation than the original ones leads to an interesting area of research, so that generalized Bernstein-Durrmeyer operators and their approximation properties were studied in [1] and [6]. Also, the modified Szasz operators were considered recently in [2].

2 Bernstein-Stancu operators

In 1968, Stancu [15] proposed the sequence of positive linear operators $S_n^{<\alpha>} : C[0,1] \to C[0,1]$ depending on a non-negative parameter α given by

$$S_n^{<\alpha>}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{<\alpha>}(x),\ x \in [0,1], \tag{2.1}$$

where

$$p_{n,k}^{<\alpha>}(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}$$

and $t^{[n,h]} := t(t-h)\cdots(t-\overline{n-1}h)$ is the n^{th} factorial power of t with increment h.

For $\alpha = 0$ these operators reduce to the classical Bernstein operators.

The values of the test function by Bernstein-Stancu operators were given by Stancu [15] as follows

Lemma 2.1. *If* $x \in [0,1]$, *then*
(i) $S_n^{<\alpha>}(e_0;x) = 1$,
(ii) $S_n^{<\alpha>}(e_1;x) = x$,
(iii) $S_n^{<\alpha>}(e_2;x) = \frac{1}{1+\alpha}\left(\frac{x(1-x)}{n} + x(x+\alpha)\right)$.

Recently, in [13] Miclăuş proposed a new technique to obtain the values of the test function, without using properties of Bernstein operators.

It is well known the following form of Bernstein operators using the divided difference

$$B_n(f;x) = \sum_{k=0}^{n} \frac{k!}{n^k}\binom{n}{k}\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] x^k. \tag{2.2}$$

Starting with the form (2.2) of the Bernstein operators, the following Stancu type operators are constructed in [7]-[8]:

$$C_n : C[0,1] \to \Pi_n,$$

$$C_n(f;x) = \sum_{k=0}^{n} \frac{k!}{n^k}\binom{n}{k}\mathbf{m}_{k,n}\left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, \quad f \in C[0,1], \tag{2.3}$$

where the real numbers $(m_{k,n})_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators and Π_n is the linear space of all real polynomials of degree $\leq n$.

Let $\mathbf{m}_{0,n} = 1$, $\lim_{n\to\infty} \mathbf{m}_{1,n} = 1$ and $\mathbf{m}_{k,n} = \frac{(a_n)_k}{k!}$, $a_n \in (0,1]$. For this special case of real sequence $(m_{k,n})_{k=0}^{\infty}$ the Bernstein-Stancu operators C_n were written in the Bernstein basis as follows (see [7], Theorem 10):

$$C_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) C_{k,n}[f], \tag{2.4}$$

where

$$C_{k,n}[f] = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)(a_n)_j(1-a_n)_{k-j}.$$

We remark that $a_n \in (0,1]$ leads to C_n linear positive operators.

The coefficients $C_{k,n}[f]$ can be written as follows

$$C_{k,n}[f] = \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n) f\left(\frac{j}{n}\right).$$

Therefore,

$$C_{k,n}[f] = S_k^{<1>}(\tilde{f}; a_n), \quad \text{where } \tilde{f}(t) = f\left(t\frac{k}{n}\right).$$

Lemma 2.2. ([7]) *The Bernstein-Stancu operators C_n verify the following identities*

(i) $C_n(e_0; x) = 1$,

(ii) $C_n(e_1; x) = a_n x$,

(iii) $C_n(e_2; x) = x^2 + \frac{x(1-x)}{n} a_n + \frac{1-a_n}{2}\left(\frac{a_n}{n} - (2 + a_n)\right) x^2$.

Let

$$\begin{aligned}\mu_{n,m}(x) &= C_n\left((t-x)^m; x\right)\\ &= \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left(\frac{j}{n} - x\right)^m, \quad n, m \in \mathbb{N},\end{aligned}$$

be the central moment operators.

Lemma 2.3. ([7]) *The central moment operators verify*

(i) $\mu_{n,2}(x) = \frac{x(1-x)}{n} a_n + x^2(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)$,

(ii) $\mu_{n,4}(x) = x^4 + \left[\frac{6(a_n)_3}{n^4}\binom{n}{3} - \frac{12(a_n)_2}{n^3}\binom{n}{2} + \frac{6a_n}{n}\right]x^3 + \left[\frac{7(a_n)_2}{n^4}\binom{n}{2} - \frac{4a_n}{n^2}\right]x^2$
$+\frac{a_n}{n^3}x + \frac{(a_n)_4}{n^4}\binom{n}{4} - \frac{4(a_n)_3}{n^3}\binom{n}{3} + \frac{6(a_n)_2}{n^2}\binom{n}{2} - 4a_n.$

In [7], Cleciu obtained the following Voronovskaya type theorem:

Theorem 2.4. ([7]) *Suppose that $x_0 \in [0,1]$ and $f''(x_0)$ exists. If $a_n \in (0,1), \lim_{n\to\infty} a_n = 1$ and $L := \lim_{n\to\infty} n(1-a_n)$ exists, then*

$$\lim_{n\to\infty} n\left[f(x_0) - C_n(f; x_0)\right] = -\frac{x_0(1-x_0)}{2} f''(x_0) + \left[x_0 f'(x_0) - \frac{x_0^2}{4} f''(x_0)\right] L.$$

3 Modified Bernstein-Stancu operators

In this section we deal with Bernstein-Stancu type generalization of (2.4). We investigate its sharp preserving and convergence properties.

We define the modified Bernstein-Stancu operators as follows:

$$C_n^{\tau}(f; x) = \sum_{k=0}^{n} p_{n,k}^{\tau}(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left(f \circ \tau^{-1}\right)\left(\frac{j}{n}\right), \tag{3.1}$$

where $p_{n,k}^{\tau}(x) = \binom{n}{k}\tau(x)^k\left(1-\tau(x)\right)^{n-k}$ and τ is any function that is continuously differentiable ∞ times on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$.

Note that these operators are positive and linear and for the case $\tau(x) = x$, these operators (3.1) reduce to the Bernstein-Stancu operators defined by Cleciu [7]-[8].

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Lemma 3.1. *The modified operators C_n^τ verify*
(i) $C_n^\tau e_0 = 1$,
(ii) $C_n^\tau \tau = a_n \tau$,
(iii) $C_n^\tau \tau^2 = \tau^2 + \frac{\tau(1-\tau)}{n} a_n + \frac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right)\tau^2$.

Let

$$\begin{aligned}\mu_{n,m}^\tau(x) &= C_n^\tau\left((\tau(t)-\tau(x))^m; x\right)\\ &= \sum_{k=0}^{n} p_{n,k}^\tau(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left(\frac{j}{n} - \tau(x)\right)^m, \quad n, m \in \mathbb{N},\end{aligned}$$

be the central moment operators.

Lemma 3.2. *The central moment operators verify*
(i) $\mu_{n,0}^\tau(x) = 1$,
(ii) $\mu_{n,1}^\tau(x) = (a_n - 1)\tau(x)$,
(iii) $\mu_{n,2}^\tau(x) = \frac{\tau(x)(1-\tau(x))}{n} a_n + \tau(x)^2(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)$,
(iv) $\mu_{n,4}(x) = \tau(x)^4 + \left[\frac{6(a_n)_3}{n^4}\binom{n}{3} - \frac{12(a_n)_2}{n^3}\binom{n}{2} + \frac{6a_n}{n}\right]\tau(x)^3$
$+\left[\frac{7(a_n)_2}{n^4}\binom{n}{2} - \frac{4a_n}{n^2}\right]\tau(x)^2 + \frac{a_n}{n^3}\tau(x) + \frac{(a_n)_4}{n^4}\binom{n}{4}$
$-\frac{4(a_n)_3}{n^3}\binom{n}{3} + \frac{6(a_n)_2}{n^2}\binom{n}{2} - 4a_n$.

Lemma 3.3. *For all $n \in \mathbb{N}$ we have*

$$\mu_{n,2}^\tau(x) \le \delta_{n,\tau}^2(x) \quad \textit{for all } x \in [0,1],$$

where $\delta_{n,\tau}^2(x) := \frac{a_n}{n}\varphi_\tau^2(x) + (1-a_n)$ *and* $\varphi_\tau^2(x) := \tau(x)(1-\tau(x))$.

Proof. We have

$$\begin{aligned}|\mu_{n,2}^\tau(x)| &= \left|\frac{\tau(x)(1-\tau(x))a_n}{n}\right| + \left|\tau^2(x)(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)\right|\\ &\le \varphi_\tau^2(x)\frac{a_n}{n} + (1-a_n) = \delta_{n,\tau}^2(x).\end{aligned}$$

□

Lemma 3.4. *If $f \in C[0,1]$, then $\|C_n^\tau f\| \le \|f\|$, where $\|\cdot\|$ is the uniform norm on $C[0,1]$.*

Proof. From the definition of the operator C_n^τ and using Lemma 3.1 it follows

$$\begin{aligned}|C_n^\tau(f;x)| &\le \sum_{k=0}^{n} p_{n,k}^\tau(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left|\left(f\circ\tau^{-1}\right)\left(\frac{j}{n}\right)\right|\\ &\le \|f\circ\tau^{-1}\| C_n^\tau(e_0;x) = \|f\|.\end{aligned}$$

□

Theorem 3.5. *Let $f \in C[0,1]$, $a_n \in (0,1]$ and $\lim_{n\to\infty} a_n = 1$. Then $C_n^\tau f$ converges to f as n tends to infinity, uniformly on $[0,1]$.*

Proof. Using the well known Korovkin theorem and Lemma 3.1 and the fact that $\{e_0, \tau, \tau^2\}$ is an extended complete Tchebychev system on $[0,1]$ it follows the uniform convergence of the operators C_n^τ. □

Let ω be the usual modulus of continuity of $f \in C[0,1]$ which is defined as

$$\omega(f;\delta) = \sup_{|h|\le\delta} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

Proposition 3.6. *Let $f \in C[0,1]$ with modulus of continuity $\omega(f,\cdot)$. Then*

$$|C_n^\tau(f;x) - f(x)| \le \left(1 + \frac{\mu_{n,2}^\tau(x)}{\delta^2}\right)\omega(f,\delta)$$

for $\delta > 0$ and $x \in [0,1]$.

Example 3.7. If we choose $\tau(x) = \frac{x^2+x}{2}$, we have $\tau(x)(1-\tau(x)) \le x(1-x)$ for all $x \in [0,1/2]$ and this inequality leads to $\mu_{n,2}^\tau(x) \le \mu_{n,2}(x)$. Therefore, the modified operators C_n^τ presents an order of approximation better than C_n in that interval.

Example 3.8. Now using a graphical example we try to illustrate these approximation processes. Let $f(x) = \sin(9x)$, $\tau(x) = \frac{x^2+x}{2}$ and $a_n = 1/2$. For $n = 20$, the approximation to the function f by C_n and C_n^τ is shown in the Figure 1.

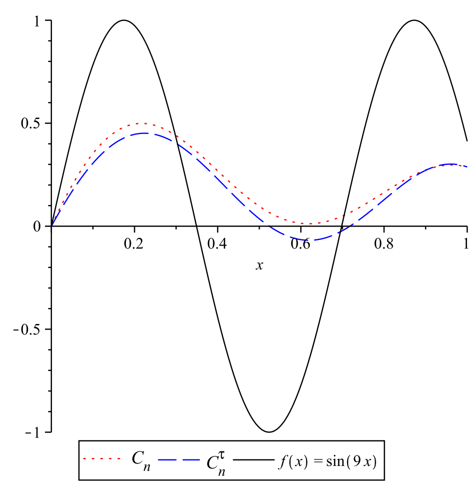

Figure 1. Approximation process by C_n and C_n^τ

Example 3.9. Let us take $f(x) = \log(x+1)$, $\tau(x) = \frac{x^2+x}{2}$ and $a_n = \frac{1}{2}$. In the Table 1 we computed the error of approximation for C_n and C_n^τ at the point $x_0 = 0.8$.

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Table 1. Error of approximation for C_n and C_n^τ

n	$\lvert C_n(f;x_0)-f(x_0)\rvert$	$\lvert C_n^\tau(f;x_0)-f(x_0)\rvert$
5	0.2800807097	0.2613318434
10	0.2762200954	0.2502212648
15	0.2749367804	0.2465367167
20	0.2742959594	0.2447029941
25	0.2739117553	0.2436063038
30	0.2736557443	0.2428768564
35	0.2734729425	0.2423567117
40	0.2733358757	0.2419671158
45	0.2732292893	0.2416644116
50	0.2731440335	0.2414224523

From the above results it follows that C_n^τ converge faster than C_n to the function $f(x) = \log(x+1)$ at the point $x_0 = 0.8$. Also, the approximation to the function f by C_n and C_n^τ is shown in the Figure 2.

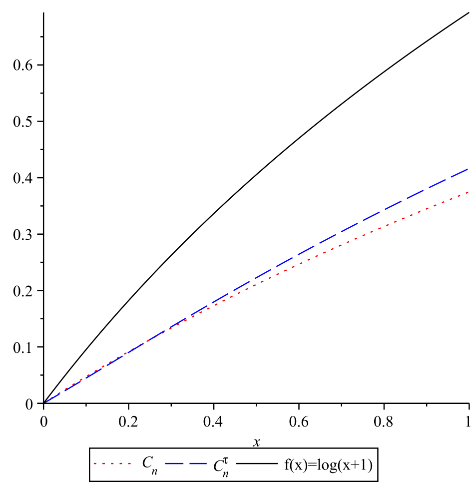

Figure 2. Approximation process by C_n and C_n^τ

4 Voronovskaya type theorem

Let $L_n : C[0,1] \to C[0,1]$, $n \geq 1$, be a positive linear operator and $L_n e_0 = e_0$. Acar et al. [1] defined a general operator $K_n : C[0,1] \to C[0,1]$ by

$$K_n g := \left(L_n(g \circ \tau^{-1})\right) \circ \tau,\ n \geq 1.$$

The authors obtained the following Voronovskaya type formula for the modified operators K_n.

Theorem 4.1. ([1]) *Let $f \in C[0,1]$ with $f''(x)$ finite for $x \in [0,1]$. If there exists $\alpha, \beta \in C[0,1]$ such that*

$$\lim_{n\to\infty} n(L_n(f,x) - f(x)) = \alpha(x) f''(x) + \beta(x) f'(x),$$

then we have

$$\lim_{n\to\infty} n\left(K_n(g,t)-g(t)\right)=\frac{\alpha(\tau(t))}{\tau'(t)^2}g''(t)+\left(\frac{\beta(\tau(t))}{\tau'(t)}-\frac{\alpha(\tau(t))\tau''(t)}{\tau'(t)^3}\right)g'(t)$$

for $g\in C[0,1]$ *with* $g''(x)$ *finite for* $x\in[0,1]$.

Using Theorem 2.4 and Theorem 4.1 we obtain a Voronovskaya type theorem for C_n^τ.

Theorem 4.2. *Let* $f\in C^2[0,1]$. *If* $a_n\in(0,1)$, $\lim_{n\to\infty}a_n=1$ *and* $L:=\lim_{n\to\infty}n(1-a_n)$ *exists, then*

$$\lim_{n\to\infty} n\left(C_n^\tau(f,x)-f(x)\right)=\frac{\alpha(\tau(x))}{\tau'(x)^2}f''(x)+\left(\frac{\beta(\tau(x))}{\tau'(x)}-\frac{\alpha(\tau(x))\tau''(x)}{\tau'(x)^3}\right)f'(x)$$

uniformly on $[0,1]$ *with* $\alpha(x)=-\frac{x(1-x)}{2}-\frac{x^2}{4}L$ *and* $\beta(x)=xL$.

5 Local Approximation

Let

$$W^2[0,1]=\left\{g\in C[0,1]:g'\in C[0,1]\right\}.$$

For $f\in C[0,1]$ and $\delta>0$, the Peetre's K-functional [14] is defined by

$$K_2(f;\delta)=\inf_{g\in W^2[0,1]}\left\{\|f-g\|+\delta\|g\|_{W^2[0,1]}\right\},$$

where

$$\|f\|_{W^2[0,1]}=\|f\|+\|f'\|+\|f''\|.$$

Throughout this paper we assume that $\inf_{x\in[0,1]}\tau'(x)\geq a$, $a\in\mathbb{R}^+$.

Theorem 5.1. *Let* $a_n\in(0,1)$ *and* $\lim_{n\to\infty}a_n=1$ *and* $f\in C[0,1]$. *For the operator* $C_n^\tau(f;\cdot)$, *there exists absolute constant* $C>0$ *such that*

$$|C_n^\tau(f;x)-f(x)|\leq CK_2\left(f;\delta^2_{n,\tau}(x)\right)+\omega\left(f;\frac{1}{a}(1-a_n)\tau(x)\right).$$

Proof. Let $g\in W^2[0,1]$ and $t\in[0,1]$. Then by Taylor's expansion, we get

$$\begin{aligned}g(t)&=\left(g\circ\tau^{-1}\right)(\tau(t))\\&=\left(g\circ\tau^{-1}\right)(\tau(x))+\left(g\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right)\\&\quad+\int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(g\circ\tau^{-1}\right)''(u)du.\end{aligned}\tag{5.1}$$

If we consider the change of variable $u=\tau(y)$, it follows

$$\int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(g\circ\tau^{-1}\right)''(u)du=\int_x^t\left(\tau(t)-\tau(y)\right)\left(g\circ\tau^{-1}\right)''(\tau(y))\,\tau'(y)dy,$$

but

$$\left(g\circ\tau^{-1}\right)''(\tau(y))=\frac{1}{\tau'(y)}\cdot\frac{g''(y)\tau'(y)-g'(y)\tau''(y)}{\left(\tau'(y)\right)^2},$$

therefore

$$\begin{aligned}&\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(g\circ\tau^{-1}\right)''(u)du\\&=\int_x^t(\tau(t)-\tau(y))\frac{g''(y)}{\tau'(y)}dy-\int_x^t(\tau(t)-\tau(y))\frac{g'(y)\tau''(y)}{\left(\tau'(y)\right)^2}dy\\&=\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^2}du-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^3}du.\end{aligned}\tag{5.2}$$

From (5.1) and (5.2) we can write

$$\begin{aligned}g(t)=g(x)&+\left(g\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x))+\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^2}du\\&-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^3}du.\end{aligned}\tag{5.3}$$

We define

$$\tilde{C_n^\tau}(f;x)=C_n^\tau(f;x)+f(x)-\left(f\circ\tau^{-1}\right)(a_n\tau(x)).$$

From Lemma 3.1 it follows

$$\tilde{C_n^\tau}(e_0;x)=1\text{ and }\tilde{C_n^\tau}(\tau;x)=C_n^\tau(\tau;x)+\tau(x)-a_n\tau(x)=\tau(x).$$

Now applying $\tilde{C_n^\tau}$ to both side of the relation (5.3) we can write

$$\begin{aligned}\tilde{C_n^\tau}(g;x)=g(x)&+C_n^\tau\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^2}du\right)\\&-\int_{\tau(x)}^{a_n\tau(x)}(a_n\tau(x)-u)\frac{g''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^2}du\\&-C_n^\tau\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^3}du\right)\\&+\int_{\tau(x)}^{a_n\tau(x)}(a_n\tau(x)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^3}du.\end{aligned}$$

Since $\inf_{x\in[0,1]}\tau'(x)\geq a, a\in\mathbb{R}^+$ and τ is strictly increasing on the interval $(0,1)$, we obtain

$$\begin{aligned}\left|\tilde{C_n^\tau}(\tau;x)-g(x)\right|&\leq\frac{1}{2}\mu_{n,2}^\tau(x)\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\quad+\frac{1}{2}\left(a_n\tau(x)-\tau(x)\right)^2\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\leq\frac{1}{2}\left[\delta_{n,\tau}^2(x)+\tau^2(x)(1-a_n)^2\right]\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\leq\delta_{n,\tau}^2(x)\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right].\end{aligned}$$

By Lemma 3.4, it follows

$$|\tilde{C}_n^\tau(g;x)| \leq |C_n^\tau(g;x)| + |g(x)| + \left|\left(g\circ\tau^{-1}\right)(a_n\tau(x))\right| \leq 3\|g\|.$$

For $f\in C[0,1]$ and $g\in W_2[0,1]$, we can write

$$\begin{aligned}
&|C_n^\tau(f;x)-f(x)|\\
&= \left|\tilde{C}_n^\tau(f;x)-f(x)+\left(f\circ\tau^{-1}\right)(a_n\tau(x))-f(x)\right|\\
&\leq |\tilde{C}_n^\tau(f-g;x)|+|\tilde{C}_n^\tau(g;x)-g(x)|+|g(x)-f(x)|\\
&\quad+\left|\left(f\circ\tau^{-1}\right)(a_n\tau(x))-\left(f\circ\tau^{-1}\right)(\tau(x))\right|\\
&\leq 4\|f-g\|+\frac{\delta_{n,\tau}^2(x)}{a^2}\|g''\|+\frac{\delta_{n,\tau}^2(x)}{a^3}\|\tau''\|\,\|g'\|+\omega\left(f\circ\tau^{-1};(1-a_n)\tau(x)\right).
\end{aligned}$$

Let $C:=\max\left\{4,\frac{1}{a^2},\frac{\|\tau''\|}{a^3}\right\}$. Then

$$|C_n^\tau(f;x)-f(x)|\leq C\left\{\|f-g\|+\delta_{n,\tau}^2(x)\|g\|_{W^2[0,1]}\right\}+\omega\left(f\circ\tau^{-1};(1-a_n)\tau(x)\right).$$

Using the following result (see [1]) $\omega\left(f\circ\tau^{-1};t\right)\leq\omega\left(f;\frac{t}{a}\right)$, the theorem is proved. □

To describe our next result, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [9]. Let $\varphi_\tau(x):=\sqrt{\tau(x)(1-\tau(x))}$ and $f\in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_{\varphi_\tau}(f;t)=\sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\varphi_\tau(x)}{2}\right)-f\left(x-\frac{h\varphi_\tau(x)}{2}\right)\right|, x\pm\frac{h\varphi_\tau(x)}{2}\in[0,1]\right\}. \tag{5.4}$$

Further, the corresponding K-functional to (5.4) is defined by

$$K_{\varphi_\tau}(f;t)=\inf_{g\in W_{\varphi_\tau}[0,1]}\{||f-g||+t||\varphi_\tau g'||\}\ (t>0), \tag{5.5}$$

where $W_{\varphi_\tau}[0,1]=\{g:g\in AC[0,1],\|\varphi_\tau g'\|<\infty\}$ and $AC[0,1]$ is the class of all absolutely continuous functions on $[0,1]$. It is well known ([9], p.11) that there exists a constant $C>0$ such that

$$K_{\varphi_\tau}(f;t)\leq C\omega_{\varphi_\tau}(f;t). \tag{5.6}$$

Now, we establish a direct approximation theorem by means of Ditzian-Totik modulus of smoothness.

Theorem 5.2. *Let $f\in C[0,1]$ and $\varphi_\tau(x)=\sqrt{\tau(x)(1-\tau(x))}$, then for every $x\in(0,1)$, we have*

$$|C_n^\tau(f;x)-f(x)|\leq\tilde{C}\omega_{\varphi_\tau}\left(f;\frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)}\right),$$

where $\tilde{C}$ is a constant independent of n and x.

Proof. Using the representation

$$g(t) = \left(g \circ \tau^{-1}\right)(\tau(t)) = \left(g \circ \tau^{-1}\right)(\tau(x)) + \int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du,$$

we get

$$|C_n^\tau(g;x) - g(x)| = \left|C_n^\tau\left(\int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du\right)\right|. \tag{5.7}$$

But,

$$\begin{aligned}\left|\int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du\right| &= \left|\int_x^t \frac{g'(y)}{\tau'(y)}\tau'(y)dy\right| = \left|\int_x^t \frac{\varphi_\tau(y)}{\varphi_\tau(y)} \cdot \frac{g'(y)}{\tau'(y)}\tau'(y)dy\right| \\ &\leq \frac{\|\varphi_\tau g'\|}{a}\left|\int_x^t \frac{\tau'(y)}{\varphi_\tau(y)}dy\right|, \end{aligned}\tag{5.8}$$

and

$$\begin{aligned}\left|\int_x^t \frac{\tau'(y)}{\varphi_\tau(y)}dy\right| &\leq \left|\int_x^t \left(\frac{1}{\sqrt{\tau(y)}} + \frac{1}{\sqrt{1-\tau(y)}}\right)\tau'(y)dy\right| \\ &\leq 2\left(\left|\sqrt{\tau(t)} - \sqrt{\tau(x)}\right| + \left|\sqrt{1-\tau(t)} - \sqrt{1-\tau(x)}\right|\right) \\ &= 2\left|\tau(t) - \tau(x)\right|\left(\frac{1}{\sqrt{\tau(t)} + \sqrt{\tau(x)}} + \frac{1}{\sqrt{1-\tau(t)} + \sqrt{1-\tau(x)}}\right) \\ &< 2|\tau(t) - \tau(x)|\left(\frac{1}{\sqrt{\tau(x)}} + \frac{1}{\sqrt{1-\tau(x)}}\right) \\ &\leq \frac{2\sqrt{2}|\tau(t) - \tau(x)|}{\varphi_\tau(x)}. \end{aligned}\tag{5.9}$$

From relations (5.7)-(5.9) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|C_n^\tau(g;x) - g(x)| &\leq 2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)}C_n^\tau\left(|\tau(t) - \tau(x)|;x\right) \\ &\leq 2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)}\left[C_n^\tau\left((\tau(t) - \tau(x))^2;x\right)\right]^{1/2} \\ &\leq 2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)}\delta_{n,\tau}(x). \end{aligned}\tag{5.10}$$

Using Lemma 3.4 and (5.10) it follows

$$\begin{aligned}|C_n^\tau(f;x) - f(x)| &\leq |C_n^\tau(f-g;x)| + |f(x) - g(x)| + |C_n^\tau(g;x) - g(x)| \\ &\leq C\left\{\|f-g\| + \frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)}\|\varphi_\tau g'\|\right\}, \end{aligned}$$

where $C = \max\left\{2, \frac{2\sqrt{2}}{a}\right\}$.

Taking infimum on the right hand side of the above inequality over all $g \in W_{\varphi_\tau}[0,1]$, we get

$$|C_n^\tau(f;x) - f(x)| \leq CK_{\varphi_\tau}\left(f; \frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)}\right).$$

Using the relation (5.6) this theorem is proven. □

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A NOTE ON HERMITE POLYNOMIALS

TAEKYUN KIM AND DAE SAN KIM

Abstract. In this paper, we consider linear differential equations satisfied by the generating function for Hermite polynomials and derive some new identities involving those polynomials.

1. Introduction

The Hermite polynomials form a Sheffer sequence and are given by the generating function

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{H_n\left(x\right)}{n!}t^n,\quad \left(\text{see [1–8, 10, 13, 14]}\right). \tag{1.1}$$

By using Taylor series, we get

$$\begin{aligned} H_n\left(x\right)&=\left[\left(\frac{\partial}{\partial t}\right)^n e^{\left(2xt-t^2\right)}\right]_{t=0}\\ &=\left[e^{x^2}\left(\frac{\partial}{\partial t}\right)^n e^{-(x-t)^2}\right]_{t=0}\\ &=\left(-1\right)^n e^{x^2}\left[\left(\frac{\partial}{\partial x}\right)^n e^{-(x-t)^2}\right]_{t=0}\\ &=\left(-1\right)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2},\quad \left(n\geq 0\right),\quad \left(\text{see [1–15, 18]}\right). \end{aligned}$$

The Hermite polynomials can be represented by the Contour integral as follows:

$$H_n\left(z\right)=\frac{n!}{2\pi i}\oint e^{-t^2+2tz}t^{-n-1}dt, \tag{1.2}$$

where the Contour encloses the origin and is traversed in a counterclockwise direction (see [2, 8, 11, 13]).

The probabilists' Hermite polynomials are given by the generating function

$$\begin{aligned} H_n^*\left(x\right)&=\left(-1\right)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}\\ &=\left(x-\frac{d}{dx}\right)^n\cdot 1,\quad \left(\text{see [10]}\right). \end{aligned} \tag{1.3}$$

The physicists' Hermite polynomials are also given by

$$\begin{aligned} H_n\left(x\right)&=\left(-1\right)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\\ &=\left(2x-\frac{d}{dx}\right)^n\cdot 1\quad \left(\text{see [20]}\right). \end{aligned} \tag{1.4}$$

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Thus, by (1.3) and (1.4), we get

$$H_n(x) = 2^{\frac{n}{2}} H_n^*\left(\sqrt{2}x\right), \quad H_n^*(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right), \tag{1.5}$$

where $n \geq 0$ (see [9, 11, 12, 15, 18]).

The first several Hermite polynomials are $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2-2$, $H_3(x) = 8x^2-12x$, $H_4(x) = 16x^4-48x^2+12$, $H_5(x) = 32x^5-160x^3+120x$, $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$, ...

The probabilists' Hermite polynomials are solutions of the differential equation:

$$\left(e^{-\frac{x^2}{2}} u'\right)' + \lambda e^{-\frac{1}{2}x^2} u = 0,$$

where λ is a constant, with the boundary conditions that u should be polynomially bounded at infinity.

The generating function of the probabilists' Hermite polynomials is given by

$$e^{xt-\frac{t^2}{2}} = \sum_{n=0}^{\infty} H_n^*(x) \frac{t^n}{n!}, \quad (\text{see } [12, 15, 18]). \tag{1.6}$$

The Hermite polynomials $H_n^{(\nu)}(x)$ of variance ν form an Appell sequence and are defined by the generating function

$$\sum_{k=0}^{\infty} \frac{H_k^{(\nu)}(x)}{k!} t^k = e^{xt-\frac{\nu t^2}{2}}, \quad (\text{see } [12]). \tag{1.7}$$

Thus, by (1.7), we get

$$x^{2m+1} = \sum_{l=0}^{m} \binom{2m+1}{2l+1} \frac{(2m-2l)!}{(m-l)!} \left(\frac{\nu}{2}\right)^{m-l} H_{2l+1}^{(\nu)}(x), \tag{1.8}$$

and

$$x^{2m} = \sum_{l=0}^{m} \binom{2m}{2l} \frac{(2m-2l)!}{(m-l)!} \left(\frac{\nu}{2}\right)^{m-l} H_{2l}^{(\nu)}(x), \quad (\text{see } [12]). \tag{1.9}$$

The Hermite polynomials have been studied in probability, combinatorics, numerical analysis, finite element methods, physics and system theory (see [1–15, 18]).

Recently, Kim has studied nonlinear differential equations arising from Frobenius-Euler numbers and polynomials.

In this paper, we consider linear differential equations arising from Hermite polynomials of variance ν and give some new and explicit identities for those polynomials.

2. Hermite polynomials of variance ν

Let

$$F = F(t : x, \nu) = e^{xt-\frac{\nu t^2}{2}}. \tag{2.1}$$

From (2.1), we note that

$$\begin{aligned} F^{(1)} &= \frac{d}{dt} F(t : x, \nu) \\ &= (x - \nu t) e^{xt-\frac{\nu t^2}{2}} \\ &= (x - \nu t) F, \end{aligned} \tag{2.2}$$

$$F^{(2)} = \frac{d}{dt}F^{(1)} = \left(-\nu + (x - t\nu)^2\right)F, \tag{2.3}$$

$$F^{(3)} = \frac{d}{dt}F^{(2)} = \left(-3\nu(x-\nu t) + (x-\nu t)^3\right)F, \tag{2.4}$$

and

$$F^{(4)} = \frac{d}{dt}F^{(3)} = \left(3\nu^2 - 6\nu(x-\nu t)^2 + (x-\nu t)^4\right)F. \tag{2.5}$$

Continuing this process, we set

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t:x,\nu) \\ &= \left(\sum_{i=0}^{N} a_i(N,\nu)(x-\nu t)^i\right)F, \end{aligned} \tag{2.6}$$

where $N \in \mathbb{N} \cup \{0\}$.

From (2.6), we have

$$\begin{aligned} F^{(N+1)} &= \frac{d}{dt}F^{(N)} \\ &= \sum_{i=0}^{N} a_i(N,\nu)\, i\,(x-\nu t)^{i-1}(-\nu)F \\ &\quad + \sum_{i=0}^{N} a_i(N,\nu)(x-\nu t)^i F^{(1)}. \end{aligned} \tag{2.7}$$

By (2.2) and (2.7), we easily get

$$\begin{aligned} F^{(N+1)} = \Big\{ &-\nu a_1(N,\nu) + a_N(N,\nu)(x-\nu t)^{N+1} + a_{N-1}(N,\nu)(x-\nu t)^N \\ &+ \sum_{i=1}^{N-1}\left(-(i+1)\nu a_{i+1}(N,\nu) + a_{i-1}(N,\nu)\right)(x-\nu t)^i \Big\} F. \end{aligned} \tag{2.8}$$

By replacing N by $(N+1)$ in (2.6), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1,\nu)(x-\nu t)^i\right)F. \tag{2.9}$$

From (2.8) and (2.9), we can derive the following equations:

$$a_0(N+1,\nu) = -\nu a_1(N,\nu), \tag{2.10}$$

$$a_N(N+1,\nu) = a_{N-1}(N,\nu), \tag{2.11}$$

$$a_{N+1}(N+1,\nu) = a_N(N,\nu) \tag{2.12}$$

and

$$a_i(N+1,\nu) = -(i+1)\nu a_{i+1}(N,\nu) + a_{i-1}(N,\nu), \tag{2.13}$$

where $1 \le i \le N-1$.

It is not difficult to show that

$$F = F^{(0)} = a_0(0,\nu)F. \tag{2.14}$$

Thus, by (2.14), we get

$$a_0(0,\nu) = 1. \tag{2.15}$$

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From (2.2) and (2.6), we note that

$$(x-\nu t)\,F=F^{(1)}=\left(a_0\left(1,\nu\right)+a_1\left(1,\nu\right)\left(x-\nu t\right)\right)F. \tag{2.16}$$

Thus, by comparing the coefficients on both sides of (2.16), we get

$$a_0\left(1,\nu\right)=0,\quad a_1\left(1,\nu\right)=1. \tag{2.17}$$

From (2.11), (2.12), (2.15) and (2.17), we have

$$a_N\left(N+1,\nu\right)=a_{N-1}\left(N,\nu\right)=\cdots=a_0\left(1,\nu\right)=0, \tag{2.18}$$

and

$$a_{N+1}\left(N+1,\nu\right)=a_N\left(N,\nu\right)=\cdots=a_1\left(1,\nu\right)=1. \tag{2.19}$$

Therefore, we obtain the following theorem.

Theorem 1. *The linear differential equations*

$$\begin{aligned}F^{(N)}&=\left(\frac{d}{dt}\right)^N F\left(t:x,\nu\right)\\&=\left(\sum_{i=0}^{N}a_i\left(N,\nu\right)\left(x-\nu t\right)^i\right)F,\quad\left(N\in\mathbb{N}\cup\{0\}\right)\end{aligned}$$

has a solution $F=F\left(t:x,\nu\right)=e^{xt-\frac{\nu t^2}{2}}$, *where*

$$\begin{aligned}a_0\left(N,\nu\right)&=-\nu a_1\left(N-1,\nu\right),\\a_{N-1}\left(N,\nu\right)&=a_{N-2}\left(N-1,\nu\right)=\cdots=a_1\left(2,\nu\right)=a_0\left(1,\nu\right)=0,\\a_N\left(N,\nu\right)&=a_{N-1}\left(N-1,\nu\right)=\cdots=a_1\left(1,\nu\right)=a_0\left(0,\nu\right)=1,\end{aligned}$$

and

$$a_i\left(N,\nu\right)=-\left(i+1\right)\nu a_{i+1}\left(N-1,\nu\right)+a_{i-1}\left(N-1,\nu\right),\quad\left(1\le i\le N-2\right).$$

Example.

(1) $N=3$, $i=1$. By (2.13), we get

$$\begin{aligned}a_1\left(3,\nu\right)&=-2\nu a_2\left(2,\nu\right)+a_0\left(2,\nu\right)\\&=-2\nu-\nu=-3\nu.\end{aligned}$$

(2) $N=4$, $1\le i\le 2$. By (2.13), we have

$$a_1\left(4,\nu\right)=0,\quad a_2\left(4,\nu\right)=-6\nu.$$

(3) $N=5$, $1\le i\le 3$. By (2.13), we get

$$a_1\left(5,\nu\right)=15\nu^2,\quad a_2\left(5,\nu\right)=0,\quad a_3\left(5,\nu\right)=-10\nu.$$

(4) $N=6$, $1\le i\le 4$. From (2.13), we have

$$a_1\left(6,\nu\right)=0,\quad a_2\left(6,\nu\right)=45\nu^2,\quad a_3\left(6,\nu\right)=0,\quad a_4\left(6,\nu\right)=-15\nu.$$

Thus, we obtain the following result.

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Remark. The matrix $\left(a_i\left(j,\nu\right)\right)_{0\le i,j\le 6}$ is given by

$$\begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array} & \left[\begin{array}{ccccccc} 1 & 0 & -\nu & 0 & 3\nu^2 & 0 & -15\nu^3 \\ & 1 & 0 & -3\nu & 0 & 15\nu^2 & 0 \\ & & 1 & 0 & -6\nu & 0 & 45\nu^2 \\ & & & 1 & 0 & -10\nu & 0 \\ & & & & 1 & 0 & -15\nu \\ & & 0 & & & 1 & 0 \\ & & & & & & 1 \end{array}\right] \end{array}.$$

From (1.7), we note that

$$F = F\left(t:x,\nu\right) = e^{xt-\frac{\nu t^2}{2}} = \sum_{k=0}^{\infty} H_k^{(\nu)}\left(x\right)\frac{t^k}{k!}. \tag{2.20}$$

Thus, by (2.20), we get

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F\left(t:x,\nu\right) \\ &= \sum_{k=N}^{\infty} H_k^{(\nu)}\left(x\right)\left(k\right)_N \frac{t^{k-N}}{k!} \\ &= \sum_{k=0}^{\infty} H_{k+N}^{(\nu)}\left(x\right)\left(k+N\right)_N \frac{t^k}{(n+k)!} \\ &= \sum_{k=0}^{\infty} H_{k+N}^{(\nu)}\left(x\right)\frac{t^k}{k!}. \end{aligned} \tag{2.21}$$

By Theorem 1, we easily get

$$\begin{aligned} &F^{(N)} \\ &= \left(\sum_{i=0}^{N} a_i\left(N,\nu\right)\left(x-\nu t\right)^i\right)F \\ &= \sum_{i=0}^{N} a_i\left(N,\nu\right)\sum_{m=0}^{\infty}\left(i\right)_m x^{i-m}\left(-\nu\right)^m\frac{t^m}{m!}\sum_{l=0}^{\infty}H_l^{(\nu)}\left(x\right)\frac{t^l}{l!} \\ &= \sum_{k=0}^{\infty}\left\{\sum_{i=0}^{N} a_i\left(N,\nu\right)\sum_{l=0}^{k}\binom{k}{l}\left(i\right)_{k-l}\left(-\nu\right)^{k-l}x^{i+l-k}H_l^{(\nu)}\left(x\right)\right\}\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left\{\sum_{i=0}^{N} a_i\left(N,\nu\right)\sum_{l=\max\{0,k-i\}}^{k}\binom{k}{l}\left(i\right)_{k-l}\left(-\nu\right)^{k-l}x^{i+l-k}H_l^{(\nu)}\left(x\right)\right\}\frac{t^k}{k!}. \end{aligned} \tag{2.22}$$

Therefore, by (2.21) and (2.22), we obtain the following theorem.

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Theorem 2. *For $k, N \in \mathbb{N} \cup \{0\}$, we have*

$$H_{k+N}^{(\nu)}(x) = \sum_{i=0}^{N} a_i(N,\nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l} (-\nu)^{k-l} x^{i+l-k} H_l^{(\nu)}(x).$$

It is easy to show that

$$H_{k+1}^{(\nu)}(x) = \left(x - \nu \frac{\partial}{\partial x}\right) H_k^{(\nu)}(x). \tag{2.23}$$

Thus, by (2.23), we have

$$H_{k+N}^{(\nu)}(x) = \left(x - \nu \frac{\partial}{\partial x}\right)^N H_k^{(\nu)}(x), \quad (N \in \mathbb{N} \cup \{0\}). \tag{2.24}$$

From Theorem 2, we note that

$$\begin{aligned} &\left(x - \nu \frac{\partial}{\partial x}\right)^N H_k^{(\nu)}(x) \\ &= \sum_{i=0}^{N} a_i(N,\nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l} (-\nu)^{k-l} x^{i+l-k} H_l^{(\nu)}(x), \end{aligned} \tag{2.25}$$

where $\frac{\partial}{\partial x} x - x \frac{\partial}{\partial x} = \text{identity}$.

Now, we observe explicit determination of $a_i(j,\nu)$.

From (2.12) and (2.13), we can derive the following equations:

$$a_N(N,\nu) = 1, \tag{2.26}$$

$$\begin{aligned} a_{N-2}(N,\nu) &= -(N-1)\nu a_{N-1}(N-1,\nu) + a_{N-3}(N-1,\nu) \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad + a_{N-4}(N-2,\nu) \\ &\vdots \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad - \cdots - 2\nu a_2(2,\nu) + a_0(2,\nu) \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad - \cdots - 2\nu a_2(2,\nu) - \nu a_1(1,\nu) \\ &= -\nu \sum_{i=1}^{N-1} i a_i(i,\nu), \end{aligned} \tag{2.27}$$

$$\begin{aligned} a_{N-4}(N,\nu) &= -(N-3)\nu a_{N-3}(N-1,\nu) + a_{N-5}(N-1,\nu) \\ &= -(N-3)\nu a_{N-3}(N-1,\nu) - (N-4)\nu a_{N-4}(N-2,\nu) \\ &\quad + a_{N-6}(N-2,\nu) \\ &\vdots \\ &= -(N-3)\nu a_{N-3}(N-1,\nu) - (N-4)\nu a_{N-4}(N-2,\nu) \end{aligned} \tag{2.28}$$

$$\begin{aligned}&-\cdots-2\nu a_2(4,\nu)+a_0(4,\nu)\\&=-(N-3)\nu a_{N-3}(N-1,\nu)-(N-4)\nu a_{N-4}(N-2,\nu)\\&\quad-\cdots-2\nu a_2(4,\nu)-\nu a_1(3,\nu)\\&=-\nu\sum_{i=0}^{N-3} i a_i(i+2,\nu),\end{aligned}$$

and

$$\begin{aligned}a_{N-6}(N,\nu)&=-(N-5)\nu a_{N-5}(N-1,\nu)+a_{N-7}(N-1,\nu)\\&=-(N-5)\nu a_{N-5}(N-1,\nu)-(N-6)\nu a_{N-6}(N-2,\nu)\\&\quad+a_{N-8}(N-2,\nu)\\&\;\vdots\\&=-(N-5)\nu a_{N-5}(N-1,\nu)-(N-6)\nu a_{N-6}(N-2,\nu)\\&\quad-\cdots-2\nu a_2(6,\nu)-\nu a_1(5,\nu)\\&=-\nu\sum_{i=1}^{N-5} i a_i(i+4,\nu).\end{aligned}\tag{2.29}$$

Continuing in this fashion, for l with $1\leq l\leq\left[\frac{N-1}{2}\right]$,

$$a_{N-2l}(N,\nu)=-\nu\sum_{i=1}^{N-2l+1} i a_i(i+2l-2,\nu).\tag{2.30}$$

By (2.26), (2.27), (2.28), (2.29) and (2.30), we get

$$a_{N-2}(N,\nu)=-\nu\sum_{i_1=1}^{N-1} i_1,\tag{2.31}$$

$$\begin{aligned}a_{N-4}(N,\nu)&=-\nu\sum_{i_2=1}^{N-3} i_2 a_{i_2}(i_2+2,\nu)\\&=(-\nu)^2\sum_{i_2=1}^{N-3}\sum_{i_1=1}^{i_2+1} i_2 i_1,\end{aligned}\tag{2.32}$$

$$\begin{aligned}a_{N-6}(N,\nu)&=-\nu\sum_{i_3=1}^{N-5} i_3 a_{i_3}(i_3+4,\nu)\\&=(-\nu)^3\sum_{i_3=1}^{N-5}\sum_{i_2=1}^{i_3+1}\sum_{i_1=1}^{i_2+1} i_3 i_2 i_1,\end{aligned}\tag{2.33}$$

and

$$a_{N-2l}(N,\nu)=(-\nu)^l\sum_{i_l=1}^{N-2l+1}\sum_{i_{l-1}=1}^{i_l+1}\cdots\sum_{i_1=1}^{i_2+1} i_l\cdot i_{l-1}\cdots i_1,\tag{2.34}$$

where $1\leq l\leq\left[\frac{N-1}{2}\right]$.

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By (2.11) and (2.13), we easily get

(2.35)
$$a_{N-1}(N,\nu)=a_{N-2}(N-1,\nu)=a_{N-3}(N-2,\nu)=\cdots=a_0(1,\nu)=0,$$

(2.36)
$$\begin{aligned}a_{N-3}(N,\nu)&=-(N-2)\nu a_{N-2}(N-1,\nu)+a_{N-4}(N-1,\nu)\\&=a_{N-4}(N-1,\nu)\\&\vdots\\&=a_0(3,\nu)=-\nu a_1(2,\nu)=-\nu a_0(1,\nu)=0,\end{aligned}$$

(2.37)
$$\begin{aligned}a_{N-5}(N,\nu)&=-(N-4)\nu a_{N-4}(N-1,\nu)+a_{N-6}(N-1,\nu)=a_{N-6}(N-1,\nu)\\&\vdots\\&=a_0(5,\nu)=-\nu a_1(4,\nu)=0,\end{aligned}$$

(2.38)
$$\begin{aligned}a_{N-7}(N,\nu)&=-(N-6)\nu a_{N-6}(N-1,\nu)+a_{N-8}(N-1,\nu)\\&\vdots\\&=a_0(7,\nu)=-\nu a_1(6,\nu)=0,\end{aligned}$$

and

$$a_{N-(2l-1)}(N,\nu)=0,\quad\left(1\le l\le\left[\frac{N}{2}\right]\right). \tag{2.39}$$

Therefore, we obtain the following theorem.

Theorem 3. *For $N\in\mathbb{N}\cup\{0\}$, we have*

$$a_{N-2l}(N,\nu)=(-\nu)^l\sum_{i_l=1}^{N-2l+1}\sum_{i_{l-1}=1}^{i_l+1}\cdots\sum_{i_1=1}^{i_2+1}i_li_{l-1}\cdots i_1,$$

where $1\le l\le\left[\frac{N-1}{2}\right]$.

Also,

$$a_{N-(2l-1)}(N,\nu)=0,\quad \textit{if } 1\le l\le\left[\frac{N}{2}\right].$$

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On new λ^2-convergent difference BK-spaces

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In this paper, we introduce the spaces $c(\lambda^2, \Delta)$ and $c_0(\lambda^2, \Delta)$, which are BK-spaces of non-absolute type and we prove that these spaces are linearly isomorphic to the spaces c and c_0, respectively. Moreover, we give some inclusion relations and compute the $\alpha-$, $\beta-$ and $\gamma-$duals of these spaces. We also determine the Schauder basis of the $c(\lambda^2, \Delta)$ and $c_0(\lambda^2, \Delta)$. Lastly we give some matrix transformations between of these spaces and others.

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Key words: λ^2-convergence, BK-spaces, $\alpha-, \beta-$ and $\gamma-$ duals, matrix mappings, difference sequence spaces

1 Introduction

A sequence space is defined to be a linear space of real or complex sequences. Let w denote the spaces of all complex sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$.

Let X be a sequence space. If X is a Banach space and

$$\tau_k : X \to C, \quad \tau_k(x) = x_k \quad (k = 1, 2, ...)$$

is a continuous for all k, X is called a $BK-$space.

We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively, which are $BK-$spaces with the norm given by $\|x\|_\infty = \sup_k |x_k|$ for all $k \in \mathbb{N}$.

For a sequence space X, the matrix domain X_A of an infinite matrix A defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1}$$

which is a sequence space. We denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

M. Mursaleen and A. K. Noman [9] introduced the sequence spaces ℓ_∞^λ, c^λ and c_0^λ as the sets of all $\lambda-$ *bounded*, $\lambda-$ *convergent* and $\lambda-$ *null* sequences as follows;

$$\begin{aligned} \ell_\infty^\lambda &= \{x \in w : \sup_n |\Lambda_n(x)| < \infty\} \\ c^\lambda &= \{x \in w : \lim_{n\to\infty} \Lambda_n(x) \ exists\} \\ c_0^\lambda &= \{x \in w : \lim_{n\to\infty} \Lambda_n(x) = 0\} \end{aligned}$$

where $\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k,\ k \in \mathbb{N}$. Also they generalized c^λ and c_0^λ spaces defining $c^\lambda(\Delta)$, $c_0^\lambda(\Delta)$ spaces using the difference operator. They studied some properties of these spaces in [8]. N. L. Braha and F. Başar introduced the infinite matrix $A(\lambda) = \{a_{nk}(\lambda)\}_{n,k=0}^{\infty}$ such as;

$$a_{nk}(\lambda) = \begin{cases} \frac{\Delta^2 \lambda_k}{\Delta \lambda_k}, & 0 \le k \le n; \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$ and they defined $A_\lambda(\ell_\infty)$, $A_\lambda(c)$ and $A_\lambda(c_0)$ spaces in [11] as follows;

$$\begin{aligned} A_\lambda(\ell_\infty) &= \left\{x \in w : \sup_n |(A_\lambda x)_n| < \infty\right\}, \\ A_\lambda(c) &= \left\{x \in w : \exists l \in \mathbb{C} \ni \lim_n (A_\lambda x)_n = l\right\}, \\ A_\lambda(c_0) &= \left\{x \in w : \lim_n (A_\lambda x)_n = 0\right\} \end{aligned}$$

where $(A_\lambda x)_n = \frac{1}{\Delta\lambda_n} \sum_{k=0}^{n} (\Delta^2 \lambda_k) x_k$. They examined some properties of these spaces. In literature, some authors have constructed new sequence spaces by using matrix domain of infinite matrix and have introduced some topological properties. (see [2], [4], [12])

2 The sequence spaces $c(\lambda^2,\Delta)$ and $c_0(\lambda^2,\Delta)$

In this section, we define the sequence spaces $c(\lambda^2,\Delta)$ and $c_0(\lambda^2,\Delta)$ as follows;

$$c(\lambda^2,\Delta)=\left\{x\in w:\lim_{n\to\infty}\Lambda_n^2(x)\ exists\right\}$$

$$c_0(\lambda^2,\Delta)=\left\{x\in w:\lim_{n\to\infty}\Lambda_n^2(x)=0\right\}$$

where $\Lambda_n^2(x)=\frac{1}{\Delta\lambda_n}\sum_{k=0}^{n}\left(\Delta^2\lambda_k\right)(x_k-x_{k-1})$ for all $k,n\in\mathbb{N}$. Δ denotes the difference operator. i.e., $\Delta^0\lambda_n=\lambda_n$, $\Delta\lambda_n=\lambda_n-\lambda_{n-1}$, $\Delta^2\lambda_n=\lambda_n-2\lambda_{n-1}+\lambda_{n-2}$ and $\Delta x_k=x_k-x_{k-1}$. $\lambda=(\lambda_k)_{k=0}^{\infty}$ is a strictly increasing sequence of positive reals tending to infinity, that is $0<\lambda_0<\lambda_1<...$and $\lambda_k\to\infty$ as $k\to\infty$ and $\lambda_{n+1}\geq 2\lambda_n$ for all $n\in\mathbb{N}$. Here and in sequel, we use the convention that any term with a negative subscript is equal to naught. e.g. $\lambda_{-1}=\lambda_{-2}=0$ and $x_{-1}=0$. On the other hand, we define the matrix $\Lambda^2=\left(\lambda_{nk}^2\right)$ for all $k,n\in\mathbb{N}$ by

$$\lambda_{nk}^2=\begin{cases}\frac{\Delta^2(\lambda_k-\lambda_{k+1})}{\Delta\lambda_k}; & k<n,\\ \frac{\Delta^2\lambda_n}{\Delta\lambda_n}; & n=k,\\ 0; & n>k.\end{cases}\tag{2}$$

The equality can be eaisly seen from

$$\Lambda_n^2(x)=\frac{1}{\Delta\lambda_n}\sum_{k=0}^{n}\left(\Delta^2\lambda_k\right)(x_k-x_{k-1})\tag{3}$$

for all $m,n\in\mathbb{N}$ and every $x=(x_k)\in w$. Then it leads us together with (1) to the fact that

$$c_0\left(\lambda^2,\Delta\right)=(c_0)_{\Lambda^2}\ \text{ and } c\left(\lambda^2,\Delta\right)=(c)_{\Lambda^2}\,.\tag{4}$$

The matrix $\Lambda^2=\lambda_{nk}^2$ is a triangle, i.e., $\lambda_{nn}^2\neq 0$ and $\lambda_{nk}^2=0$ $(k>n)$ for all $n,k\in\mathbb{N}$. Further, for any sequence $x=(x_k)$ we define the sequence $y\left(\lambda^2\right)=\left\{y_k\left(\lambda^2\right)\right\}$ as the Λ^2-transform of x, i.e., $y\left(\lambda^2\right)=\Lambda^2(x)$ and so we have that

$$y_k\left(\lambda^2\right)=\sum_{j=0}^{k-1}\frac{\Delta^2\left(\lambda_j-\lambda_{j+1}\right)}{\Delta\lambda_k}x_j+\frac{\Delta^2\lambda_k}{\Delta\lambda_k}x_k\tag{5}$$

for $k\in\mathbb{N}$. Here and in what follows, the summation running from 0 to $k-1$ is equal to zero when $k=0$. Also it can be written from (3) with (5) for $k\in\mathbb{N}$ such as;

$$y_k\left(\lambda^2\right)=\sum_{j=0}^{k}\frac{\Delta^2\lambda_j}{\Delta\lambda_k}\left(x_j-x_{j-1}\right).$$

Theorem 1 *$c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ are BK-spaces with the norm*

$$\|x\|_{(c_0)_{\Lambda^2}}=\|x\|_{(c)_{\Lambda^2}}=\sup_n\left|\Lambda_n^2(x)\right|.$$

Proof. We know that c and c_0 are $BK-$spaces with their natural norms from [6]. (4) holds and $\Lambda^2=\lambda_{nk}^2$ is a triangle matrix and from Theorem 4.3.12 of Wilansky [1], we derive that $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ are $BK-$spaces. This completes the proof. ■

Remark 2 *The absolute property does not hold on the $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ spaces. For instance, if we take $|x|=(|x_k|)$ we hold $\|x\|_{(c)_{\Lambda^2}}\neq\||x|\|_{(c)_{\Lambda^2}}$ .Thus, the space $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ are BK-space of non-absolute type.*

Theorem 3 *The sequence spaces $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ of non-absolute type are linearly isomorphic to the spaces c_0 and c, respectively, that is $c_0(\lambda^2,\Delta)\cong c_0$ and $c(\lambda^2,\Delta)\cong c$.*

Proof: We only consider $c_0(\lambda^2,\Delta)\cong c_0$ and others will prove similarly. To prove the theorem we must show the existence of linear bijection operator between $c_0(\lambda^2,\Delta)$ and c_0. Hence, let define the linear operator with the notation (5), from $c_0(\lambda^2,\Delta)$ and c_0 by $x\to y\left(\lambda^2\right)=Tx$.

Then $Tx=y\left(\lambda^2\right)=\Lambda^2\left(x\right)\in c_0$ for every $x\in c_0(\lambda^2,\Delta)$. Also, the linearity of T is clear. Further, it is trivial that $x=0$ whenever $Tx=0$. Hence T is injective.

Let $y=(y_k)\in c_0$ and define the sequence $x=\left\{x\left(\lambda^2\right)\right\}$ by

$$x_k\left(\lambda^2\right)=\sum_{j=0}^{k}\sum_{i=j-1}^{j}(-1)^{j-i}\frac{\Delta\lambda_i}{\Delta^2\lambda_j}y_i. \tag{6}$$

and we have

$$x_k\left(\lambda^2\right)-x_{k-1}\left(\lambda^2\right)=\sum_{i=k-1}^{k}(-1)^{k-i}\frac{\Delta\lambda_i}{\Delta^2\lambda_k}y_i.$$

Thus, for every $k\in\mathbb{N}$, we have by (5) that

$$\Lambda_n^2(x)=\frac{1}{\Delta\lambda_n}\sum_{k=0}^{n}\left[\Delta\left(\lambda_k y_k-\lambda_{k-1}y_{k-1}\right)\right]=y_n$$

This shows that $\Lambda^2(x)=y$ and since $y\in c_0$, we obtain that $\Lambda^2(x)\in c_0$. Thus we deduce that $x\in c_0(\lambda^2,\Delta)$ and $Tx=y$. Hence T is surjective.

Further, we have for every $x\in c_0(\lambda^2,\Delta)$ that

$$\|Tx\|_{c_0}=\|Tx\|_{\ell_\infty}=\left\|y\left(\lambda^2\right)\right\|_{\ell_\infty}=\left\|\Lambda^2(x)\right\|_{\ell_\infty}=\|x\|_{(c_0)_{\Lambda^2}}$$

which means that $c_0(\lambda^2,\Delta)$ and c_0 are linearly isomorphic.

3 Some inclusion relations

Theorem 4 *The inclusion $c_0\left(\lambda^2,\Delta\right)\subset c\left(\lambda^2,\Delta\right)$ strictly holds.*

Proof. $c_0\left(\lambda^2,\Delta\right)\subset c\left(\lambda^2,\Delta\right)$ is clear. To show strict, consider the sequence $x=(x_k)$ defined by $x_k=k+1$ for all $k\in\mathbb{N}$. Then we obtain that

$$\Lambda_n^2(x)=\frac{1}{\Delta\lambda_n}\sum_{k=0}^{n}\left(\Delta^2\lambda_k\right)\left(x_k-x_{k-1}\right)=1;\quad(n\in\mathbb{N})$$

for $n\in\mathbb{N}$ which shows that $\Lambda^2(x)\in c-c_0$. Thus, the sequence x is in $c\left(\lambda^2,\Delta\right)$ but not in $c_0\left(\lambda^2,\Delta\right)$. Hence the inclusion $c_0\left(\lambda^2,\Delta\right)\subset c\left(\lambda^2,\Delta\right)$ is strict and this completes the proof. ■

Theorem 5 *The inclusion $c\subset c_0\left(\lambda^2,\Delta\right)$ strictly holds.*

Proof. Let $x\in c$. Then, $\Lambda^2(x)\in c_0$. This shows that $x\in c_0\left(\lambda^2,\Delta\right)$. Hence, the inclusion $c\subset c_0\left(\lambda^2,\Delta\right)$ holds. Then, consider the sequence $y=(y_k)$ defined by $y_k=\sqrt{k+1}$ for $k\in\mathbb{N}$. It is trivial that $y\notin c$. On the other hand, it can easily be seen that $\Lambda^2(y)\in c_0$ and $y\in c_0\left(\lambda^2,\Delta\right)$.Consequently, the sequence y is in $c_0\left(\lambda^2,\Delta\right)$ but not in c. We therefore deduce that the inclusion $c\subset c_0\left(\lambda^2,\Delta\right)$ is strict. ■

Corollary 6 *$c_0\subset c_0\left(\lambda^2,\Delta\right)$ and $c\subset c\left(\lambda^2,\Delta\right)$ strictly hold.*

Theorem 7 *Although the spaces ℓ_∞ and $c_0\left(\lambda^2,\Delta\right)$ overlap, the space ℓ_∞ does not include the space $c_0\left(\lambda^2,\Delta\right)$.*

Proof. It can be seen from the sequence y, which was defined in Theorem 5, is in $c_0\left(\lambda^2,\Delta\right)$ but not in ℓ_∞. ■

Lemma 8 *$A \in (\ell_\infty : c_0)$ if and only if* $\lim_n \sum_k |a_{nk}| = 0.$

Theorem 9 *The inclusion $\ell_\infty \subset c_0\left(\lambda^2, \Delta\right)$ strictly holds if and only if $z \in A_\lambda\left(c_0\right)$ where the sequence $z = (z_k)$ is defined by*

$$z_k = \left|1 - \frac{\Delta^2 \lambda_{k+1}}{\Delta^2 \lambda_{k-1}}\right|; \quad (k \in \mathbb{N}).$$

Proof. Let $\ell_\infty \subset c_0\left(\lambda^2, \Delta\right)$. Then, we obtain that $\Lambda^2(x) \in c_0$ for every $x \in \ell_\infty$ and the matrix $\Lambda^2 = \left(\lambda^2_{nk}\right)$ is in the class $(\ell_\infty : c_0)$. It follows by Lemma 8

$$\lim_n \sum_k \left|\lambda^2_{nk}\right| = 0. \tag{7}$$

From definition of $\Lambda^2 = \left(\lambda^2_{nk}\right)$ given in (2) we have

$$\sum_k \left|\lambda^2_{nk}\right| = \frac{1}{\Delta\lambda_n} \sum_{k=0}^{n-1} \left|\left(\Delta^2 \lambda_k - \Delta^2 \lambda_{k-1}\right)\right| + \frac{\Delta^2 \lambda_n}{\Delta \lambda_n}. \tag{8}$$

From (7)

$$\lim_n \frac{\Delta^2 \lambda_n}{\Delta \lambda_n} = 0 \tag{9}$$

and

$$\lim_n \frac{1}{\Delta\lambda_n} \sum_{k=0}^{n-1} \left|\Delta^2\left(\lambda_k - \lambda_{k+1}\right)\right| = 0. \tag{10}$$

We have

$$\frac{1}{\Delta\lambda_n} \sum_{k=0}^{n-1} \left|\Delta^2\left(\lambda_k - \lambda_{k+1}\right)\right| = \frac{\Delta\lambda_{n-1}}{\Delta\lambda_n}\left[\frac{1}{\Delta\lambda_{n-1}} \sum_{k=0}^{n-1} \left(\Delta^2 \lambda_k\right) z_k\right]$$

and since $\lim_n \frac{\Delta\lambda_{n-1}}{\Delta\lambda_n} = 1$ by (9); we have from (10) that

$$\lim_n \frac{1}{\Delta\lambda_n} \sum_{k=0}^{n-1} \left(\Delta^2 \lambda_k\right) z_k = 0 \tag{11}$$

which shows that $z = (z_k) \in A_\lambda(c_0)$. ∎

Conversely, let $z = (z_k) \in A_\lambda(c_0)$. Then we have that (11) holds. Also we obtain that

$$\begin{aligned}
\frac{1}{\Delta\lambda_n} \sum_{k=0}^{n} \left|\Delta^2\left(\lambda_k - \lambda_{k+1}\right)\right| &= \frac{1}{\Delta\lambda_n} \sum_{k=0}^{n-1} \Delta^2 \lambda_k z_k \\
&\leq \frac{1}{\Delta\lambda_{n-1}} \sum_{k=0}^{n-1} \Delta^2 \lambda_k z_k.
\end{aligned}$$

This and (11) provides (10). On the other hand, we have that

$$\begin{aligned}
\left|\frac{\Delta^2 \lambda_n - \lambda_0}{\Delta\lambda_n}\right| &= \left|\frac{2\lambda_{n-1} - \left(\lambda_n + \lambda_{n-2} - \lambda_0\right)}{\Delta\lambda_n}\right| \\
&= \left|\frac{1}{\Delta\lambda_n} \sum_{k=0}^{n-1} \Delta^2\left(\lambda_k - \lambda_{k+1}\right)\right| \\
&\leq \frac{1}{\Delta\lambda_n} \sum_{k=0}^{n-1} \left|\Delta^2\left(\lambda_k - \lambda_{k+1}\right)\right|.
\end{aligned}$$

From (10), we derive that

$$\lim_n \frac{\Delta^2 \lambda_n}{\Delta\lambda_n} = \lim_n \frac{\Delta^2 \lambda_n - \lambda_0}{\Delta\lambda_n} = 0.$$

This provides (9). Hence, we obtain from (8) that (7) holds. From Lemma 8 $\Lambda^2 \in (\ell_\infty : c_0)$. Hence, the inclusion $\ell_\infty \subset c_0\left(\lambda^2, \Delta\right)$ holds. This inclusion is strict from Theorem 7. The proof is completed.

Corollary 10 *If* $\lim_n \frac{\Delta^2\lambda_{n+1}}{\Delta^2\lambda_n} = 1$, *then the inclusion* $\ell_\infty \subset c_0\left(\lambda^2, \Delta\right)$ *is strict.*

4 The bases for the spaces $c\left(\lambda^2, \Delta\right)$ and $c_0\left(\lambda^2, \Delta\right)$

If a normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence (α_n) of scalars such that

$$\lim_n \|x - (\alpha_0 b_0 + \alpha_1 b_1 + ... + \alpha_n b_n)\| = 0.$$

Then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$.

Theorem 11 *Define the sequence* $b^{(k)}\left(\lambda^2\right) \in c_0\left(\lambda^2, \Delta\right)$ *for every fixed* $k \in \mathbb{N}$ *and by*

$$b_n^{(k)}(\lambda^2) = \begin{cases} \frac{\Delta\lambda_k}{\Delta^2\lambda_k} - \frac{\Delta\lambda_k}{\Delta^2\lambda_{k+1}}; & n > k, \\ \frac{\Delta\lambda_k}{\Delta^2\lambda_k}; & n = k, \\ 0; & n < k. \end{cases}$$

(i) The sequence $\left\{b_n^{(k)}\left(\lambda^2\right)\right\}_{k=0}^{\infty}$ is a Schauder basis for the space $c_0\left(\lambda^2, \Delta\right)$ and every $x \in c_0\left(\lambda^2, \Delta\right)$ has a unique representation of the form

$$x = \sum_k \alpha_k\left(\lambda^2\right) b^{(k)}\left(\lambda^2\right)$$

(ii) The sequence $\left\{b, b_n^{(0)}\left(\lambda^2\right), b_n^{(1)}\left(\lambda^2\right), ...\right\}$ is a Schauder basis for the space $c\left(\lambda^2, \Delta\right)$ and every $x \in c\left(\lambda^2, \Delta\right)$ has a unique representation of the form

$$x = lb + \sum_k \left[\alpha_k\left(\lambda^2\right) - l\right] b_n^{(k)}\left(\lambda^2\right)$$

where $\alpha_k\left(\lambda^2\right) = \Lambda^2(x)$ for all $k \in \mathbb{N}$ and the sequence $b = (b_k)$ is defined by $b_k = k + 1$.

Corollary 12 *The difference sequence spaces* $c\left(\lambda^2, \Delta\right)$ *and* $c_0\left(\lambda^2, \Delta\right)$ *are seperable.*

5 The $\alpha-, \beta-$ and $\gamma-$duals of the spaces $c\left(\lambda^2, \Delta\right)$ and $c_0\left(\lambda^2, \Delta\right)$

In this section, we introduce and prove the theorems determining the $\alpha-, \beta-$ and $\gamma-$ duals of the difference sequence spaces $c\left(\lambda^2, \Delta\right)$ and $c_0\left(\lambda^2, \Delta\right)$ of non-absolute type. For arbitrary sequence spaces X and Y ,the set $M(X, Y)$ defined by

$$M(X, Y) = \{a = (a_k) \in w : ax = (a_k x_k) \in Y \text{ for all } x = (x_k) \in X\} \tag{12}$$

is called the multipier space of X and Y. With the notation of (12); the $\alpha-, \beta-$ and $\gamma-$duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ are defined by

$$X^\alpha = M(X, \ell_1), \ X^\beta = M(X, cs) \text{ and } X^\gamma = M(X, bs).$$

Now, we may begin with lemmas which are given in [10]. We are needed them in proving theorems.

Lemma 13 $A \in (c_0 : \ell_1) = (c : \ell_1)$ *if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \left|\sum_{k \in K} a_{nk}\right| < \infty.$$

Lemma 14 *$A \in (c_0 : c)$ if and only if*

$$\lim_n a_{nk} \text{ exists for each } k \in \mathbb{N}, \tag{13}$$

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{14}$$

Lemma 15 *$A \in (c : c)$ if and only if (13) and (14) hold, and*

$$\lim_n \sum_k a_{nk} \text{ exists.} \tag{15}$$

Lemma 16 *$A \in (c_0 : \ell_\infty) = (c : \ell_\infty)$ if and only if (14) holds.*

Lemma 17 *$A \in (\ell_\infty : c)$ if and only if (13) holds and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k |\alpha_k| .$$

Theorem 18 *The $\alpha-$dual of the space $c\left(\lambda^2, \Delta\right)$ and $c_0\left(\lambda^2, \Delta\right)$ is the set*

$$h_1 = \left\{ a = (a_k) \in w : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} b_{nk}\left(\lambda^2\right) \right| < \infty \right\};$$

where the matrix $B^{\lambda^2} = \left(b_{nk}^{\lambda^2}\right)$ is defined via the sequence $a = (a_k)$ by

$$b_{nk}^{\lambda^2} = \begin{cases} \left(\frac{\Delta\lambda_k}{\Delta^2\lambda_k} - \frac{\Delta\lambda_k}{\Delta^2\lambda_{k+1}}\right) a_n; & n > k, \\ \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n; & n = k, \\ 0; & n < k. \end{cases}$$

Proof. We prove the theorem for the space $c_0\left(\lambda^2, \Delta\right)$. Let $a = (a_k) \in w$. Then, we obtain the equality

$$a_k x_k = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\Delta\lambda_j}{\Delta^2\lambda_k} y_j a_n = B_n^{\lambda^2}(y); \quad (n \in \mathbb{N}). \tag{16}$$

Thus, we observe by (16) that $ax = (a_k x_k) \in \ell_1$ whenever $x = (x_k) \in c_0\left(\lambda^2, \Delta\right)$ or $c\left(\lambda^2, \Delta\right)$ if and only if $B^{\lambda^2} y \in \ell_1$ whenever $y = (y_k) \in c_0$ or c. This means that the sequence $a = (a_k)$ is in the $\alpha-$dual of the spaces $c_0\left(\lambda^2, \Delta\right)$ or $c\left(\lambda^2, \Delta\right)$ if and only if $B^\lambda \in (c_0 : \ell_1) = (c : \ell_1)$. We therefore obtain by Lemma 13 with B^λ instead of A that $a \in \left\{c_0\left(\lambda^2, \Delta\right)\right\}^\alpha = \left\{c\left(\lambda^2, \Delta\right)\right\}^\alpha$ if and only if

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} b_{nk}^{\lambda^2} \right| < \infty.$$

Which leads us to the consequence that $\left\{c_0\left(\lambda^2, \Delta\right)\right\}^\alpha = \left\{c\left(\lambda^2, \Delta\right)\right\}^\alpha = h_1$. This concludes proof. ■

Theorem 19 *Define the sets*

$$h_2 = \left\{ a = (a_k) \in w : \sum_{j=k}^{\infty} a_j \text{ exists for each } k \in \mathbb{N}. \right\}$$

$$h_3 = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n-1} |g_k(n)| < \infty. \right\}$$

$$h_4 = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n \right| < \infty. \right\}$$

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$$h_5 = \left\{ a = (a_k) \in w : \sum_k (k+1)\, a_k \text{ converges.} \right\}$$

where

$$g_k(n) = \Delta\lambda_k \left(\frac{a_k}{\Delta\lambda_k} + (\frac{1}{\Delta\lambda_k} - \frac{1}{\Delta\lambda_{k+1}}) \sum_{j=k+1}^{n} a_j \right)$$

for $k < n$. Then $\left\{ c\left(\lambda^2, \Delta\right) \right\}^{\beta} = h_3 \cap h_4 \cap h_5$ and $\left\{ c_0\left(\lambda^2, \Delta\right) \right\}^{\beta} = h_2 \cap h_3 \cap h_4$.

Proof. We have from (6) that

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \left(\sum_{i=j-1}^{j} (-1)^{j-i} \frac{\Delta\lambda_i}{\Delta^2\lambda_j} y_i \right) \right] a_k \\
&= \sum_{k=0}^{n-1} \Delta\lambda_k \left[\frac{a_k}{\Delta^2\lambda_k} + \left(\frac{1}{\Delta^2\lambda_k} - \frac{1}{\Delta^2\lambda_{k+1}} \right) \sum_{j=k+1}^{n} a_j \right] y_k + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n y_n \qquad (17) \\
&= \sum_{k=0}^{n-1} g_k(n)\, y_k + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n y_n \\
&= T_n(y)\,; \quad (n \in \mathbb{N})
\end{aligned}$$

where the matrix $T = (t_{nk})$

$$t_{nk} = \begin{cases} g_k(n)\,; & k < n, \\ \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n; & k = n, \\ 0; & k > n. \end{cases} \quad (k, n \in \mathbb{N})\,.$$

Then we derive that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0\left(\lambda^2, \Delta\right)$ if and only if $Ty \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in \left\{ c_0\left(\lambda^2, \Delta\right) \right\}^{\beta}$ if and only if $T \in (c_0 : c)$. Therefore, by using Lemma 14, we obtain from (13) and (14) that

$$\sum_{j=k}^{\infty} a_j \text{ exists for each } k \in \mathbb{N}, \qquad (18)$$

$$\sup_n \sum_{k=0}^{n-1} |g_k(n)| < \infty, \qquad (19)$$

$$\sup_k \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_k < \infty. \qquad (20)$$

Hence we conclude that $\left\{ c_0\left(\lambda^2, \Delta\right) \right\}^{\beta} = h_2 \cap h_3 \cap h_4$. We can derive from Lemma 15 and 16 that $a = (a_k) \in \left\{ c\left(\lambda^2, \Delta\right) \right\}^{\beta}$ if and only if $T \in (c : c)$. Therefore, we have from (13) and (14) that (18), (19) and (20) hold. It can be seen that the equality

$$\sum_{k=0}^{n} (k+1)\, a_k = \sum_{k=0}^{n-1} g_k(n) + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n; \ (n \in \mathbb{N})$$

holds, which can be written as follows;

$$\sum_{k=0}^{n} (k+1)\, a_k = \sum_k t_{nk}; \quad (n \in \mathbb{N})\,.$$

Consequently, we have from (15) that

$$\{(k+1)\, a_k\} \in cs.$$

Hence (18) is redundant. We conclude that $\left\{ c\left(\lambda^2, \Delta\right) \right\}^{\beta} = h_3 \cap h_4 \cap h_5$. ■

Theorem 20 $\left\{c_0\left(\lambda^2,\Delta\right)\right\}^\gamma=\left\{c\left(\lambda^2,\Delta\right)\right\}^\gamma=h_3\cap h_4.$

Proof. It can be proved similarly as the proof of the Theorem 19 with Lemma 16 instead of Lemma 14. ■

6 Some matrix transformations

In this section, we state some matrix classes of matrix mappings on the $c_0\left(\lambda^2,\Delta\right)$ and $c\left(\lambda^2,\Delta\right)$. Let $x,y\in w$ be connected by the relation $y=\Lambda^2(x)$ like given in (5). For an infinite matrix $A=(a_{nk})$, we have by (17)

$$\sum_{k=0}^{m}a_{nk}x_k=\sum_{k=0}^{m-1}g_{nk}(m)\,y_k+\frac{\Delta\lambda_m}{\Delta^2\lambda_m}a_{nm}y_m \tag{21}$$

where

$$g_{nk}(m)=\Delta\lambda_k\left[\frac{a_{nk}}{\Delta\lambda_k}+\left(\frac{1}{\Delta\lambda_k}-\frac{1}{\Delta\lambda_{k+1}}\right)\sum_{j=k+1}^{m}a_{nj}\right].$$

Let $x\in c\left(\lambda^2,\Delta\right)$ and $A_n=(a_{nk})_{k=0}^{\infty}\in\left(c\left(\lambda^2,\Delta\right)\right)^\beta$ for all $n\in\mathbb{N}$. By passing limits in (21) as $m\to\infty$

$$\begin{aligned}\sum_k a_{nk}x_k&=\sum_k g_{nk}y_k+la_n\\&=\sum_k g_{nk}(y_k-l)+l\left(\sum_k g_{nk}+a_n\right)\end{aligned}\tag{22}$$

where $l=\lim_{k\to\infty}y_k$ and $a_n=\lim_{k\to\infty}\left(\frac{\Delta\lambda_k}{\Delta^2\lambda_k}a_{nk}\right)$ for all $n\in\mathbb{N}$. Let consider following conditions;

$$\sup_{F\in\mathcal{F}}\sum_n\left|\sum_{k\in F}g_{nk}\right|^p<\infty, \tag{23}$$

$$\sup_m\sum_{k=0}^{m-1}|g_{nk}(m)|<\infty, \tag{24}$$

$$\{(k+1)\,a_{nk}\}_{k=0}^{\infty}\in cs, \tag{25}$$

$$\lim_k\frac{\Delta\lambda_k}{\Delta^2\lambda_k}a_{nk}=a_n, \tag{26}$$

$$\sum_n|a_n|^p<\infty, \tag{27}$$

$$\sup_n\sum_k|g_{nk}|<\infty, \tag{28}$$

$$\sup_n|a_n|<\infty, \tag{29}$$

$$\sum_{j=k}^{\infty}a_{nj}\ exists, \tag{30}$$

$$\left\{\frac{\Delta\lambda_k}{\Delta^2\lambda_k}a_{nk}\right\}_{k=0}^{\infty}\in\ell_\infty, \tag{31}$$

$$\lim_n a_n=a, \tag{32}$$

$$\lim_n g_{nk}=\alpha_k, \tag{33}$$

$$\lim_n\sum_k g_{nk}=\alpha, \tag{34}$$

$$\lim_n a_n = 0, \tag{35}$$

$$\lim_n g_{nk} = 0, \tag{36}$$

$$\lim_n \sum_k g_{nk} = 0. \tag{37}$$

Using Theorem 19 and the results given in [10] with (21) and (22), we derive the following result:

Theorem 21

(a) Let $1 \leq p < \infty$. Then $A \in \left(c\left(\lambda^2, \Delta\right) : \ell_p\right)$ if and only if (23), (24), (25), (26) and (27).
(b) $A \in \left(c\left(\lambda^2, \Delta\right) : \ell_p\right)$ if and only if (25), (26), (28), (29).
(c) Let $1 \leq p < \infty$. Then $A \in \left(c_0\left(\lambda^2, \Delta\right) : \ell_p\right)$ if and only if (23), (24), (30) and (31).
(d) $A \in \left(c_0\left(\lambda^2, \Delta\right) : \ell_\infty\right)$ if and only if (28), (30) and (31).
(e) $A \in \left(c\left(\lambda^2, \Delta\right) : c\right)$ if and only if (25), (26), (28), (32), (33) and (34).
(f) $A \in \left(c\left(\lambda^2, \Delta\right) : c_0\right)$ if and only if (25), (26), (28), (35), (36) and (37).
(g) $A \in \left(c_0\left(\lambda^2, \Delta\right) : c\right)$ if and only if (28), (30), (31) and (33).
(h) $A \in \left(c_0\left(\lambda^2, \Delta\right) : c_0\right)$ if and only if (28), (30), (31) and (36).

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Stable cubic sets

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Abstract. The notions of (almost) stable cubic set, stable element, evaluative set and stable degree are introduced, and related properties are investigated. Regarding internal (external) cubic sets and the complement of cubic set, their (almost) stableness and unstableness are discussed. Regarding the P-union, R-union, P-intersection and R-intersection of cubic sets, their (almost) stableness and unstableness are investigated.

1. **Introduction**

Fuzzy sets are initiated by Zadeh [14]. In [15], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. In traditional fuzzy logic, to represent, e.g., the expert's degree of certainty in different statements, numbers from the interval $[0, 1]$ are used. It is often difficult for an expert to exactly quantify his or her certainty; therefore, instead of a real number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Interval-valued fuzzy sets have been actively used in real-life applications. For example, Sambuc [8] in Medical diagnosis in thyroidian pathology, Kohout [7] also in Medicine, in a system CLINAID, Gorzalczany [10] in Approximate reasoning, Turksen [10, 11] in Interval-valued logic, in preferences modelling [12], etc. These works and others show the importance of these sets. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [4] introduced a new notion, called a (internal, external) cubic set, and investigated several properties. They dealt with P-union, P-intersection, R-union and R-intersection of cubic sets, and investigated several related properties. Cubic set theory is applied to CI-algebras (see [1]), B-algebras (see [9]), BCK/BCI-algebras (see [5, 6]), KU-Algebras (see [2, 13]), and semigroups (see [3]).

In this paper, we introduce the notions of (almost) stable cubic set, stable element, evaluative set and stable degree. We investigate related properties. Regarding internal (external) cubic sets and the complement of cubic set, we investigate their (almost) stableness and unstableness.

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Regarding the P-union, R-union, P-intersection and R-intersection of cubic sets, we deal with their (almost) stableness and unstableness.

2. **Preliminaries**

A *fuzzy set* in a set X is defined to be a function $\lambda : X \to [0,1]$. Denote by I^X the collection of all fuzzy sets in a set X. Define a relation $\leq$ on I^X as follows:

$$(\forall \lambda, \mu \in I^X)\,(\lambda \leq \mu \iff (\forall x \in X)(\lambda(x) \leq \mu(x))).$$

The join ($\vee$) and meet ($\wedge$) of λ and μ are defined by $(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}$, and $(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$, respectively, for all $x \in X$. The complement of λ, denoted by λ^c, is defined by $(\forall x \in X)\,(\lambda^c(x) = 1 - \lambda(x))$. For a family $\{\lambda_i \mid i \in \Lambda\}$ of fuzzy sets in X, we define the join ($\vee$) and meet ($\wedge$) operations as follows: $\left(\bigvee_{i\in\Lambda} \lambda_i\right)(x) = \sup\{\lambda_i(x) \mid i \in \Lambda\}$, $\left(\bigwedge_{i\in\Lambda} \lambda_i\right)(x) = \inf\{\lambda_i(x) \mid i \in \Lambda\}$, respectively, for all $x \in X$.

Let $D[0,1]$ be the set of all closed subintervals of the unit interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^-, M^+]$, where M^- and M^+ are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for every $a \in (0,1)$. We also note that

(i) $(\forall M, N \in D[0,1])\ (M = N \Leftrightarrow M^- = N^-,\ M^+ = N^+)$.
(ii) $(\forall M, N \in D[0,1])\ (M \leq N \Leftrightarrow M^- \leq N^-,\ M^+ \leq N^+)$.

For every $M \in D[0,1]$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^+, 1 - M^-]$.

Let X be a nonempty set. A function $A : X \to D[0,1]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^-$ and $A(x)^+$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. Denote by D^X the collection of all interval-valued fuzzy sets in a set X. In particular, for any $a \in [0,1]$, the IVF set whose value is $\mathbf{a} = [a,a]$ for all $x \in X$ is denoted by simply $\tilde{a}$.

For every $A, B \in D^X$, we define

$$A = B \Leftrightarrow (\forall x \in X)\,(A(x)^- = B(x)^-,\ A(x)^+ = B(x)^+),$$

$$A \subseteq B \Leftrightarrow (\forall x \in X)\,(A(x)^- \leq B(x)^-,\ A(x)^+ \leq B(x)^+).$$

The *complement* A^c of A is defined by $(\forall x \in X)\,(A^c(x)^- = 1 - A(x)^+,\ A^c(x)^+ = 1 - A(x)^-)$. For a family $\{A_i \mid i \in \Lambda\}$ of IVF sets where Λ is an index set, the *union* $G = \bigcup_{i\in\Lambda} A_i$ and the

intersection $F = \bigcap_{i\in\Lambda} A_i$ are defined by

$$(\forall x \in X)\ \big(G(x)^- = \sup_{i\in\Lambda} A_i(x)^-,\ G(x)^+ = \sup_{i\in\Lambda} A_i(x)^+\big),$$

$$(\forall x \in X)\ \big(F(x)^- = \inf_{i\in\Lambda} A_i(x)^-,\ F(x)^+ = \inf_{i\in\Lambda} A_i(x)^+\big),$$

respectively.

Definition 2.1 ([4])**.** Let X be a nonempty set. By a *cubic set* in X we mean a structure

$$\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$$

in which A is an IVF set in X and λ is a fuzzy set in X.

A cubic set $\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$ is simply denoted by $\mathscr{A} = \langle A, \lambda\rangle$. Note that a cubic set is a generalization of an intuitionistic fuzzy set.

Definition 2.2 ([4])**.** Let X be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is said to be an *internal cubic set* (briefly, ICS) if $A(x)^- \leq \lambda(x) \leq A(x)^+$ for all $x \in X$.

Definition 2.3 ([4])**.** Let X be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is said to be an *external cubic set* (briefly, ECS) if $\lambda(x) \notin (A(x)^-, A(x)^+)$ for all $x \in X$.

Theorem 2.4 ([4])**.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X. If $\mathscr{A}$ is both an ICS and an ECS, then* $(\forall x \in X)$ $(\lambda(x) \in U(A) \cup L(A))$ *where* $U(A) = \{A(x)^+ \mid x \in X\}$ *and* $L(A) = \{A(x)^- \mid x \in X\}$.

Definition 2.5 ([4])**.** Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in X. Then we define

(a) (Equality) $\mathscr{A} = \mathscr{B} \Leftrightarrow A = B$ and $\lambda = \nu$.
(b) (P-order) $\mathscr{A} \sqsubseteq \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \leq \nu$.
(c) (R-order) $\mathscr{A} \Subset \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \geq \nu$.

Definition 2.6 ([4])**.** Let $\mathscr{A} = \langle A, \lambda\rangle$, $\mathscr{B} = \langle B, \mu\rangle$ and $\mathscr{A}_i = \{\langle x, A_i(x), \lambda_i(x)\rangle \mid x \in X\}$, $i \in \Lambda$, be cubic sets in X for $i \in \Lambda$. The *complement, P-union, P-intersection, R-union and R-intersection* are defined as follows;

(a) (Complement) $\mathscr{A}^c = \{\langle x, A^c(x), 1 - \lambda(x)\rangle \mid x \in X\}$.
(b) (P-union) $\mathscr{A} \sqcup \mathscr{B} = \{\langle x, (A \cup B)(x), (\lambda \vee \nu)(x)\rangle \mid x \in X\}$ and
$\sqcup\mathscr{A}_i = \{\langle x, (\bigcup A_i)(x), (\bigvee \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
(c) (P-intersection) $\mathscr{A} \sqcap \mathscr{B} = \{\langle x, (A \cap B)(x), (\lambda \wedge \nu)(x)\rangle \mid x \in X\}$ and
$\sqcap\mathscr{A}_i = \{\langle x, (\bigcap A_i)(x), (\bigwedge \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
(d) (R-union) $\mathscr{A} \Cup \mathscr{B} = \{\langle x, (A \cup B)(x), (\lambda \wedge \nu)(x)\rangle \mid x \in X\}$ and
$\Cup\mathscr{A}_i = \{\langle x, (\bigcup A_i)(x), (\bigwedge \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
(e) (R-intersection) $\mathscr{A} \Cap \mathscr{B} = \{\langle x, (A \cap B)(x), (\lambda \vee \nu)(x)\rangle \mid x \in X\}$ and
$\Cap\mathscr{A}_i = \{\langle x, (\bigcap A_i)(x), (\bigvee \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.

3. (Almost) stable cubic sets

In what follows, let X denote a nonempty set unless otherwise specified.

Definition 3.1. Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X. Then the *evaluative set* of $\mathscr{A} = \langle A, \lambda\rangle$ is defined to be a structure

$$\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\} \tag{3.1}$$

where $E_{\mathscr{A}}(x) = \langle l(E_{\mathscr{A}}(x)), r(E_{\mathscr{A}}(x))\rangle$ with $l(E_{\mathscr{A}}(x)) = \lambda(x) - A(x)^-$ and $r(E_{\mathscr{A}}(x)) = A(x)^+ - \lambda(x)$ which are called the *left evaluative point* and the *right evaluative point*, respectively, of $\mathscr{A} = \langle A, \lambda\rangle$ at $x \in X$. We say that $E_{\mathscr{A}}(x)$ is the *evaluative point* of $\mathscr{A} = \langle A, \lambda\rangle$ at $x \in X$.

Example 3.2. Let $\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in I\}$ be a cubic set in $I = [0, 1]$.

(1) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.4$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle 0.1, 0.3\rangle) \mid x \in I\}$.
(2) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.2$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle -0.1, 0.5\rangle) \mid x \in I\}$.
(3) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.8$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle 0.5, -0.1\rangle) \mid x \in I\}$.

Example 3.3. Let $\mathscr{B} = \{\langle x, B(x), \mu(x)\rangle \mid x \in I\}$ be a cubic set in $I = [0, 1]$ with $B(x) = [\frac{x}{4}, 1-\frac{x}{4}]$ and $\mu(x) = \frac{x}{3}$. Then $\mathbf{E}_{\mathscr{B}} = \left\{\left(x, \langle \frac{x}{12}, 1 - \frac{7x}{12}\rangle\right) \mid x \in I\right\}$, and so the evaluative point of $\mathscr{B}$ at $\frac{1}{2} \in I$ is $E_{\mathscr{B}}(\frac{1}{2}) = \langle \frac{1}{24}, \frac{17}{24}\rangle$.

Example 3.4. Let $\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in I\}$ be a cubic set in $X = \{0, a, b, c\}$ which is defined by Table 1.

TABLE 1. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
0	$[\frac{1}{8}, \frac{7}{8}]$	$\frac{7}{8} = 0.875$
a	$[\frac{1}{4}, \frac{3}{4}]$	$\frac{3}{8} = 0.375$
b	$[\frac{3}{8}, \frac{5}{8}]$	$\frac{1}{4} = 0.250$
c	$[\frac{1}{2}, \frac{1}{2}]$	$\frac{5}{8} = 0.625$

Then every evaluative point of $\mathscr{A}$ at each $x \in X$ is $E_{\mathscr{A}}(0) = \langle \frac{3}{4}, 0\rangle$, $E_{\mathscr{A}}(a) = \langle \frac{1}{8}, \frac{3}{8}\rangle$, $E_{\mathscr{A}}(b) = \langle -\frac{1}{8}, \frac{3}{8}\rangle$, and $E_{\mathscr{A}}(c) = \langle \frac{1}{8}, -\frac{1}{8}\rangle$, respectively. Hence the evaluative set of $\mathscr{A}$ is

$$\mathbf{E}_{\mathscr{A}} = \{(0, \langle \tfrac{3}{4}, 0\rangle), (a, \langle \tfrac{1}{8}, \tfrac{3}{8}\rangle), (b, \langle -\tfrac{1}{8}, \tfrac{3}{8}\rangle), (c, \langle \tfrac{1}{8}, -\tfrac{1}{8}\rangle)\}.$$

Definition 3.5. Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X with the evaluative set

$$\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}.$$

An element $a \in X$ is called a *stable element* of $\mathscr{A} = \langle A, \lambda\rangle$ in X if it satisfies: $l(E_{\mathscr{A}}(a)) = \lambda(a) - A(a)^- \geq 0$, $r(E_{\mathscr{A}}(a)) = A(a)^+ - \lambda(a) \geq 0$. Otherwise, we say that a is an *unstable element* of $\mathscr{A} = \langle A, \lambda\rangle$ in X. The set of all stable elements of $\mathscr{A} = \langle A, \lambda\rangle$ in X is called the *stable cut* of

$\mathscr{A} = \langle A, \lambda \rangle$ in X and is denoted by $S_{\mathscr{A}}$. The set of all unstable elements of $\mathscr{A} = \langle A, \lambda \rangle$ in X is called the *unstable cut* of $\mathscr{A} = \langle A, \lambda \rangle$ in X and is denoted by $U_{\mathscr{A}}$. We say that $\mathscr{A} = \langle A, \lambda \rangle$ is a *stable* cubic set if $S_{\mathscr{A}} = X$. Otherwise, $\mathscr{A} = \langle A, \lambda \rangle$ is called an *unstable* cubic set.

It is clear that $X = S_{\mathscr{A}} \cup U_{\mathscr{A}}$, $S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0,\ r(E_{\mathscr{A}}(x)) \geq 0\}$ and $U_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) < 0\} \cup \{x \in X \mid r(E_{\mathscr{A}}(x)) < 0\}$.

Example 3.6. Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in $X = \{0, a, b, c\}$ given by Table 2.

TABLE 2. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
0	$[0.2, 0.3]$	0.10
a	$[0.2, 0.3]$	0.25
b	$[0.7, 0.8]$	0.75
c	$[0.3, 0.7]$	0.80

Then a and b are stable elements of $\mathscr{A}$ in X, and 0 and c are unstable elements of $\mathscr{A}$ in X. Hence $S_{\mathscr{A}} = \{a, b\}$ and $U_{\mathscr{A}} = \{0, c\}$.

Example 3.7. (1) Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in $X = \{a, b, c\}$ defined by Table 3.

TABLE 3. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.1, 0.6]$	0.5
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	0.6

It is routine to verify that $\mathscr{A} = \langle A, \lambda \rangle$ is a stable cubic set.
(2) Let $\mathscr{B} = \langle B, \mu \rangle$ be a cubic set in $X = \{a, b, c\}$ defined by Table 4.

TABLE 4. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.5
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	0.6

Then $\mathscr{B}$ is an unstable cubic set since $E_{\mathscr{B}}(a) = (0.5 - 0.1, 0.3 - 05) = (0.4, -0.2)$.

Theorem 3.8. *Every ICS is a stable cubic set.*

Proof. Straightforward. □

The following example shows that every ECS would be stable or unstable.

Example 3.9. (1) Let $\mathscr{A} = \langle A, \lambda \rangle$ be an ECS in $X = \{a, b, c\}$ given by Table 5.

TABLE 5. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.1, 0.6]$	0.6
b	$[0.6, 0.9]$	0.5
c	$[0.1, 0.9]$	0.1

Then $\mathscr{A}$ is unstable because $E_{\mathscr{A}}(b) = (0.5 - 0.6, 0.9 - 0.5) = (-0.1, 0.4)$.
(2) Let $\mathscr{B} = \langle B, \mu \rangle$ be an ECS in $X = \{a, b, c\}$ defined by Table 6.

TABLE 6. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.1
b	$[0.6, 0.9]$	0.9
c	$[0.1, 0.9]$	0.1

Then $\mathscr{B}$ is stable since $E_{\mathscr{B}}(a) = (0, 0.2)$, $E_{\mathscr{B}}(b) = (0.3, 0)$, and $E_{\mathscr{B}}(c) = (0, 0.8)$.

We provide a condition for an ECS to be a stable cubic set.

Theorem 3.10. *If an ECS $\mathscr{A} = \langle A, \lambda \rangle$ in X satisfies the following condition*

$$(\forall x \in X) \left(\mathscr{A}^-(x) = \lambda(x) \text{ or } \mathscr{A}^+(x) = \lambda(x) \right), \tag{3.2}$$

then $\mathscr{A} = \langle A, \lambda \rangle$ is a stable cubic set.

Proof. Straightforward. □

Corollary 3.11. *Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in X. If $\mathscr{A}$ is both an ICS and an ECS, then $\mathscr{A}$ is stable.*

Proof. Straightforward. □

Theorem 3.12. *The complement of a stable cubic set is also stable.*

Proof. Let $\mathscr{A} = \langle A, \lambda \rangle$ be a stable cubic set in X. Then $X = S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0,\ r(E_{\mathscr{A}}(x)) \geq 0\}$. Hence $\lambda(x) - A(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$ for all $x \in X$. It follows that $l(E_{\mathscr{A}^c}(x)) = (1 - \lambda(x)) - (1 - A(x)^+) = A(x)^+ - \lambda(x)) \geq 0$ and $r(E_{\mathscr{A}^c}(x)) = (1 - A(x)^-) - (1 - \lambda(x)) = \lambda(x) - A(x)^- \geq 0$. Therefore $\mathscr{A}^c = \langle A^c, \lambda^c \rangle$ is a stable cubic set. □

Theorem 3.13. *The complement of an unstable cubic set is also unstable.*

Proof. Let $\mathscr{A} = \langle A, \lambda \rangle$ be an unstable cubic set in X. Then $U_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) < 0\} \cup \{x \in X \mid r(E_{\mathscr{A}}(x)) < 0\} \neq \emptyset$, and so there exist $x \in X$ such that $\lambda(x) - A(x)^- < 0$ or $A(x)^+ - \lambda(x) < 0$. It follows that $l(E_{\mathscr{A}^c}(x)) = (1 - \lambda(x)) - (1 - A(x)^+) = A(x)^+ - \lambda(x)) < 0$ or $r(E_{\mathscr{A}^c}(x)) = (1 - A(x)^-) - (1 - \lambda(x)) = \lambda(x) - A(x)^- < 0$. Hence $U_{\mathscr{A}^c} \neq \emptyset$, and therefore $\mathscr{A}^c = \langle A^c, \lambda^c \rangle$ is an unstable cubic set in X. □

The following example illustrates Theorem 3.13.

Example 3.14. Note that the cubic set $\mathscr{B} = \langle B, \mu \rangle$ in Example 3.7(2) is unstable, and its complement is represented by Table 7.

TABLE 7. Tabular representation of the cubic set $\mathscr{B}^c$

X	$B^c(x)$	$\mu^c(x)$
a	$[0.7, 0.9]$	0.5
b	$[0.1, 0.4]$	0.3
c	$[0.1, 0.9]$	0.4

Then $\mathscr{B}^c = \langle B^c, \mu^c \rangle$ is unstable since $a \in U_{\mathscr{B}^c}$.

Theorem 3.15. *The P-union and P-intersection of two stable cubic sets in X are stable cubic sets in X.*

Proof. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be stable cubic sets in X. Then $S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0,\ r(E_{\mathscr{A}}(x)) \geq 0\} = X$ and $S_{\mathscr{B}} = \{x \in X \mid l(E_{\mathscr{B}}(x)) \geq 0,\ r(E_{\mathscr{B}}(x)) \geq 0\} = X$. It follows that $\lambda(x) - A(x)^- \geq 0,\ A(x)^+ - \lambda(x) \geq 0$ for all $x \in X$ and $\mu(x) - B(x)^- \geq 0,\ B(x)^+ - \mu(x) \geq 0$ for all $x \in X$. Assume that $\lambda(x) \geq \mu(x)$ and consider four cases:

(i) $A(x)^- \geq B(x)^-$ and $A(x)^+ \geq B(x)^+$,
(ii) $A(x)^- \geq B(x)^-$ and $A(x)^+ \leq B(x)^+$,
(iii) $A(x)^- \leq B(x)^-$ and $A(x)^+ \geq B(x)^+$,
(iv) $A(x)^- \leq B(x)^-$ and $A(x)^+ \leq B(x)^+$.

The first case implies that $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq A(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq A(x)^+ = \max\{A(x)^+, B(x)^+\}$. It follows that $\lambda(x) - A(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$. From the second case, we have $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq A(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq B(x)^+ = \max\{A(x)^+, B(x)^+\}$. Hence $\lambda(x) - A(x)^- \geq 0$ and $B(x)^+ - \lambda(x) \geq A(x)^+ - \lambda(x) \geq 0$. The third case induces $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq \mu(x) \geq B(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq A(x)^+ = \max\{A(x)^+, B(x)^+\}$, and so $\lambda(x) - B(x)^- \geq \mu(x) - B(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$. For the final case, we get $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq \mu(x) \geq B(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} =$

$\lambda(x) \leq A(x)^+ \leq B(x) = \max\{A(x)^+, B(x)^+\}$. Thus $\lambda(x) - B(x)^- \geq \mu(x) - B(x)^- \geq 0$ and $B(x)^+ - \lambda(x) \geq 0$. In the case of $\mu(x) \geq \lambda(x)$, we can obtain the same results in a similar way. Therefore $\mathscr{A} \sqcup \mathscr{B}$ is a stable cubic set in X. By the similar method, we know that $\mathscr{A} \sqcap \mathscr{B}$ is a stable cubic set in X. □

The following example shows that the R-union and the R-intersection of two stable cubic sets in X may not be stable in X.

Example 3.16. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 8 and 9, respectively.

TABLE 8. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.3]$	0.20
b	$[0.7, 0.8]$	0.75
c	$[0.3, 0.7]$	0.60

TABLE 9. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.15
b	$[0.6, 0.9]$	0.70
c	$[0.1, 0.9]$	0.80

Then

$$\mathscr{A} \Cup \mathscr{B} = \{\langle a, [0.2, 0.3], 0.15\rangle, \langle b, [0.7, 0.9], 0.7\rangle, \langle c, [0.3, 0.9], 0.6\rangle\}$$

and

$$\mathscr{A} \Cap \mathscr{B} = \{\langle a, [0.1, 0.3], 0.2\rangle, \langle b, [0.6, 0.8], 0.75\rangle, \langle c, [0.1, 0.7], 0.8\rangle\}.$$

Hence we know that $E_{\mathscr{A} \Cup \mathscr{B}}(a) = \langle -0.05, 0.15\rangle$ and $E_{\mathscr{A} \Cap \mathscr{B}}(c) = \langle 0.7, -0.1\rangle$. Thus $\mathscr{A} \Cup \mathscr{B}$ and $\mathscr{A} \Cap \mathscr{B}$ are unstable.

Now, we provide conditions for the R-union (resp. R-intersection) of two ICSs to be stable.

Theorem 3.17. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in X such that*

$$(\forall x \in X) \left(\max\{A(x)^-, B(x)^-\} \leq (\lambda \wedge \mu)(x)\right). \tag{3.3}$$

Then the R-union of $\mathscr{A}$ and $\mathscr{B}$ is a stable cubic set in X.

Proof. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in X. Then $A(x)^- \leq \lambda(x) \leq A(x)^+$ and $B(x)^- \leq \mu(x) \leq B(x)^+$ for all $x \in X$. It follows from (3.3) that $\max\{A(x)^-, B(x)^-\} \leq (\lambda \wedge \mu)(x) \leq \max\{A(x)^+, B(x)^+\}$ for all $x \in X$. Hence the R-union of $\mathscr{A}$ and $\mathscr{B}$ is an ICS, and so it is stable by Theorem 3.8. □

Theorem 3.18. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in X such that*

$$(\forall x \in X)\left(\max\{A(x)^+, B(x)^+\} \leq (\lambda \vee \mu)(x)\right). \tag{3.4}$$

Then the R-intersection of $\mathscr{A}$ and $\mathscr{B}$ is a stable cubic set in X.

Proof. The proof is by the similar method to Theorem 3.17. □

Theorem 3.19. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in X such that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are ICSs in X. Then the P-union $\mathscr{A} \sqcup \mathscr{B}$ and the P-intersection $\mathscr{A} \sqcap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are stable in X.*

Proof. It is straightforward by Theorems 3.20 and 3.21 in [4] and Theorem 3.8. □

Definition 3.20. Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set with the evaluative set $\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}$ in X. Then the *stable degree* of $\mathscr{A}$ in X is denoted by $SD_{\mathscr{A}}$ and is defined by

$$SD_{\mathscr{A}} = \left(\sum_{x\in X} l(E_{\mathscr{A}}(x)),\ \sum_{x\in X} r(E_{\mathscr{A}}(x))\right). \tag{3.5}$$

Definition 3.21. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ with the evaluative set $\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}$ in X is said to be *almost stable* if there exists the stable degree $SD_{\mathscr{A}}$ in which $\sum_{x\in X} l(E_{\mathscr{A}}(x)) \geq 0$ and $\sum_{x\in X} r(E_{\mathscr{A}}(x)) \geq 0$.

Example 3.22. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 10 and 11, respectively.

TABLE 10. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.3]$	0.2
b	$[0.7, 0.8]$	0.9
c	$[0.3, 0.7]$	0.6

Then

$$\mathbf{E}_{\mathscr{A}} = \{(a, \langle 0, 0.1\rangle),\ (b, \langle 0.2, -0.1\rangle),\ (c, \langle 0.3, 0.1\rangle)\}$$

and

TABLE 11. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.2, 0.3]$	0.9
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	1

$$\mathbf{E}_{\mathscr{B}} = \{(a, \langle 0.7, -0.6\rangle),\ (b, \langle 0.1, 0.2\rangle),\ (c, \langle 0.9, -0.1\rangle)\}.$$

Thus $SD_{\mathscr{A}} = (0 + 0.2 + 0.3,\ 0.1 - 0.1 + 0.1) = (0.5, 0.1)$ and so $\mathscr{A}$ is almost stable. But $\mathscr{B}$ is not almost stable since $SD_{\mathscr{B}} = (0.7 + 0.1 + 0.9,\ -0.6 + 0.2 - 0.1) = (1.7, -0.5)$.

Theorem 3.23. *Every stable cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is almost stable.*

Proof. Straightforward. □

In Example 3.22, the almost stable cubic set $\mathscr{A} = \langle A, \lambda\rangle$ is not stable. This shows that the converse of Theorem 3.23 is not true in general.

Combining Theorems 3.8, 3.10, 3.15, 3.19 and 3.23, we know that

(1) Every ICS is almost stable.
(2) Every ESC satisfying the condition (3.2) is almost stable.
(3) The P-union and P-intersection of two stable cubic sets is almost stable.
(4) If $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are ECSs in X such that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are ICSs in X, then the P-union and the P-intersection of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable in X.

Proposition 3.24. *If $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are cubic sets in X, then either*

$$(\forall x \in X)\left(\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\} \leq \lambda(x) - A(x)^-\right) \tag{3.6}$$

or

$$(\forall x \in X)\left(\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\} \leq \mu(x) - B(x)^-\right). \tag{3.7}$$

Proof. For each $x \in X$, we consider the four cases as follows:

(1) $\max\{\lambda(x), \mu(x)\} = \lambda(x)$ and $\max\{A(x)^-, B(x)^-\} = A(x)^-$.
(2) $\max\{\lambda(x), \mu(x)\} = \lambda(x)$ and $\max\{A(x)^-, B(x)^-\} = B(x)^-$.
(3) $\max\{\lambda(x), \mu(x)\} = \mu(x)$ and $\max\{A(x)^-, B(x)^-\} = A(x)^-$.
(4) $\max\{\lambda(x), \mu(x)\} = \mu(x)$ and $\max\{A(x)^-, B(x)^-\} = B(x)^-$.

First two cases induce the inequality (3.6), and the inequality (3.7) is induced by the last two cases. □

Proposition 3.25. *If $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are cubic sets in X, then either*

$$(\forall x \in X) \left(\max\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \leq A(x)^+ - \lambda(x) \right) \tag{3.8}$$

or

$$(\forall x \in X) \left(\max\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \leq B(x)^+ - \mu(x) \right). \tag{3.9}$$

Proof. It is similar to the proof of Proposition 3.24. □

In the following example, we know that the P-union and the R-union of almost stable cubic sets may not be almost stable.

Example 3.26. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 12 and 13, respectively.

TABLE 12. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[1.0, 1.0]$	0.7
b	$[0.5, 1.0]$	0.7
c	$[0.6, 1.0]$	0.7

TABLE 13. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.5, 1.0]$	0.7
b	$[1.0, 1.0]$	0.7
c	$[0.6, 1.0]$	0.7

Then $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable cubic sets in X because

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = 0, \sum_{x \in X} r(E_{\mathscr{A}}(x)) = 0.9, \sum_{x \in X} l(E_{\mathscr{B}}(x)) = 0, \text{ and } \sum_{x \in X} r(E_{\mathscr{B}}(x)) = 0.9.$$

But the P-union $\mathscr{A} \sqcup \mathscr{B}$ and the R-union $\mathscr{A} \Cup \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are not almost stable because $\sum_{x \in X} l(E_{\mathscr{A} \sqcup \mathscr{B}}(x)) = \sum_{x \in X} (\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\}) = -0.5 \ngeq 0$ and $\sum_{x \in X} l(E_{\mathscr{A} \Cup \mathscr{B}}(x)) = \sum_{x \in X} (\min\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\}) = -0.5 \ngeq 0$.

We now provide conditions for the P-union of almost stable cubic sets to be almost stable.

Theorem 3.27. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be almost stable cubic sets in X such that*

$$(\forall x \in X) \left(\sum_{x \in X} (|\lambda(x) - \mu(x)| - A(x)^-) \geq 0, \sum_{x \in X} (|A(x)^+ - B(x)^+| - \lambda(x)) \geq 0 \right). \tag{3.10}$$

Then the P-union $\mathscr{A} \sqcup \mathscr{B} = \langle A \cup B, \lambda \vee \mu \rangle$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is almost stable in X.

Proof. Assume that $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable in X. Then there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = \sum_{x\in X}(\lambda(x) - A(x)^-) \geq 0, \sum_{x\in X} r(E_{\mathscr{A}}(x)) = \sum_{x\in X}(A(x)^+ - \lambda(x)) \geq 0,$$

$$\sum_{x\in X} l(E_{\mathscr{B}}(x)) = \sum_{x\in X}(\mu(x) - B(x)^-) \geq 0, \text{ and } \sum_{x\in X} r(E_{\mathscr{B}}(x)) = \sum_{x\in X}(B(x)^+ - \mu(x)) \geq 0.$$

Now, we have to show that $\sum_{x\in X} l(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) \geq 0$ and $\sum_{x\in X} r(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) \geq 0$ in the stable degree $SD_{\mathscr{A}\sqcup\mathscr{B}}$ of $\mathscr{A}\sqcup\mathscr{B}$. Using (3.10), we have

$$\begin{aligned}
&\sum_{x\in X} l(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) = \sum_{x\in X} \big((\lambda\vee\mu)(x) - (A\cup B)(x)^-\big) \\
&= \sum_{x\in X} \big(\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\}\big) \\
&= \sum_{x\in X} \Big(\tfrac{|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2} - \tfrac{|A(x)^- - B(x)^-|+A(x)^- + B(x)^-}{2}\Big) \\
&= \sum_{x\in X} \Big(\tfrac{|\lambda(x)-\mu(x)|-|A(x)^- - B(x)^-|+\lambda(x)-A(x)^- +\mu(x)-B(x)^-}{2}\Big) \\
&= \tfrac{1}{2}\sum_{x\in X} \big(|\lambda(x) - \mu(x)| - |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^-\big) \\
&= \tfrac{1}{2}\sum_{x\in X} \big(|\lambda(x) - \mu(x)| - |A(x)^- - B(x)^-|\big) \\
&\qquad\qquad + \tfrac{1}{2}\sum_{x\in X} \big(\lambda(x) - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X} \big(\mu(x) - B(x)^-\big) \\
&\geq \tfrac{1}{2}\sum_{x\in X} \big(|\lambda(x) - \mu(x)| - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X} \big(\lambda(x) - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X} \big(\mu(x) - B(x)^-\big) \\
&\geq 0.
\end{aligned}$$

Similarly, we have $\sum_{x\in X} r(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) \geq 0$. Therefore $\mathscr{A}\sqcup\mathscr{B} = \langle A\cup B, \lambda\vee\mu\rangle$ is almost stable in X. □

Theorem 3.28. *The complement of an almost stable cubic set is also almost stable.*

Proof. Let $\mathscr{A} = \langle A, \lambda\rangle$ be an almost stable cubic set in X. Then there exists a stable degree $SD_{\mathscr{A}}$ such that

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = \sum_{x\in X}(\lambda(x) - A(x)^-) \geq 0, \text{ and } \sum_{x\in X} r(E_{\mathscr{A}}(x)) = \sum_{x\in X}(A(x)^+ - \lambda(x)) \geq 0.$$

It follows that $\sum_{x\in X} l(E_{\mathscr{A}^c}(x)) = \sum_{x\in X} ((1-\lambda(x)) - (1-A(x)^+)) = \sum_{x\in X} (A(x)^+ - \lambda(x)) \geq 0$ and $\sum_{x\in X} r(E_{\mathscr{A}^c}(x)) = \sum_{x\in X} ((1-A(x)^-) - (1-\lambda(x))) = \sum_{x\in X} (\lambda(x) - A(x)^-) \geq 0$. Therefore $\mathscr{A}^c = \langle A^c, \lambda^c\rangle$ is almost stable. □

We now provide conditions for the R-union of almost stable cubic sets to be almost stable.

Theorem 3.29. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be almost stable cubic sets in X such that*

$$\sum_{x\in X} (|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-|) \leq \left(\sum_{x\in X} \lambda(x) - A(x)^-\right) + \sum_{x\in X} (\mu(x) - B(x)^-) \tag{3.11}$$

and

$$\sum_{x\in X} (|\lambda(x) - \mu(x)| + |A(x)^+ - B(x)^+|) \geq \sum_{x\in X} (\lambda(x) - A(x)^+) + \sum_{x\in X} (\mu(x) - B(x)^+) \tag{3.12}$$

for all $x \in X$. Then the R-union $\mathscr{A} \Cup \mathscr{B} = \langle A\cup B, \lambda\wedge\mu\rangle$ is almost stable in X.

Proof. Assume that $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable in X. Then there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = \sum_{x\in X} (\lambda(x) - A(x)^-) \geq 0, \sum_{x\in X} r(E_{\mathscr{A}}(x)) = \sum_{x\in X} (A(x)^+ - \lambda(x)) \geq 0,$$
$$\sum_{x\in X} l(E_{\mathscr{B}}(x)) = \sum_{x\in X} (\mu(x) - B(x)^-) \geq 0, \text{ and } \sum_{x\in X} r(E_{\mathscr{B}}(x)) = \sum_{x\in X} (B(x)^+ - \mu(x)) \geq 0.$$

It follows from (3.11) that

$$\begin{aligned}
&\sum_{x\in X} l(E_{\mathscr{A}\Cup\mathscr{B}}(x)) = \sum_{x\in X} \big((\lambda\wedge\mu)(x) - (A\cup B)(x)^-\big) \\
&= \sum_{x\in X} \big(\min\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\}\big) \\
&= \sum_{x\in X} \Big(\tfrac{-|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2} - \tfrac{|A(x)^- - B(x)^-| + A(x)^- + B(x)^-}{2}\Big) \\
&= \sum_{x\in X} \Big(\tfrac{-|\lambda(x)-\mu(x)| - |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^-}{2}\Big) \\
&= -\tfrac{1}{2}\sum_{x\in X} \big(|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-|\big) \\
&\qquad + \tfrac{1}{2}\sum_{x\in X} \big(\lambda(x) - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X} \big(\mu(x) - B(x)^-\big) \\
&\geq -\tfrac{1}{2}\left(\sum_{x\in X} \big(\lambda(x) - A(x)^-\big) + \sum_{x\in X} \big(\mu(x) - B(x)^-\big)\right) \\
&+ \tfrac{1}{2}\sum_{x\in X} \big(\lambda(x) - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X} \big(\mu(x) - B(x)^-\big) = 0.
\end{aligned}$$

Using (3.12), we have

$$\begin{aligned}\sum_{x\in X} r(E_{\mathscr{A}\Cup\mathscr{B}}(x)) &= \sum_{x\in X}\big((A\cup B)(x)^{+} - (\lambda\wedge\mu)(x)\big)\\
&= \sum_{x\in X}\big(\max\{A(x)^{-}, B(x)^{-}\} - \min\{\lambda(x),\mu(x)\}\big)\\
&= \sum_{x\in X}\Big(\tfrac{|A(x)^{+}-B(x)^{+}|+A(x)^{+}+B(x)^{+}}{2} - \tfrac{-|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2}\Big)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(|\lambda(x)-\mu(x)| + |A(x)^{+}-B(x)^{+}|\big)\\
&- \tfrac{1}{2}\left(\sum_{x\in X}\big(\lambda(x)-A(x)^{+}\big) + \sum_{x\in X}\big(\mu(x)-B(x)^{+}\big)\right) \geq 0.\end{aligned}$$

Hence $\mathscr{A}\Cup\mathscr{B} = \langle A\cup B, \lambda\wedge\mu\rangle$ is almost stable in X. $\square$

The following examples show that the P-intersection and the R-intersection of almost stable cubic sets may not be almost stable.

Example 3.30. (1) Let $\mathscr{A} = \langle A,\lambda\rangle$ and $\mathscr{B} = \langle B,\mu\rangle$ be cubic sets in $X = \{a,b,c\}$ defined by Tables 14 and 15, respectively.

TABLE 14. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.7, 1.0]$	0.4
b	$[0.5, 1.0]$	0.8
c	$[0.6, 1.0]$	0.7

TABLE 15. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.5, 1.0]$	0.8
b	$[0.6, 1.0]$	0.7
c	$[0.7, 1.0]$	0.4

Then $\mathscr{A} = \langle A,\lambda\rangle$ and $\mathscr{B} = \langle B,\mu\rangle$ are almost stable cubic sets in X because

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = 0.1,\ \sum_{x\in X} r(E_{\mathscr{A}}(x)) = 1.1,\ \sum_{x\in X} l(E_{\mathscr{B}}(x)) = 0.1, \text{ and } \sum_{x\in X} r(E_{\mathscr{B}}(x)) = 1.1.$$

But the P-intersection $\mathscr{A} \sqcap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is not almost stable because

$$\sum_{x \in X} l(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) = \sum_{x \in X} \big(\min\{\lambda(x), \mu(x)\} - \min\{A(x)^-, B(x)^-\} \big) = -0.1 \ngeq 0.$$

(2) Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 16 and 17, respectively.

Table 16. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.7]$	0.8
b	$[0.3, 0.6]$	0.5
c	$[0.1, 0.5]$	0.5

Table 17. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.2, 0.7]$	0.6
b	$[0.3, 0.6]$	0.7
c	$[0.1, 0.5]$	0.5

Then $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable cubic sets in X because

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = 1.2, \ \sum_{x \in X} r(E_{\mathscr{A}}(x)) = 0, \ \sum_{x \in X} l(E_{\mathscr{B}}(x)) = 1.2, \text{ and } \sum_{x \in X} r(E_{\mathscr{B}}(x)) = 0.$$

But the R-intersection $\mathscr{A} \Cap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is not almost stable since

$$\sum_{x \in X} r(E_{\mathscr{A} \Cap \mathscr{B}}(x)) = \sum_{x \in X} \big(\min\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \big) = -0.2 \ngeq 0.$$

We now provide conditions for the P-intersection and the R-intersection of almost stable cubic sets to be almost stable.

Theorem 3.31. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be almost stable cubic sets in X.*
(i) *Assume that the following condition is valid.*

$$(\forall x \in X) \left(\begin{array}{l} \sum\limits_{x \in X} (|A(x)^- - B(x)^-| - |\lambda(x) - \mu(x)|) \geq 0, \\ \sum\limits_{x \in X} (|\lambda(x) - \mu(x)| - |A(x)^+ - B(x)^+|) \geq 0 \end{array} \right). \tag{3.13}$$

Then the P-intersection $\mathscr{A} \sqcap \mathscr{B} = \langle A \cap B, \lambda \wedge \mu \rangle$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is almost stable in X.

(ii) *If $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ satisfy the following condition*

$$(\forall x \in X)\left(\sum_{x\in X}\big(|\lambda(x) - \mu(x)| + |A(x)^{+} - B(x)^{+}|\big) = 0\right), \tag{3.14}$$

then the R-intersection $\mathscr{A} \Cap \mathscr{B} = \langle A \cap B, \lambda \vee \mu\rangle$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ is almost stable in X.

Proof. Since $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable in X, there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\begin{aligned}
&\sum_{x\in X} l(E_{\mathscr{A}}(x)) = \sum_{x\in X}(\lambda(x) - A(x)^{-}) \geq 0, \sum_{x\in X} r(E_{\mathscr{A}}(x)) = \sum_{x\in X}(A(x)^{+} - \lambda(x)) \geq 0,\\
&\sum_{x\in X} l(E_{\mathscr{B}}(x)) = \sum_{x\in X}(\mu(x) - B(x)^{-}) \geq 0, \text{and} \sum_{x\in X} r(E_{\mathscr{B}}(x)) = \sum_{x\in X}(B(x)^{+} - \mu(x)) \geq 0.
\end{aligned}$$

(i) We have to show that $\sum_{x\in X} l(E_{\mathscr{A}\sqcap\mathscr{B}}(x)) \geq 0$ and $\sum_{x\in X} r(E_{\mathscr{A}\sqcap\mathscr{B}}(x)) \geq 0$ in the stable degree $SD_{\mathscr{A}\sqcap\mathscr{B}}$ of $\mathscr{A} \sqcap \mathscr{B}$. Using (3.13), we have

$$\begin{aligned}
&\sum_{x\in X} l(E_{\mathscr{A}\sqcap\mathscr{B}}(x)) = \sum_{x\in X}\big((\lambda \wedge \mu)(x) - (A\cap B)(x)^{-}\big)\\
&= \sum_{x\in X}\big(\min\{\lambda(x), \mu(x)\} - \min\{A(x)^{-}, B(x)^{-}\}\big)\\
&= \sum_{x\in X}\left(\tfrac{-|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2} + \tfrac{|A(x)^{-}-B(x)^{-}|-A(x)^{-}-B(x)^{-}}{2}\right)\\
&= \sum_{x\in X}\left(\tfrac{-|\lambda(x)-\mu(x)|+|A(x)^{-}-B(x)^{-}|+\lambda(x)-A(x)^{-}+\mu(x)-B(x)^{-}}{2}\right)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(-|\lambda(x) - \mu(x)| + |A(x)^{-} - B(x)^{-}| + \lambda(x) - A(x)^{-} + \mu(x) - B(x)^{-}\big)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(|A(x)^{-} - B(x)^{-}| - |\lambda(x) - \mu(x)|\big)\\
&\qquad + \tfrac{1}{2}\sum_{x\in X}\big((\lambda(x) - (A(x)^{-}\big) + \tfrac{1}{2}\sum_{x\in X}\big((\mu(x)) - B(x)^{-})\big) \geq 0.
\end{aligned}$$

Similarly, we have $\sum_{x\in X} r(E_{\mathscr{A}\sqcap\mathscr{B}}(x)) \geq 0$. Therefore $\mathscr{A} \sqcap \mathscr{B} = \langle A \cap B, \lambda \wedge \mu\rangle$ is almost stable in X.

(ii) We have

$$
\begin{aligned}
&\sum_{x\in X} l(E_{\mathscr{A}\Cap\mathscr{B}}(x)) = \sum_{x\in X}\big((\lambda\vee\mu)(x) - (A\cap B)(x)^{-}\big)\\
&= \sum_{x\in X}\big(\max\{\lambda(x),\mu(x)\} - \min\{A(x)^{-}, B(x)^{-}\}\big)\\
&= \sum_{x\in X}\Big(\tfrac{|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2} + \tfrac{|A(x)^{-}-B(x)^{-}|-A(x)^{-}-B(x)^{-}}{2}\Big)\\
&= \sum_{x\in X}\Big(\tfrac{|\lambda(x)-\mu(x)|+|A(x)^{-}-B(x)^{-}|+\lambda(x)-A(x)^{-}+\mu(x)-B(x)^{-}}{2}\Big)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(|\lambda(x)-\mu(x)| + |A(x)^{-}-B(x)^{-}|\big)\\
&\qquad\qquad + \tfrac{1}{2}\sum_{x\in X}\big(\lambda(x)-A(x)^{-}\big) + \tfrac{1}{2}\sum_{x\in X}\big(\mu(x)-B(x)^{-}\big)\\
&\geq \tfrac{1}{2}\left(\sum_{x\in X}\big(\lambda(x)-A(x)^{-}\big) + \sum_{x\in X}\big(\mu(x)-B(x)^{-}\big)\right) \geq 0.
\end{aligned}
$$

Using (3.14), we have

$$
\begin{aligned}
&\sum_{x\in X} r(E_{\mathscr{A}\Cap\mathscr{B}}(x)) = \sum_{x\in X}\big((A\cap B)(x)^{+} - (\lambda\vee\mu)(x)\big)\\
&= \sum_{x\in X}\big(\min\{A(x)^{+}, B(x)^{+}\} - \max\{\lambda(x),\mu(x)\}\big)\\
&= \sum_{x\in X}\Big(\tfrac{-|A(x)^{+}-B(x)^{+}|+A(x)^{+}+B(x)^{+}}{2} - \tfrac{|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2}\Big)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(-|\lambda(x)-\mu(x)| - |A(x)^{+}-B(x)^{+}|\big)\\
&\qquad\qquad + \tfrac{1}{2}\left(\sum_{x\in X}\big(A(x)^{+}-\lambda(x)\big) + \sum_{x\in X}\big(B(x)^{+}-\mu(x)\big)\right)\\
&= \tfrac{1}{2}\left(\sum_{x\in X}\big(A(x)^{+}-\lambda(x)\big) + \sum_{x\in X}\big(B(x)^{+}-\mu(x)\big)\right) \geq 0.
\end{aligned}
$$

Hence $\mathscr{A}\Cap\mathscr{B} = \langle A\cap B, \lambda\vee\mu\rangle$ is almost stable in X. □

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SOME IDENTITIES OF CHEBYSHEV POLYNOMIALS ARISING FROM NON-LINEAR DIFFERENTIAL EQUATIONS

TAEKYUN KIM, DAE SAN KIM, JONG-JIN SEO, AND DMITRY V. DOLGY

Abstract. In this paper, we investigate some properties of Chebyshev polynomials arising from non-linear differential equations. From our investigation, we derive some new and interesting identities on Chebyshev polynomials.

1. Introduction

As is well known, the Chebyshev polynomials of the first kind, $T_n(x)$, $(n \geq 0)$, are defined by the generating function

$$\frac{1-t^2}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}, \quad (\text{see } [1, 3, 5, 8, 17, 21]). \tag{1.1}$$

The higher-order Chebyshev polynomials are given by the generating function

$$\left(\frac{1-t^2}{1-2xt+t^2} \right)^{\alpha} = \sum_{n=0}^{\infty} T_n^{(\alpha)}(x) t^n, \tag{1.2}$$

and Chebyshev polynomials of the second kind are denoted by U_n and given by generating function

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x) t^n, \quad (\text{see } [1, 7, 12, 17]). \tag{1.3}$$

The higher-order Chebyshev polynomials of the second kind are also defined by

$$\left(\frac{1}{1-2xt+t^2} \right)^{\alpha} = \sum_{n=0}^{\infty} U_n^{(\alpha)}(x) t^n. \tag{1.4}$$

The Chebyshev polynomials of the third kind are defined by the generating function

$$\frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty} V_n(x) t^n, \quad (\text{see } [1, 7, 8, 17]). \tag{1.5}$$

and the higher-order Chebyshev polynomials of the third kind are also given by the generating function

$$\left(\frac{1-t}{1-2xt+t^2} \right)^{\alpha} = \sum_{n=0}^{\infty} V_n^{(\alpha)}(x) t^n. \tag{1.6}$$

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Finally, we introduce the Chebyshev polynomials of the fourth kind defined by the generating function

$$\frac{1+t}{1-2xt+t^2}=\sum_{n=0}^{\infty}W_n\left(x\right)t^n. \tag{1.7}$$

The higher-order Chebyshev polynomials of the fourth kind are defined by

$$\left(\frac{1+t}{1-2xt+t^2}\right)^{\alpha}=\sum_{n=0}^{\infty}W_n^{(\alpha)}\left(x\right)t^n. \tag{1.8}$$

It is well known that the Legendre polynomials are defined by the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}p_n\left(x\right)t^n,\quad \left(\text{see [2, 20]}\right). \tag{1.9}$$

Chebyshev polynomials are important in approximation theory because the roots of the Chebyshev polynomials of the first kind, which are also called Chebyshev nodes, are used as nodes in polynomial nodes (see [19]).

The Chebyshev polynomials of the first kind and of the second kind are solutions of the following Chebyshev differential equations

$$\left(1-x^2\right)y''-xy'+n^2y=0, \tag{1.10}$$

and

$$\left(1-x^2\right)y''-3xy'+n\left(n+2\right)y=0. \tag{1.11}$$

These equations are special cases of the Strum-Liouville differential equation (see [1–3]).

The Chebyshev polynomials of the first kind can be defined by the contour integral

$$T_n\left(z\right)=\frac{1}{4\pi i}\oint\frac{\left(1-t^2\right)}{1-2tz+t^2}t^{-n-1}dt, \tag{1.12}$$

where the contour encloses the origin and is traversed in a counterclockwise direction (see [1, 19, 21]). The formula for $T_n\left(x\right)$ is given by

$$T_n\left(x\right)=\sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n}{2m}x^{n-2m}\left(x^2-1\right)^m. \tag{1.13}$$

From (1.3), we note that

$$2\left(x-t\right)\left(1-2xt+t^2\right)^{-2}=\sum_{n=0}^{\infty}nU_n\left(x\right)t^{n-1}. \tag{1.14}$$

Thus, by (1.14), we get

$$\left(2xt-2t^2\right)\left(1-2xt+t^2\right)^{-2}=\sum_{n=0}^{\infty}nU_n\left(x\right)t^n. \tag{1.15}$$

From (1.3) and (1.15), we can derive the following equation:

$$\frac{\left(2xt-2t^2\right)+\left(1-2xt+t^2\right)}{\left(1-2xt+t^2\right)^2}=\frac{1-t^2}{\left(1-2xt+t^2\right)^2} \tag{1.16}$$

$$= \sum_{n=0}^{\infty} (n+1) U_n(x) t^n.$$

Note that

$$\begin{aligned} (1.17) \qquad & \frac{1-t^2}{(1-2xt+t^2)^2} \\ &= \left(\frac{1-t^2}{1-2xt+t^2}\right)\left(\frac{1}{1-2xt+t^2}\right) \\ &= \left(\sum_{l=0}^{\infty} T_l(x) t^l\right)\left(\sum_{m=0}^{\infty} U_m(x) t^m\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} T_l(x) U_{n-l}(x)\right) t^n. \end{aligned}$$

From (1.16) and (1.17), we have

$$U_n(x) = \frac{1}{n+1}\sum_{l=0}^{n} T_l(x) U_{n-l}(x).$$

The Chebyshev polynomials have been studied by many authors in the several areas (see [1–21]).

In [11], Kim-Kim studied non-linear differential equations arising from Changhee polynomials and numbers related to Chebyshev poynomials.

In this paper, we study non-linear differential equations arising from Chebyshev polynomials and give some new and explicit formulas for those polynomials.

2. Differential equations arising from Chebyshev polynomials and their applications

Let

$$F = F(t,x) = \frac{1}{1-2tx+t^2}. \tag{2.1}$$

Then, by (1.1), we get

$$F^{(1)} = \frac{d}{dt}F(t,x) = 2(x-t)F^2. \tag{2.2}$$

From (2.2), we note that

$$2F^2 = (x-t)^{-1} F^{(1)}. \tag{2.3}$$

By using (2.3) and (2.2), we obtain the following equations:

$$2^2 \cdot 2F^3 = (x-t)^{-3} F^{(1)} + (x-t)^{-2} F^{(2)}, \tag{2.4}$$

$$2^3 \cdot 2 \cdot 3F^4 = 3(x-t)^{-5} F^{(1)} + 3(x-t)^{-4} F^{(2)} + (x-t)^{-3} F^{(3)} \tag{2.5}$$

and

$$\begin{aligned} 2^4 \cdot 2 \cdot 3 \cdot 4F^5 = {} & 3\cdot 5(x-t)^{-6} F^{(1)} + 3\cdot 5(x-t)^{-6} F^{(2)} \\ & + (3\cdot 2)(x-t)^{-5} F^{(3)} + (x-t)^{-4} F^{(4)}, \end{aligned} \tag{2.6}$$

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where

$$F^N = \underbrace{F \times \cdots \times F}_{N-\text{times}} \quad \text{and} \quad F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x).$$

Continuing this process, we set

$$2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} F^{(i)}, \tag{2.7}$$

where $N \in \mathbb{N}$.

From (2.7), we note that

$$\begin{aligned} &2^N N! F^N (N+1) F^{(1)} \\ &= \sum_{i=1}^{N} a_i(N)(2N-i)(x-t)^{i-2N-1} F^{(i)} + \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} F^{(i+1)}. \end{aligned} \tag{2.8}$$

By (2.2) and (2.8), we get

$$\begin{aligned} &2^N N!(N+1) F^N \left(2(x-t) F^2\right) \\ &= \sum_{i=1}^{N} a_i(N)(2N-i)(x-t)^{i-2N-1} F^{(i)} \\ &\quad + \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} F^{(i+1)}. \end{aligned} \tag{2.9}$$

Thus, from (2.9), we have

$$\begin{aligned} &2^{N+1}(N+1)! F^{N+2} \\ &= \sum_{i=1}^{N} a_i(N)(2N-i)(x-t)^{i-2(N+1)} F^{(i)} \\ &\quad + \sum_{i=2}^{N+1} a_{i-1}(N)(x-t)^{i-2(N+1)} F^{(i)}. \end{aligned} \tag{2.10}$$

On the other hand, by replacing N by $N+1$, in (2.7), we get

$$2^{N+1}(N+1)! F^{N+2} = \sum_{i=1}^{N+1} a_i(N+1)(x-t)^{i-2(N+1)} F^{(i)}. \tag{2.11}$$

Comparing the coefficients on both sides of (2.10) and (2.11), we have

$$a_1(N+1) = (2N-1) a_1(N), \tag{2.12}$$

$$a_{N+1}(N+1) = a_N(N), \tag{2.13}$$

and

$$a_i(N+1) = a_{i-1}(N) + (2N-i) a_i(N), \quad (2 \le i \le N). \tag{2.14}$$

Moreover, by (2.4) and (2.7), we get

$$2F^2 = (x-t)^{-1} F^{(1)} = a_1(1)(x-t)^{-1} F^{(1)}. \tag{2.15}$$

By comparing the coefficients on both sides of (2.15), we get

$$a_1(1) = 1. \tag{2.16}$$

Now, by (2.12) and (2.16), we have

$$\begin{aligned} a_1(N+1) &= (2N-1)\,a_1(N) \\ &= (2N-1)(2N-3)\,a_1(N-1) \\ &= (2N-1)(2N-3)(2N-5)\,a_1(N-2) \\ &\;\vdots \\ &= (2N-1)(2N-3)(2N-5)\cdots 1\cdot a_1(1) \\ &= (2N-1)!!, \end{aligned} \tag{2.17}$$

where $(2N-1)!!$ is Arfken's double factorial.

From (2.13), we easily note that

$$a_{N+1}(N+1) = a_N(N) = \cdots = a_1(1) = 1. \tag{2.18}$$

For $2 \le i \le N$, from (2.14), we can derive the following equation:

$$\begin{aligned} a_i(N+1) &= a_{i-1}(N) + (2N-i)\,a_i(N) \\ &= a_{i-1}(N) + (2N-i)\,a_{i-1}(N-1) + (2N-i)(2N-2-i)\,a_i(N-1) \\ &\;\vdots \\ &= \sum_{k=0}^{N-i}\left(\prod_{l=0}^{k-1}\left(2(N-l)-i\right)\right)a_{i-1}(N-k) + \prod_{l=0}^{N-i}\left(2(N-l)-i\right)a_i(i) \\ &= \sum_{k=0}^{N-i} 2^k\left(N-\frac{i}{2}\right)_k a_{i-1}(N-k) + 2^{N-i+1}\left(N-\frac{i}{2}\right)_{N-i+1} \\ &= \sum_{k=0}^{N-i+1} 2^k\left(N-\frac{i}{2}\right)_k a_{i-1}(N-k), \end{aligned} \tag{2.19}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \ge 1)$ and $(x)_0 = 1$.

As the above is also valid for $i = N+1$, by (2.19), we get

$$a_i(N+1) = \sum_{k=0}^{N+1-i} 2^k\left(N-\frac{i}{2}\right)_k a_{i-1}(N-k), \tag{2.20}$$

where $2 \le i \le N+1$.

Now, we give an explicit expression for $a_i(N+1)$.

From (2.17) and (2.20), we can derive the following equations:

$$\begin{aligned} a_2(N+1) &= \sum_{k_1=0}^{N-1} 2^{k_1}\left(N-\frac{2}{2}\right)_{k_1} a_1(N-k_1) \\ &= \sum_{k_1=0}^{N-1} 2^{k_1}\left(N-\frac{2}{2}\right)_{k_1} \left(2(N-k_1-1)-1\right)!!, \end{aligned} \tag{2.21}$$

(2.22)

$$\begin{aligned}a_3(N+1) &= \sum_{k_2=0}^{N-2} 2^{k_2}\left(N-\frac{3}{2}\right)_{k_2} a_2(N-k_2)\\ &= \sum_{k_2=0}^{N-2}\sum_{k_1=0}^{N-2-k_2} 2^{k_1+k_2}\left(N-\frac{3}{2}\right)_{k_2}\left(N-k_2-\frac{4}{2}\right)_{k_1}(2(N-2-k_1-k_2)-1)!!,\end{aligned}$$

and

(2.23)

$$\begin{aligned}a_4(N+1) &= \sum_{k_3=0}^{N-3} 2^{k_3}\left(N-\frac{4}{2}\right)_{k_3} a_3(N-k_3)\\ &= \sum_{k_3=0}^{N-3}\sum_{k_2=0}^{N-3-k_3}\sum_{k_1=0}^{N-3-k_3-k_2} 2^{k_1+k_2+k_3}\left(N-\frac{4}{2}\right)_{k_3}\left(N-k_3-\frac{5}{2}\right)_{k_2}\left(N-k_3-k_2-\frac{6}{2}\right)_{k_1}\\ &\quad\times(2(N-3-k_1-k_2-k_3)-1)!!.\end{aligned}$$

Thus, we see that, for $2\le i\le N+1$,

(2.24)

$$\begin{aligned}a_i(N+1) &= \sum_{k_{i-1}=0}^{N-i+1}\sum_{k_{i-2}=0}^{N-i+1-k_{i-1}}\cdots\sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2} 2^{\sum_{j=1}^{i-1}k_j}\\ &\quad\times\prod_{j=2}^{i}\left(N-\sum_{l=j}^{i-1}k_l-\frac{2i-j}{2}\right)_{k_{j-1}}\left(2\left(N-i+1-\sum_{j=1}^{i-1}k_j\right)-1\right)!!.\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 1. *The nonlinear differential equations*

$$2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N}F^{(i)}, \quad (N\in\mathbb{N})$$

has a solution $F=F(t,x)=\frac{1}{1-2tx+t^2}$, *where*

$a_1(N)=(2N-3)!!$,

$$\begin{aligned}a_i(N) &= \sum_{k_{i-1}=0}^{N-i}\sum_{k_{i-2}=0}^{N-i-k_{i-1}}\cdots\sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2} 2^{\sum_{j=1}^{i-1}k_j}\\ &\quad\times\prod_{j=2}^{i}\left(N-\sum_{l=j}^{i-1}k_l-\frac{2i+2-j}{2}\right)_{k_{j-1}}\left(2\left(N-i-\sum_{j=1}^{i-1}k_j\right)-1\right)!!\end{aligned}$$

$(2\le i\le N)$.

From (1.3) and (1.9), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}U_n(x)t^n &= \frac{1}{1-2xt+t^2}\end{aligned} \tag{2.25}$$

$$
\begin{aligned}
&= \left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^2 \\
&= \left(\sum_{l=0}^{\infty} p_l(x)\, t^l\right)\left(\sum_{m=0}^{\infty} p_m(x)\, t^m\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} p_l(x)\, p_{n-l}(x)\right) t^n.
\end{aligned}
$$

Thus, from (2.25), we have

$$U_n(x) = \sum_{l=0}^{n} p_l(x)\, p_{n-l}(x).$$

From (1.4), we obtain

$$2^N N! F^{N+1} = 2^N N! \sum_{n=0}^{\infty} U_n^{(N+1)}(x)\, t^n. \tag{2.26}$$

On the other hand, by Theorem 1, we get

(2.27)

$$
\begin{aligned}
2^N N! F^{N+1} &= \sum_{i=1}^{N} a_i(N)\,(x-t)^{i-2N} F^{(i)} \\
&= \sum_{i=1}^{N} a_i(N)\left(\sum_{m=0}^{\infty}\binom{2N+m-i-1}{m} x^{i-2N-m} t^m\right)\left(\sum_{l=0}^{\infty} U_{i+l}(x)\,(l+i)_i\, t^l\right) \\
&= \sum_{i=1}^{N} a_i(N) \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\binom{2N+n-l-i-1}{n-l} x^{i-2N-n+l} U_{l+i}(x)\,(l+i)_i\right\} t^n \\
&= \sum_{n=0}^{\infty}\left\{\sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n}\binom{2N+n-l-i-1}{n-l} x^{i+l-2N-n} U_{i+l}(x)\,(l+i)_i\right\} t^n.
\end{aligned}
$$

Comparing the coefficients on the both sides of (2.26) and (2.27), we obtain the following theorem.

Theorem 2. *For $N \in \mathbb{N}$, and $n \in \mathbb{N}\cup\{0\}$, the following identity holds.*

$$U_n^{(N+1)}(x) = \frac{1}{2^N N!}\sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n}\binom{2N+n-l-i-1}{n-l} U_{l+i}(x)\, x^{i+l-2N-n}\,(l+i)_i.$$

The higher-order Legendre polynomials are given by the generating function

$$\left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^{\alpha} = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n. \tag{2.28}$$

Thus, by 1.4 and (2.27), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} U_n^{(\alpha)}(x)\, t^n \\
&= \left(\frac{1}{1-2xt+t^2}\right)^{\alpha}
\end{aligned}
\tag{2.29}
$$

$$
\begin{aligned}
&= \left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^{2\alpha} \\
&= \left(\sum_{l=0}^{\infty} p_l^{(\alpha)}(x)\, t^l\right)\left(\sum_{m=0}^{\infty} p_m^{(\alpha)}(x)\, t^m\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} p_l^{(\alpha)}(x)\, p_{n-l}^{(\alpha)}(x)\right) t^n.
\end{aligned}
$$

From (2.29), we note that

$$
U_n^{(\alpha)}(x) = \sum_{l=0}^{n} p_l^{(\alpha)}(x)\, p_{n-l}^{(\alpha)}(x). \tag{2.30}
$$

Therefore, we obtian the following corollaries.

Corollary 3. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n} p_l^{(N+1)} p_{n-l}^{(N+1)}(x) \\
&= \frac{1}{2^N N!}\sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \binom{2N+n-l-i-1}{n-l} U_{l+i}(x)\,(l+i)_i\, x^{i+l-2N-n}.
\end{aligned}
$$

Corollary 4. *For $N \in \mathbb{N}$ and $n \in \mathbb{N}$, we have*

$$
\begin{aligned}
&U_n^{(N+1)}(x) \\
&= \frac{1}{2^N N!}\sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n}\sum_{j=0}^{l+i} \binom{2N+n-l-i-1}{n-l} x^{i+l-2N-n}\,(l+i)_i\,(x)\, p_{l+i-j}(x).
\end{aligned}
$$

By (1.6), we get

$$
\begin{aligned}
&2^N N! F^{N+1} \\
&= 2^N N!\,(1-t)^{-N-1}\left(\frac{1-t}{1-2xt+t^2}\right)^{N+1} \\
&= 2^N N!\left(\sum_{m=0}^{\infty}\binom{N+m}{m} t^m\right)\left(\sum_{l=0}^{\infty} V_l^{(N+1)}(x)\, t^l\right) \\
&= 2^N N!\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{N+n-l}{n-l} V_l^{(N+1)}(x)\right) t^n.
\end{aligned} \tag{2.31}
$$

On the other hand, by Theorem 1, we have

$$
\begin{aligned}
2^N N! F^{N+1} &= \sum_{i=1}^{N} a_i(N)\,(x-t)^{i-2N} F^{(i)} \\
&= \sum_{i=1}^{N} a_i(N)\,(x-t)^{i-2N}\left(\frac{d}{dt}\right)^i\left(\frac{1}{1-t}\cdot\frac{1-t}{1-xt+t^2}\right).
\end{aligned} \tag{2.32}
$$

From Leibniz formula, we note that

$$
\left(\frac{d}{dt}\right)^i\left(\frac{1-t}{1-2xt+t^2}\cdot\frac{1}{1-t}\right) \tag{2.33}
$$

$$
\begin{aligned}
&= \sum_{l=0}^{i} \binom{i}{l} \left(\left(\frac{d}{dt} \right)^{i-l} \frac{1}{1-t} \right) \left(\left(\frac{d}{dt} \right)^{l} \frac{1-t}{1-2xt+t^2} \right) \\
&= \sum_{l=0}^{i} \binom{i}{l} (i-l)! \, (1-t)^{-i+l-1} \left(\frac{d}{dt} \right)^{l} \left(\frac{1-t}{1-2xt+t^2} \right) \\
&= \sum_{l=0}^{i} \binom{i}{l} (i-l)! \sum_{s=0}^{\infty} \binom{i-l+s}{s} t^s \sum_{p=0}^{\infty} V_{p+l}(x) \, (p+l)_l \, t^p \\
&= \sum_{l=0}^{i} \frac{i!}{l!} \sum_{s=0}^{\infty} \binom{i-l+s}{s} t^s \sum_{p=0}^{\infty} V_{p+l}(x) \, (p+l)_l \, t^p.
\end{aligned}
$$

By (2.32) and (2.33), we get

$$
\begin{aligned}
&2^N N! F^{N+1} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} \binom{2N+m-i-1}{m} \binom{i-l+s}{s} \right. \\
&\quad \left. \times (p+l)_l \, x^{i-2N-m} V_{p+l}(x) \right\} t^n.
\end{aligned} \tag{2.34}
$$

Therefore, by (2.31) and (2.34), we obtain the following theorem.

Theorem 5. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have the following identity:*

$$
\begin{aligned}
&\sum_{l=0}^{n} \binom{N+n-l}{n-l} V_l^{(N+1)}(x) \\
&= \frac{1}{2^N N!} \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} \binom{2N+m-i-1}{m} \binom{i-l+s}{s} (p+l)_l \\
&\quad \times x^{i-2N-m} V_{p+l}(x).
\end{aligned}
$$

From (1.8), we note that

$$
\begin{aligned}
&2^N N! F^{N+1} \\
&= 2^N N! \, (1+t)^{-N-1} \left(\frac{1+t}{1-2xt+t^2} \right)^{N+1} \\
&= 2^N N! \left(\sum_{m=0}^{\infty} \binom{N+m}{m} (-1)^m t^m \right) \left(\sum_{l=0}^{\infty} W_l^{(N+1)}(x) \, t^l \right) \\
&= 2^N N! \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} (-1)^{n-l} \binom{N+n-l}{n-l} W_l^{(N+1)}(x) \right) t^n.
\end{aligned} \tag{2.35}
$$

On the other hand, by Theorem 1, we get

$$
2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N) \, (x-t)^{i-2N} \left(\frac{d}{dt} \right)^{i} \left\{ \frac{1}{1+t} \cdot \frac{1+t}{1-2xt+t^2} \right\}. \tag{2.36}
$$

Now, we observe that

$$
\left(\frac{d}{dt} \right)^{i} \left\{ \left(\frac{1}{1+t} \right) \left(\frac{1+t}{1-2xt+t^2} \right) \right\} \tag{2.37}
$$

$$
\begin{aligned}
&= \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} (i-l)! \left(\frac{1}{1+t}\right)^{i-l+1} \left(\frac{d}{dt}\right)^{l} \left(\frac{1+t}{1-2xt+t^2}\right) \\
&= \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} (i-l)! \sum_{s=0}^{\infty} \binom{i-l+s}{s} (-1)^{s} t^{s} \sum_{p=0}^{\infty} W_{p+l}(x) (p+l)_{l} t^{p}.
\end{aligned}
$$

From (2.36) and (2.37), we have

$$
\begin{aligned}
&2^N N! F^{N+1} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{i} (-1)^{i-l} \frac{i!}{l!} \sum_{m+s+p=n} (-1)^{s} \binom{2N+m-i-1}{m} \right. \\
&\quad \left. \times \binom{i-l+s}{s} (p+l)_{l} x^{i-2N-m} W_{p+l}(x) \right\} t^{n}.
\end{aligned} \tag{2.38}
$$

Therefore, by (2.35) and (2.38), we obtain the following theorem.

Theorem 6. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, the following identity is valid:*

$$
\begin{aligned}
&\sum_{l=0}^{n} (-1)^{n-l} \binom{N+n-l}{n-l} W_{l}^{(N+1)}(x) \\
&= \frac{1}{2^N N!} \sum_{i=1}^{N} \sum_{l=0}^{i} (-1)^{i-l} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} (-1)^{s} \binom{2N+m-i-1}{m} \\
&\quad \times \binom{i-l+s}{s} (p+l)_{l} x^{i-2N-m} W_{p+l}(x).
\end{aligned}
$$

From (1.1), we have

$$
\begin{aligned}
&2^N N! F^{N+1} \\
&= 2^N N! \left(\frac{1}{1-t^2} \cdot \frac{1-t^2}{1-2xt+t^2}\right)^{N+1} \\
&= 2^N N! \left(\frac{1}{1-t}\right)^{N+1} \left(\frac{1}{1+t}\right)^{N+1} \left(\frac{1-t^2}{1-2xt+t^2}\right)^{N+1} \\
&= 2^N N! \left(\sum_{l=0}^{\infty} \binom{N+l}{l} t^{l}\right) \left(\sum_{m=0}^{\infty} \binom{m+N}{m} (-1)^{m} t^{m}\right) \left(\sum_{p=0}^{\infty} T_{p}^{(N+1)}(x) t^{p}\right) \\
&= 2^N N! \sum_{n=0}^{\infty} \left(\sum_{l+m+p=n} \binom{N+l}{l} \binom{m+N}{m} (-1)^{m} T_{p}^{(N+1)}(x) \right) t^{n}.
\end{aligned} \tag{2.39}
$$

On the other hand, by Theorem 1, we get

$$
\begin{aligned}
&2^N N! F^{N+1} \\
&= \sum_{i=1}^{N} a_i(N) (x-t)^{i-2N} F^{(i)}
\end{aligned} \tag{2.40}
$$

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$$= \frac{1}{2}\sum_{i=1}^{N} a_i(N)(x-t)^{i-2N}\left(\frac{d}{dt}\right)^i\left\{\left(\frac{1}{1-t}+\frac{1}{1+t}\right)\frac{1-t^2}{1-2xt+t^2}\right\}.$$

From Leibniz formula, we note that the following equations:

$$\begin{aligned}(2.41)\qquad &\left(\frac{d}{dt}\right)^i\left\{\left(\frac{1}{1-t}\right)\cdot\left(\frac{1-t^2}{1-2xt+t^2}\right)\right\}\\ &=\sum_{l=0}^{i}\binom{i}{l}(i-l)!\sum_{s=0}^{\infty}\binom{i+s-l}{s}t^s\sum_{p=0}^{\infty}T_{p+l}(x)(p+l)_l\,t^p,\end{aligned}$$

and

$$\begin{aligned}(2.42)\qquad &\left(\frac{d}{dt}\right)^i\left\{\left(\frac{1}{1+t}\right)\left(\frac{1-t^2}{1-2xt+t^2}\right)\right\}\\ &=\sum_{l=0}^{i}\binom{i}{l}(i-l)!(-1)^{i-l}\sum_{s=0}^{\infty}\binom{i-l+s}{s}(-1)^s t^s\sum_{p=0}^{\infty}T_{p+l}(x)(p+l)_l\,t^p.\end{aligned}$$

By (2.40), (2.41), and (2.42), we obtain

(2.43)

$$\begin{aligned}&2^N N! F^{N+1}\\ &=\frac{1}{2}\sum_{i=1}^{N}a_i(N)(x-t)^{i-2N}\sum_{l=0}^{i}\binom{i}{l}(i-l)!\sum_{s=0}^{\infty}\binom{i+s-l}{s}t^s\sum_{k=0}^{\infty}T_{p+l}(x)(p+l)_l\,t^p\\ &+\frac{1}{2}\sum_{i=1}^{N}a_i(N)(x-t)^{i-2N}\sum_{l=0}^{i}\binom{i}{l}(i-l)!(-1)^{i-l}\sum_{s=0}^{\infty}\binom{i-l+s}{s}(-1)^s t^s\\ &\times\sum_{p=0}^{\infty}T_{p+l}(x)(p+l)_l\,t^p\\ &=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{i=1}^{N}\sum_{l=0}^{i}a_i(N)\frac{i!}{l!}\sum_{m+s+p=n}\binom{2N+m-i-1}{m}\binom{i+s-l}{s}(p+l)_l\\ &\times x^{i-2N-m}T_{p+l}(x)t^n+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{i=1}^{N}\sum_{l=0}^{i}a_i(N)\frac{i!}{l!}(-1)^{i-l}\\ &\times\sum_{m+s+p=n}(-1)^s\binom{2N+m-i-1}{m}\binom{i+s-l}{s}(p+l)_l\,x^{i-2N-m}T_{p+l}(x)t^n.\end{aligned}$$

Therefore, by (2.39) and (2.43), we obtain the following theorem.

Theorem 7. *For $n\in\mathbb{N}\cup\{0\}$ and $N\in\mathbb{N}$, we have the following identity*

$$\begin{aligned}&2^{N+1}N!\sum_{s+m+p=n}\binom{N+s}{s}\binom{m+N}{m}(-1)^m T_p^{(N+1)}(x)\\ &=\sum_{i=1}^{N}\sum_{l=0}^{i}a_i(N)\frac{i!}{l!}\sum_{m+s+p=n}\binom{2N+m-i-1}{m}\binom{i+s-l}{s}(p+l)_l\end{aligned}$$

$$\times x^{i-2N-m} T_{p+l}(x) + \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} (-1)^{i-l} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m}$$
$$\times \binom{i+s-l}{s} (p+l)_l \, x^{i-2N-m} T_{p+l}(x).$$

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Blowup singularity for a degenerate and singular parabolic equation with nonlocal boundary *

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Abstract

In this paper, we are interested in the blowup behavior of the solution to a degenerate and singular parabolic equation

$$u_t = (x^\alpha u_x)_x + \int_0^l u^p dx - ku^q, \quad (x,t) \in (0,l) \times (0,+\infty)$$

with nonlocal boundary condition

$$u(0,t) = \int_0^l f(x)\, u(x,t)dx, \quad u(l,t) = \int_0^l g(x)\, u(x,t)dx, \quad t \in (0,+\infty),$$

where p, $q \in [1,\infty)$, $\alpha \in [0,1)$ and $k \in (0,\infty)$. In view of comparison principle, we investigate the conditions on the global existence and blowup of the solutions. Moreover, under some suitable hypotheses, we discuss the global blowup and the uniform blowup profile of the blowup solution.

Keywords: Degenerate and singular parabolic equation; Nonlocal boundary; Global existence; Blowup singularity

Mathematics Subject Classification(2000) : 35K50, 35K55, 35K65

1 Introduction

The main purpose of this paper is to deal with the blowup singularity of the following degenerate and singular parabolic equation with nonlocal source and nonlocal boundary condition

$$\begin{cases} u_t = (x^\alpha u_x)_x + \int_0^l u^p dx - ku^q, & (x,t) \in (0,l) \times (0,+\infty), \\ u(0,t) = \int_0^l f(x)\, u(x,t)dx, & t \in (0,+\infty), \\ u(l,t) = \int_0^l g(x)\, u(x,t)dx, & t \in (0,+\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in [0,l], \end{cases} \tag{1.1}$$

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where $0 \leq \alpha < 1$, p, $q \geq 1$, $k > 0$, the weight functions $f(x)$ and $g(x)$ in the boundary condition are nonnegative continuous on $[0, l]$ and not identically zero, and the initial value $u_0(x) \in C^{2+\delta}(0, l) \cap C[0, l]$ with $0 < \delta < 1$, and satisfies the compatibility conditions. It is obvious that the equation in problem (1.1) is singular and degenerate because the coefficients of u_x and u_{xx} may tend to ∞ and 0 as $x \to 0$.

This type equation in problem (1.1) can be viewed as a model which describes the conduction of heat related to the geometric shape of the body (see [1] and the references therein for more details of the physical background). On the other hand, lots of physical phenomena were formulated into nonlocal mathematical models, for example, Day [4, 5] derived a heat equation with nonlocal boundary in the study of the heat conduction with thermoelastcity. From then on, a lot of mathematicians devoted to studying the blowup behavior of the solutions of various parabolic problems with nonlocal boundary conditions (see [6, 7, 8, 9, 10, 11, 13, 15, 16, 21]).

The blowup phenomenon related to problem (1.1) attracted extensive attention of mathematicians in the past several decades (see [2, 3, 12, 18, 20, 22, 23]), but most of them considered the problems with null Dirichlet boundary conditions. Inspired by the works mentioned above, we consider problem (1.1), and our main attention is focused on evaluating the effects of the weighted nonlocal boundary, the nonlocal source and absorption term on the asymptotic blowup behavior of the solution $u(x, t)$ of problem (1.1). Compared with [3] and [18], we need more skills to handle the difficulties, which are produced by the degeneration and singularity of problem (1.1), and the appearance of the nonlinear nonlocal boundary condition.

Before stating our main results, for the sake of convenience, we denote

$$\mathcal{N} = \max\left\{\int_0^l f(x)dx, \int_0^l g(x)dx\right\},$$

and let λ_1 be the first eigenvalue and $\zeta_1(x)$ be the corresponding eigenfunction of the following eigenvalue problem

$$-\left(x^\alpha \zeta_x\right)_x = \lambda_1 \zeta, \quad 0 < x < l; \quad \zeta(0) = \zeta(l) = 0. \tag{1.2}$$

Indeed, from [3, 14], we know that the principle eigenvalue λ_1 of the eigenvalue problem (1.2) is the first zero of

$$J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda}}{2-\alpha} l^{\frac{2-\alpha}{2}}\right) = 0,$$

and $\zeta_1(x)$ can be expressed in an explicit form as follows

$$\zeta_1(x) = a x^{\frac{1-\alpha}{2}} J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}}\right), \tag{1.3}$$

where $J_{\frac{1-\alpha}{2-\alpha}}$ is Bessel function of the first kind of order $\frac{1-\alpha}{2-\alpha}$, and a is an appropriate positive parameter such that $\|\zeta_1(x)\|_{L^1([0,l])} = 1$. Furthermore, we know easily that $\zeta_1(x)$ is a positive smooth function in $(0, l)$, and in light of

$$\frac{d}{d\tau} J_\vartheta(\tau) = \frac{\vartheta}{2} J_\vartheta(\tau) - J_{\vartheta+1}(\tau),$$

we can deduce that, for $x \in (0, l)$,

$$\frac{d}{dx}\zeta_1(x) = \frac{a(1-\alpha)}{2}\left(1 + \frac{\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}}\right) x^{-\frac{1+\alpha}{2}} J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}}\right) - a\sqrt{\lambda_1} x^{\frac{1-2\alpha}{2}} J_{\frac{3-2\alpha}{2-\alpha}}\left(\frac{\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}}\right).$$

And hence, by making use of

$$J_\vartheta(\tau) \to \frac{1}{\Gamma(\vartheta+1)}\left(\frac{\tau}{2}\right)^\vartheta \text{ as } \tau \to 0,$$

where $\Gamma(\cdot)$ is the Gamma function, we find that

$$\lim_{x\to 0^+} \zeta_1(x) = 0$$

and

$$\lim_{x\to 0^+} \frac{d}{dx}\zeta_1(x) = \frac{a(1-\alpha)}{2\Gamma\left(\frac{3-2\alpha}{2-\alpha}\right)}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha}\right)^{\frac{1-\alpha}{2-\alpha}},$$

which imply that

$$\sup_{x\in[0,l]} \zeta_1(x) < \infty \text{ and } \sup_{x\in[0,l]} \frac{d}{dx}\zeta_1(x) < \infty. \tag{1.4}$$

The main results of this paper are stated as follows.

Theorem 1.1. *Assume that $q > p \geq 1$, then all the solutions of problem (1.1) exist globally.*

Theorem 1.2. *Assume that $p > q \geq 1$, then problem (1.1) admits blowup solutions as well as global solutions. More precisely,*

(i) if $\mathcal{N} \leq 1$, then the solution exists globally provided that $u_0(x) \leq \left(\frac{k}{l}\right)^{\frac{1}{p-q}}$;

(ii) if $\mathcal{N} > 1$, then the solution of problem (1.1) blows up in finite time provided that $u_0(x) > \eta_1$, where $\eta_1 > 1$ is an appropriate constant;

(iii) there is a suitable positive small constant η_2 such that the solution $u(x,t)$ of problem (1.1) blows up in finite time for any $f(x)$ and $g(x)$ provided that

$$u_0(x) > \eta_2^{-\xi}\left(\frac{l}{2-\alpha}x^{1-\alpha} - \frac{1}{2-\alpha}x^{2-\alpha}\right),$$

where $\xi > \frac{1}{p-1}$.

Theorem 1.3. *Assume that $p = q > 1$. The solution $u(x,t)$ of problem (1.1) exists globally provided that $\mathcal{N} < 1$ and $u_0(x) \leq \epsilon_1\mathcal{N}$, where ϵ_1 is given by (3.13). For any nonnegative weight functions $f(x)$ and $g(x)$, the solution $u(x,t)$ of problem (1.1) blows up in finite time provided that the initial value $u_0(x)$ is sufficiently large.*

Remark 1.1. *If $p = q = 1$, one can show that problem (1.1) has no blowup solution.*

The remaining part is devote to discussing the global blowup and the uniform blowup profile of the blowup solution, to this end, we assume that $p > q \geq 1$ (or $p = q > 1$), $\mathcal{N} \leq 1$ and $u_0(x)$ is large enough in some suitable sense. Moreover, we assume that $u_0(x)$ satisfies extra

$$(x^\alpha u_{0x})_x + \int_0^l u_0^p dx - ku_0^q \geq 0 \text{ for } x \in (0,l), \tag{1.5}$$

$$(x^\alpha u_{0x})_x \leq 0 \text{ in } (0,l), \tag{1.6}$$

and

$$\lim_{x\to 0^+}\left[(x^\alpha u_{0x})_x + \int_0^l u_0^p dx - ku_0^q\right] = \lim_{x\to l^-}\left[(x^\alpha u_{0x})_x + \int_0^l u_0^p dx - ku_0^q\right] = 0. \tag{1.7}$$

Theorem 1.4. *Assume that $p > q \geq 1$ and $\mathcal{N} \leq 1$. Suppose that hypotheses (1.5), (1.6) and (1.7) hold. Then*

$$u(x,t) \sim [l(p-1)(T-t)]^{-\frac{1}{p-1}} \qquad a.e.\ in\ (0,l)\ \ as\ t \to T,$$

where T is the blowup time.

Corollary 1.1. *Under the assumptions of Theorem 1.4, we see that the blowup set of the blowup solution $u(x,t)$ of problem (1.1) is the whole interval $(0,l)$.*

Theorem 1.5. *Assume that $p = q > 1$, $\mathcal{N} \leq 1$ and $0 < k < l$. Suppose that hypotheses (1.5), (1.6) and (1.7) hold. Then*

$$u(x,t) \sim [l(p-1)(T-t)]^{-\frac{1}{p-1}} \qquad a.e.\ in\ (0,l)\ \ as\ t \to T,$$

where T is the blowup time.

Corollary 1.2. *Under the assumptions of Theorem 1.5, we know that the blowup set of the blowup solution $u(x,t)$ of problem (1.1) is the whole interval $(0,l)$.*

The rest of this paper is organized as follows. In Section 2, we shall state the comparison principle and local existence theorem for problem (1.1). In section 3, we shall concern with the conditions on the global existence of solution and prove Theorems 1.1, 1.2 and 1.3. In section 4, we shall estimate the uniform blowup profile and give the proofs of Theorems 1.4 and 1.5.

2 Comparison principle and local existence

In this section, we will establish a suitable comparison principle for problem (1.1) and state the existence and uniqueness result on the local solution. For the sake of simplify, we denote $I_T = (0,l) \times (0,T)$ and $\overline{I}_T = [0,l] \times [0,T)$. First, we give the definitions of the super-solution and sub-solution to problem (1.1).

Definition 2.1. *A nonnegative function $\overline{u}(x,t)$ is called a super-solution of problem (1.1) if $\overline{u}(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ satisfies*

$$\begin{cases} \overline{u}_t \geq (x^\alpha \overline{u}_x)_x + \int_0^l \overline{u}^p dx - k\overline{u}^q, & (x,t) \in I_T, \\ \overline{u}(0,t) \geq \int_0^l f(x)\,\overline{u}(x,t)dx, & t \in (0,T), \\ \overline{u}(l,t) \geq \int_0^l g(x)\,\overline{u}(x,t)dx, & t \in (0,T), \\ \overline{u}(x,0) \geq \overline{u}_0(x), & x \in [0,l]. \end{cases} \tag{2.1}$$

Similarly, $\underline{u}(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ is called a sub-solution of problem (1.1) if it satisfies all the reversed inequalities in (2.1). We say that $u(x,t)$ is a solution of problem (1.1) if it is both a sub-solution and a super-solution of problem (1.1).

Now, by using the similar arguments as those in [6] (or [10]), we give directly the following maximum principle.

Lemma 2.1. *Let $\omega(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ satisfy*

$$\begin{cases} \omega_t - (x^\alpha \omega_x)_x \geq \theta_1(x,t)\,\omega + \int_0^l \theta_1(x,t)\,\omega(x,t)\,dx, & (x,t) \in I_T, \\ \omega(0,t) \geq \int_0^l \theta_3(x)\,\omega(x,t)dx, & t \in (0,T), \\ \omega(l,t) \geq \int_0^l \theta_4(x)\,\omega(x,t)dx, & t \in (0,T), \end{cases} \tag{2.2}$$

where $\theta_i(x,t)$, $i = 1,2,3,4$, are bounded functions, $\theta_2(x,t)$ is nonnegative for $(x,t) \in I_T$, $\theta_3(x)$ and $\theta_4(x)$ are nonnegative, nontrivial in $(0,l)$. Then $\omega(x,0) > 0$ in $[0,l]$ implies that $\omega(x,t) > 0$ for $(x,t) \in I_T$. Moreover, if one of the following conditions holds, (i) $\theta_3(x) = \theta_4(x) \equiv 0$ for $x \in (0,l)$; (ii) $\theta_3(x), \theta_4(x) \geq 0$ for $x \in (0,l)$ and $\max\left\{\int_0^l \theta_3(x)\,dx, \int_0^l \theta_4(x)\,dx\right\} \leq 1$, then $\omega(x,0) \geq 0$ in $[0,l]$ leads to $\omega(x,t) \geq 0$ for $(x,t) \in I_T$.

Based on the idea of [10], we can establish the comparison principle for problem (1.1) as follows, which is the main tool of establishing the conditions on the global existence and blowup of the solution.

Proposition 2.1 (Comparison principle)**.** *Let $\overline{u}(x,t)$ and $\underline{u}(x,t)$ be a nonnegative super-solution and sub-solution of problem (1.1), respectively. Then $\overline{u}(x,t) \geq \underline{u}(x,t)$ holds in $\overline{I}_T$ if $\overline{u}(x,0) \geq \underline{u}(x,0)$ for $x \in [0,l]$.*

Next, we state the result on the existence and uniqueness of the local solution of problem (1.1) at the end of this section.

Theorem 2.1 (Local existence and uniqueness)**.** *Assume that (1.5) holds, then there exists a small positive real number T such that problem (1.1) admits a unique nonnegative solution $u(x,t) \in C(\overline{I}_T) \cap C^{2,1}(I_T)$.*

Remark 2.1. *We can get the proof of Theorem 2.1 by using regularization method and Schauder's fixed point theorem. For more details, we refer the readers to [3, 23].*

3 Global existence of solution

The main goal of this section is to discuss the global existence and blowup property of the solution $u(x,t)$ to the problem (1.1). To this end, by Proposition 2.1, we only need to construct some suitable global super-solutions (or blowup sub-solutions).

Proof of Theorem 1.1. Let T be any positive number and $\overline{u}_1(x,t)$ be defined as

$$\overline{u}_1(x,t) = \frac{\chi_2}{\chi_1 \zeta_1(x) + 1} e^{\chi_3 t}$$

where χ_1 is large enough such that

$$\int_0^l \frac{1}{1 + \chi_1 \zeta_1(x)} dx \leq \max\left\{\max_{x\in[0,l]} f(x), \max_{x\in[0,l]} g(x)\right\},$$

and

$$\chi_2 = \max\left\{\max_{x\in[0,l]} (u_0(x) + 1)(\chi_1 \zeta_1(x) + 1), \max_{x\in[0,l]} \left[\frac{(\chi_1 \zeta_1(x) + 1)^q}{k} \int_0^l \frac{1}{(1 + \chi_1 \zeta_1(x))^p} dx\right]^{\frac{1}{q-p}}\right\},$$

$$\chi_3 = \lambda_1 + \max_{x\in[0,l]} \frac{2l^\alpha \chi_1^2}{(\chi_1\zeta_1(x)+1)^2} \left|\frac{d\zeta_1(x)}{dx}\right|^2.$$

By the direct calculation, one has

$$\begin{aligned} P\overline{u}_1 :&= \overline{u}_{1t} - (x^\alpha \overline{u}_{1x})_x - \int_0^l \overline{u}_1^p dx + k\overline{u}_1^q \\ &= \overline{u}_1 \left[\chi_3 - \left(\frac{\lambda_1\chi_1\zeta_1(x)}{1+\chi_1\zeta_1(x)} + \frac{2x^\alpha\chi_1^2}{(\chi_1\zeta_1(x)+1)^2}\left|\frac{d}{dx}\zeta_1(x)\right|^2\right)\right] \\ &\quad + k\left(\frac{\chi_2 e^{\chi_3 t}}{1+\chi_1\zeta_1(x)}\right)^q - \left(\chi_2 e^{\chi_3 t}\right)^p \int_0^l \frac{1}{(1+\chi_1\zeta_1(x))^p}dx \\ &\geq 0, \end{aligned} \tag{3.1}$$

and

$$\overline{u}_1(x,0) = \frac{\chi_2}{1+\chi_1\zeta_1(x)} \geq \frac{\max\limits_{x\in[0,l]}(u_0(x)+1)(1+\chi_1\zeta_1(x))}{1+\chi_1\zeta_1(x)} > u_0(x). \tag{3.2}$$

On the other hand, we can verify that

$$\overline{u}_1(0,t) = \chi_2 e^{\chi_3 t} \geq \chi_2 e^{\chi_3 t} \max_{x\in[0,l]} f(x) \int_0^l \frac{1}{1+\chi_1\zeta_1(x)}dx \geq \int_0^l \frac{f(x)\chi_2 e^{\chi_3 t}}{1+\chi_1\zeta_1(x)}dx = \int_0^l f(x)\overline{u}_1(x,t)\,dx, \tag{3.3}$$

and

$$\overline{u}_1(l,t) \geq \int_0^l g(x)\overline{u}_1(x,t)\,dx. \tag{3.4}$$

Combining now from (3.1) to (3.4), we know that $\overline{u}_1(x,t)$ is a global super-solution of (1.1) in I_T and the solution $u(x,t)$ of (1.1) exists globally by Proposition 2.1. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. **(i)** If $p > q$ and $\mathcal{N} > 1$, then it is easy to check that $\overline{u}_2(x) = \left(\frac{k}{l}\right)^{\frac{1}{p-q}}$ is a global super-solution of problem (1.1) provided that $u_0(x) \leq \left(\frac{k}{l}\right)^{\frac{1}{p-q}}$.

(ii) Consider the following ordinary differential equation

$$\begin{cases} \underline{v}_1{}'(t) = l\underline{v}_1^p - k\underline{v}_1^q, & t > 0, \\ \underline{v}_1(0) = \underline{v}_{10}. \end{cases} \tag{3.5}$$

From $p > q \geq 1$, it follows that $\underline{v}_1^q \leq \underline{v}_1^p + 1$, and hence, we have

$$l\underline{v}_1^p - k\underline{v}_1^q \geq (l-k)\underline{v}_1^p - k,$$

which tells us that the solution $\underline{v}_1(t)$ of (3.5) is a super-solution of the following problem

$$\begin{cases} \underline{v}_2{}'(t) = (l-k)\underline{v}_2^p - k, & t > 0, \\ \underline{v}_2(0) = \underline{v}_{10} \end{cases} \tag{3.6}$$

provided $l > k$. Noticing that $(l-k)\underline{v}_2^p$ is convex, then there exists $\eta_1 > 1$ such that $(l-k)\underline{v}_2^p \geq 2k$ holds for $\underline{v}_2 \geq \eta_1$. It follows easily that if $\underline{v}_2(0) = \underline{v}_{10} > \eta_1$, then $\underline{v}_2(t)$ is increasing on its interval of the existence and

$$\underline{v}_2{}'(t) \geq \frac{l-k}{2}\underline{v}_2^p. \tag{3.7}$$

From the above inequality it follows that

$$\underline{v}_2(t) \to \infty \text{ as } t \to \frac{2}{(l-k)(p-1)\underline{v}_{10}^{p-1}}, \tag{3.8}$$

which leads to that $\underline{v}_1(t)$ will become infinite in a finite time. Recalling that $\mathcal{N} > 1$, then $\underline{v}_1(t)$ is a blowup sub-solution of problem (1.1) when $u_0(x) \geq \underline{v}_{10} > \eta$, so the solution $u(x,t)$ of problem (1.1) blows up in finite time for sufficiently large initial value.

(iii) Let $v(x,t)$ be the solution of the following auxiliary problem

$$\begin{cases} v_t = (x^\alpha v_x)_x + \int_0^l v^p(x,t)dx - kv^q, & 0 < x < l, t > 0, \\ v(0,t) = v(l,t) = 0, & t > 0, \\ v(x,0) = u_0(x), & 0 < x < l, \end{cases} \tag{3.9}$$

then $v(x,t)$ is a sub-solution of problem (1.1). Set

$$\underline{v}_3(x,t) = (\eta_2 - t)^{-\xi}\left(\frac{l}{2-\alpha}x^{1-\alpha} - \frac{1}{2-\alpha}x^{2-\alpha}\right) := (\eta_2 - t)^{-\xi}\mu(x),$$

where η_2 and $\xi > 0$ will be chosen later. Calculating directly, we have

$$\begin{aligned} P\underline{v}_3 :&= \underline{v}_{3t} - (x^\alpha \underline{v}_{3x})_x - \int_0^l \underline{v}_3^p(x,t)dx + k\underline{v}_3^q \\ &= (\eta_2 - t)^{-\xi p}\left[\xi(\eta_2 - t)^{\xi p - \xi - 1}\mu(x) + (\eta_2 - t)^{\xi(p-1)} + k(\eta_2 - t)^{\xi(p-q)}\mu^q(x) - \int_0^l \mu^p(x)\,dx\right]. \end{aligned}$$

Since $p > q \geq 1$, we can take ξ large enough such that $\xi p - \xi - 1 > 0$, then we have $P\underline{v}_3 \leq 0$ with $\eta_2 - t$ small enough, which implies that $\underline{v}_3(x,t)$ is a blowup sub-solution to problem (3.9) provided that $v(x,0) = u_0(x) > \mu(x)\eta_2^{-\xi}$. And hence, Proposition 2.1 tells us that the solution $u(x,t)$ of problem (1.1) blows up in finite time for large initial value. The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. For any given constant

$$\epsilon_0 \in \left(0, \frac{(1-\mathcal{N})(2-\alpha)^{3-\alpha}}{l^{2-\alpha}(1-\alpha)^{1-\alpha}}\right), \tag{3.10}$$

let $\sigma(x)$ be the unique positive solution of the following ordinary differential equation

$$\begin{cases} -(x^\alpha \sigma_x)_x = \epsilon_0, & 0 < x < l, \\ \sigma(0) = \sigma(l) = \mathcal{N}. \end{cases} \tag{3.11}$$

In fact, we can solve the explicit expression of $\sigma(x)$ as follows

$$\sigma(x) = \frac{l\epsilon_0}{2-\alpha}x^{1-\alpha} - \frac{\epsilon_0}{2-\alpha}x^{2-\alpha} + \mathcal{N}, \quad x \in [0,l].$$

Moreover, according to $\mathcal{N} < 1$, we can verify that

$$0 < \min_{x\in[0,l]}\sigma(x) = \mathcal{N} < \max_{x\in[0,l]}\sigma(x) = \mathcal{N} + \frac{\epsilon_0 l^{2-\alpha}(1-\alpha)^{1-\alpha}}{(2-\alpha)^{3-\alpha}} < 1. \tag{3.12}$$

Define

$$\overline{u}_3\left(x,t\right)=\epsilon_1\sigma\left(x\right),$$

where

$$\epsilon_1=\begin{cases}\left(\frac{\epsilon_0}{k\mathcal{N}^p-l}\right)^{\frac{1}{p-1}}, & \text{if } k\mathcal{N}^p-l>0,\\ \text{any fixed positive constant}, & \text{if } k\mathcal{N}^p-l\leq 0.\end{cases}\tag{3.13}$$

Calculating directly, one has

$$\begin{aligned}P\overline{u}_3:&=\overline{u}_{3t}-\left(x^\alpha\overline{u}_{3x}\right)_x-\int_0^l\overline{u}_3^p dx+k\overline{u}_3^p\\&=\epsilon_0\epsilon_1-\epsilon_1^p\int_0^l\sigma^p dx+k\epsilon_1^p\sigma^p\\&\geq\epsilon_0\epsilon_1-l\epsilon_1^p\left[\max_{x\in[0,l]}\sigma\left(x\right)\right]^p+k\epsilon_1^p\left[\min_{x\in[0,l]}\sigma\left(x\right)\right]^p\\&>\epsilon_0\epsilon_1-\epsilon_1^p\left(k\mathcal{N}^p-l\right)\\&\geq 0.\end{aligned}\tag{3.14}$$

Meanwhile, we can prove that

$$\overline{u}_3\left(0,t\right)=\epsilon_1\mathcal{N}\geq\int_0^l\epsilon_1 f\left(x\right)dx>\int_0^l\epsilon_1\sigma\left(x\right)f\left(x\right)dx=\int_0^l\overline{u}_3\left(x,t\right)f\left(x\right)dx\tag{3.15}$$

and

$$\overline{u}_3\left(l,t\right)>\int_0^l\overline{u}_3\left(x,t\right)f\left(x\right)dx.\tag{3.16}$$

Then $\overline{u}_3\left(x,t\right)$ is a global super-solution of problem (1.1) if $u_0\left(x\right)\leq\epsilon_1\mathcal{N}$, and hence, we obtain our global existence result by Proposition 2.1.

The proof of blowup conclusion in this case is similar to the arguments of (iii) in Theorem 1.2, we omit the details here. The proof of Theorem 1.3 is completed. □

4 Global blowup set and uniform blowup profile

This section is mainly about the global blowup and the uniform blowup profile of the blowup solution for problem (1.1). Throughout this section, we assume that $p>q\geq 1$ (or $p=q>1$), $\mathcal{N}\leq 1$ and $u_0\left(x\right)$ is large enough in some suitable sense. From Theorems 1.2 and 1.3, it follows that the solution $u\left(x,t\right)$ of problem (1.1) blows up in finite. For convenience, we denote T the blowup time.

From the assumptions on the initial value $u_0\left(x\right)$ and (1.5), (1.6) and (1.7), we can find a sufficiently small positive constant ε_1 and a nonnegative function $w_{0\varepsilon}\left(x\right)$ such that

(1) $w_{0\varepsilon}\in C^{2+\delta}\left(\varepsilon,l-\varepsilon\right)\cap C\left[\varepsilon,l-\varepsilon\right]$ with $\delta\in(0,1)$ and $\varepsilon\in(0,\varepsilon_1]$.

(2) $w_{0\varepsilon}\left(\varepsilon\right)=\int_\varepsilon^{l-\varepsilon}f\left(x\right)w_{0\varepsilon}\left(x\right)dx$ and $w_{0\varepsilon}\left(l-\varepsilon\right)=\int_\varepsilon^{l-\varepsilon}g\left(x\right)w_{0\varepsilon}\left(x\right)dx$.

(3) $w_{0\varepsilon}\left(x\right)<u_0\left(x\right)$ for $x\in(\varepsilon,2\varepsilon)\cup(l-2\varepsilon,l-\varepsilon)$, and $w_{0\varepsilon}\left(x\right)=u_0\left(x\right)$ for $x\in[2\varepsilon,l-2\varepsilon]$.

(4) $\left(x^\alpha w_{0\varepsilon x}\right)_x\leq 0$ for $x\in(\varepsilon,l-\varepsilon)$.

(5) $(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - k w_{0\varepsilon}^q \geq 0$ for $\varepsilon \in (0, \varepsilon_1]$ and $x \in (\varepsilon, l-\varepsilon)$.

(6) $w_{0\varepsilon}$ is non-increasing with respect to ε in $(0, \varepsilon_1]$. Moreover

$$\lim_{x\to\varepsilon^+}\left[(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - k w_{0\varepsilon}^q\right] = \lim_{x\to(l-\varepsilon)^-}\left[(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - k w_{0\varepsilon}^q\right] = 0.$$

It is obvious that

$$\lim_{\varepsilon\to 0^+} w_{0\varepsilon}(x) = u_0(x).$$

Now, we consider the following regularized problem

$$\begin{cases} w_{\varepsilon t} = (x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p dx - k w_\varepsilon^q, & (x,t) \in (\varepsilon, l-\varepsilon)\times(0,+\infty), \\ w_\varepsilon(\varepsilon, t) = \int_\varepsilon^{l-\varepsilon} f(x)\, w_\varepsilon(x,t) dx, & t\in(0,+\infty), \\ w_\varepsilon(l-\varepsilon, t) = \int_\varepsilon^{l-\varepsilon} g(x)\, w_\varepsilon(x,t) dx, & t\in(0,+\infty), \\ w_\varepsilon(x,0) = w_{0\varepsilon}(x), & x\in[0,l]. \end{cases} \tag{4.1}$$

Then it is not difficult to show that there exists a unique solution $w_\varepsilon(x,t)$ for problem (4.1). In addition, from the arguments of Section 2 in [23], it follows that

$$\lim_{\varepsilon\to 0^+} w_\varepsilon(x,t) = u(x,t),$$

where $u(x,t)$ is the solution of problem (1.1).

Lemma 4.1. *Suppose that hypotheses* (1.5), (1.6) *and* (1.7) *hold, and assume that* $p \geq q > 1$ *and* $\mathcal{N} \leq 1$. *Then* $(x^\alpha u_x)_x \leq 0$ *holds for* $(x,t)\in I_T$.

Proof. Taking $\eta = (x^\alpha w_{\varepsilon x})_x$, then from (4.1), we have

$$\eta_t = \left\{x^\alpha\left[(x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p dx - k w_\varepsilon^q\right]_x\right\}_x = (x^\alpha \eta_x)_x - kq w_\varepsilon^{q-1}\eta - kq(q-1) w_\varepsilon^{q-2}|w_{\varepsilon x}|^2 \tag{4.2}$$

holds for any $(x,t)\in(\varepsilon, l-\varepsilon)\times(0,T)$, which tells us that

$$\eta_t - (x^\alpha\eta_x)_x + kq w_\varepsilon^{q-1}\eta \leq 0. \tag{4.3}$$

On the other hand, for any $t\in(0,T)$, we have

$$\begin{aligned} \eta(\varepsilon,t) &= \int_\varepsilon^{l-\varepsilon} f(x)\, w_{\varepsilon t}(x,t)\, dx - \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\, dx + k\left(\int_\varepsilon^{l-\varepsilon} f(x)\, w_\varepsilon(x,t)\, dx\right)^q \\ &= \int_\varepsilon^{l-\varepsilon} f(x)\left((x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\, dx - k w_\varepsilon^q\right) dx \\ &\quad - \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\, dx + k\left(\int_\varepsilon^{l-\varepsilon} f(x)\, w_\varepsilon(x,t)\, dx\right)^q \\ &= \int_\varepsilon^{l-\varepsilon} f(x)\,\eta(x,t)\, dx + \left(\int_\varepsilon^{l-\varepsilon} f(x)\, dx - 1\right)\int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\, dx \\ &\quad - k\left[\int_\varepsilon^{l-\varepsilon} f(x)\, w_\varepsilon^q(x,t)\, dx - \left(\int_\varepsilon^{l-\varepsilon} f(x)\, w_\varepsilon(x,t)\, dx\right)^q\right]. \end{aligned} \tag{4.4}$$

It follows from Jensen's inequality that

$$
\begin{aligned}
&\int_{\varepsilon}^{l-\varepsilon} f(x)\, w_{\varepsilon}^{q} dx - \left(\int_{\varepsilon}^{l-\varepsilon} f(x)\, w_{\varepsilon}(x,t)\, dx\right)^{q} \\
\geq& \int_{\varepsilon}^{l-\varepsilon} f(x)\, dx \left(\frac{\int_{\varepsilon}^{l-\varepsilon} f(x)\, w_{\varepsilon}(x,t)\, dx}{\int_{\varepsilon}^{l-\varepsilon} f(x)\, dx}\right)^{q} - \left(\int_{\varepsilon}^{l-\varepsilon} f(x)\, w_{\varepsilon}(x,t)\right)^{q} \\
\geq& 0.
\end{aligned}
$$

Exploiting the above inequality and the assumption $\mathcal{N} \leq 1$ to (4.4), we can claim that

$$
\eta(\varepsilon,t) \leq \int_{\varepsilon}^{l-\varepsilon} f(x)\, \eta(x,t)\, dx, \quad t \in (0,T). \tag{4.5}
$$

By the analogous arguments, one can also show that

$$
\eta(l-\varepsilon,t) \leq \int_{\varepsilon}^{l-\varepsilon} g(x)\, \eta(x,t)\, dx \tag{4.6}
$$

holds for all $t \in (0,T)$.

Moreover, noticing that $\eta(x,0) = (x^{\alpha} w_{0\varepsilon x})_x \leq 0$ holds for $x \in (\varepsilon, l-\varepsilon)$. Then, maximum principle tells us that $\eta(x,t) = (x^{\alpha} w_{\varepsilon x})_x \leq 0$ holds for all $(x,t) \in (\varepsilon, l-\varepsilon) \times (0,T)$. In addition, by the arbitrariness of ε, we know that $(x^{\alpha} u_x)_x \leq 0$ holds in I_T. The proof of Lemma 4.1 is complete. □

In what follows, for the sake of simplicity, we denote

$$
\psi(t) = \int_0^l u^p(x,t)\, dx \text{ and } \Psi(t) = \int_0^t \psi(\tau)\, d\tau.
$$

Lemma 4.2. *Assume that* (1.5), (1.6) *and* (1.7) *hold,* $p > q \geq 1$ *and* $\mathcal{N} \leq 1$, *then there exists a positive constant* C *such that*

$$
\sup_{x \in K_d} (\Psi(t) - u(x,t)) \leq \frac{C}{d^2}\left(1 + Z(t) + \int_0^t \Psi(\tau)\, d\tau\right)
$$

in $[0,l] \times [\frac{T}{2}, T)$, *where*

$$
Z(t) = o(\Psi(t)) \ \text{ as } t \to T,
$$

and

$$
K_d = \{x \in (0,l) : dist(x,0) \geq d, dist(x,l) \geq d\} \subset (0,l).
$$

Proof. Put

$$
\mathfrak{F}(t) = \int_0^l (\Psi(t) - u(x,t))\, \zeta_1(x)\, dx, \tag{4.7}
$$

where $\zeta_1(x)$ is given by (1.3). Taking the derivative of $\mathfrak{F}(t)$ with respect to t, we arrive at

$$\begin{aligned}
\mathfrak{F}'(t) &= \int_0^l (\psi(t) - u_t)\,\zeta_1(x)\,dx \\
&= \int_0^l \left(-(x^\alpha u_x)_x + ku^q\right)\zeta_1(x)\,dx \\
&= \lambda_1 \int_0^l u(x,t)\,\zeta_1(x)\,dx + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx \\
&\quad + l^\alpha \zeta_{1x}\mid_{x=l} \int_0^l g(x)\,u(x,t)\,dx \\
&\le \lambda_1 \int_0^l u(x,t)\,\zeta_1(x)\,dx + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx \\
&= -\lambda_1 \mathfrak{F}(t) + \lambda_1 \Psi(t) + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx.
\end{aligned} \tag{4.8}$$

On the other hand, it follows from Lemma 4.1 that

$$u_t \le \psi(t) - ku^q,$$

which implies that

$$-\max_{x\in[0,l]} u_0(x) \le \Psi(t) - u(x,t). \tag{4.9}$$

Then (4.9) and (4.8) lead to

$$\mathfrak{F}'(t) \le \lambda_1 \max_{x\in[0,l]} u_0(x) + \lambda_1 \Psi(t) + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx.$$

Integrating above inequality over from 0 to t, one has

$$\mathfrak{F}(t) \le \max\left\{\lambda_1, k\max_{x\in[0,l]}\zeta_1(x), \mathfrak{F}(0) + \lambda_1 T \max_{x\in[0,l]} u_0(x)\right\}\left(1 + \int_0^t \Psi(\tau)\,d\tau + \int_0^t\int_0^l u^q(x,\tau)\,dxd\tau\right). \tag{4.10}$$

Further, since $p > q \ge 1$, Hölder's inequality implies that

$$\int_0^t\int_0^l u^q(y,\tau)\,dyd\tau \le (lT)^{\frac{p-q}{p}} \left(\int_0^t\int_0^l u^p(y,\tau)\,dyd\tau\right)^{\frac{q}{p}} := Z(t). \tag{4.11}$$

It is not difficult to verify that

$$Z(t) = o(\Psi(t)) \text{ as } t \to T. \tag{4.12}$$

Combining (4.13), (4.11) with (4.12), we see that

$$\mathfrak{F}(t) \le \max\left\{\lambda_1, k\max_{x\in[0,l]}\zeta_1(x), \mathfrak{F}(0) + \lambda_1 T \max_{x\in[0,l]} u_0(x)\right\}\left(1 + Z(t) + \int_0^t \Psi(\tau)\,d\tau\right). \tag{4.13}$$

Now, by Lemma 4.5 in [17], we can claim that

$$\sup_{x\in K_d} (\Psi(t) - u(x,t)) \le \frac{C}{d^2}\left(1 + \int_0^t \Psi(\tau)\,d\tau + o(\Psi(t))\right)$$

holds for $(x,t) \in [0,l] \times \left[\frac{T}{2}, T\right)$, where C is an appropriate positive constant. The proof of Lemma 4.2 is complete. $\square$

In view of Lemma 4.2, and by a slight variant of the proof of Lemma 4.5 in [17], we have the following Lemma.

Lemma 4.3. *Assume that* (1.6) *and* (1.7) *hold,* $p > q \geq 1$ *and* $\mathcal{N} \leq 1$, *then*

$$\lim_{t\to T} \sup_{[0,l]} |u(\cdot,t)| = +\infty \tag{4.14}$$

is equivalent to

$$\lim_{t\to T} \Psi(t) = +\infty \tag{4.15}$$

Moreover, if (4.14) or (4.15) is fulfilled, then

$$\lim_{t\to T} \frac{u(x,t)}{\Psi(t)} = \lim_{t\to T} \frac{|u(\cdot,t)|_\infty}{\Psi(t)} = 1 \tag{4.16}$$

uniformly on any compact subset of $(0,l)$.

Next, we give the proofs of Theorems 1.4 and Theorem 1.5, respectively.

Proof of Theorem 1.4. It follows from (4.16) that

$$u^p(x,t) \sim \Psi^p(t), \quad t \to T.$$

By the Lebesgue's dominated convergence theorem, we have

$$\Psi'(t) = \psi(t) = \int_0^l u^p(x,t)\,dx \sim l\Psi^p(t), \quad t \to T.$$

Therefore, by integrating the above equality, we can claim that

$$\Psi(t) \sim (l(p-1)(T-t))^{-\frac{1}{p-1}}. \tag{4.17}$$

Combining (4.16) with (4.17), we find that

$$u(x,t) \sim (l(p-1)(T-t))^{-\frac{1}{p-1}}, \quad t \to T, \tag{4.18}$$

which means that

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}} u(x,t) = \lim_{t\to T} (T-t)^{\frac{1}{p-1}} |u(\cdot,t)|_\infty = (l(p-1))^{-\frac{1}{p-1}}.$$

The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. Denote

$$\varphi(t) = \int_0^l u^p(y,t)\,dy - k\left(\max_{x\in[0,l]} u(x,t)\right)^p \text{ and } \Phi(t) = \int_0^t g(\tau)\,d\tau.$$

Similar to Lemma 4.3, we can get

$$\lim_{t\to T} \frac{u(x,t)}{\Phi(t)} = \lim_{t\to T} \frac{|u(\cdot,t)|_\infty}{\Phi(t)} = 1, \tag{4.19}$$

uniformly on any compact subset of $(0,l)$.

Since, the remaining arguments are the same as those in the proof of Theorem 1.4, we omit it here. The proof of Theorem 1.5 is complete. □

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Competing interests

The authors declare that they have no competing interests.

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Approximation properties of Kantorovich-type q-Bernstein-Stancu-Schurer operators

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Abstract. In this paper, we introduce a Kantorovich-type Bernstein-Stancu-Schurer operators $K_{n,p,q}^{\alpha,\beta}$ based on the concept of q-integers. We investigate statistical approximation properties and establish a local approximation theorem, we also give a convergence theorem for the Lipschitz continuous functions. Finally, we give some graphics to illustrate the convergence properties of operators to some functions.

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1 Introduction

In 2013, Özarslan and Vedi [7] introduced the q-Bernstein-Schurer-Kantorovich operators as follows:

$$K_n^p(f;q;x) = \sum_{r=0}^{n+p} \left[\begin{array}{c} n+p \\ r \end{array} \right]_q x^r \prod_{s=o}^{n+p-r-1} (1-q^s x) \int_0^1 f\left(\frac{[r]_q}{[n+1]_q} + \frac{1+(q-1)[r]_q}{[n+1]_q} t \right) d_q t$$

for any real number $0 < q < 1$, fixed $p \in \mathbb{N}_0$ and $f \in C[0, p+1]$. They gave the Korovkin-type approximation theorem, obtained the rate of convergence of the operators and so on. In 2014, Ren and Zeng [8] introduced two kinds of Kantorovich-type q-Bernstein-Stancu operators based on q-Jackson integral and Riemann-type q-integral respectively and got some approximation properties. In 2015, Acu [1] introduced and studied q analogue of Stancu-Schurer-Kantorovich operators. They proved a convergence theorem, established the rate of convergence, obtained a Voronovskaya type result and so on, they constructed

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the operators as follows:

$$K_{n,p}^{\alpha,\beta}(f;x)=\sum_{k=0}^{n+p}\left[\begin{array}{c}n+p\\k\end{array}\right]_q x^k(1-x)_q^{n+p-k}\int_0^1 f\left(\frac{[k]_q+q^kt+\alpha}{[n+1]_q+\beta}\right)d_qt.$$

In 2015, Agrawal, Finta and Kumar [2] introduced a new Kantorovich-type generalization of the q-Bernstein-Schurer operators, they gave the basic convergence theorem, obtained the local direct results, estimated the rate of convergence and so on. The operators are defined as

$$K_{n,p}(f;q;x)=[n+1]_q\sum_{k=0}^{n+p}b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}}f(t)d_q^Rt, \tag{1}$$

where $b_{n+p,k}(q;x)$ is defined by

$$b_{n+p,k}(q;x)=\left[\begin{array}{c}n+p\\k\end{array}\right]_q x^k(1-x)_q^{n+p-k}. \tag{2}$$

Motivated by above investigations, it seems there have no papers mentioned about the Stancu-type of the operators defined in (1). In present paper, we will introduce the Kantorovich-type q-Bernstein-Stancu-Schurer operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ which will be defined in (4). We will investigate statistical approximation properties, establish a local approximation theorem and give a convergence theorem for the Lipschitz continuous functions. Furthermore, we will give some graphics to illustrate the convergence properties of operators to some functions.

Before introducing the operators, we mention certain definitions based on q-integers, detail can be found in [5, 6]. For any fixed real number $0<q\leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q=\begin{cases}\frac{1-q^k}{1-q}, & q\neq 1;\\ k, & q=1.\end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q!=\begin{cases}[k]_q[k-1]_q...[1]_q, & k=1,2,...;\\ 1, & k=0,\end{cases},\quad \left[\begin{array}{c}n\\k\end{array}\right]_q=\frac{[n]_q!}{[k]_q![n-k]_q!},\quad (n\geq k\geq 0).$$

For $x\in[0,1]$ and $n\in\mathbb{N}_0$, we recall that

$$(1-x)_q^n=\begin{cases}1, & n=0;\\ \prod_{j=0}^{n-1}\left(1-q^jx\right)=(1-x)(1-qx)...\left(1-q^{n-1}x\right), & n=1,2,...\end{cases}.$$

The Riemann-type q-integral is defined by

$$\int_a^b f(t)d_q^Rt=(1-q)(b-a)\sum_{j=0}^{\infty}f\left(a+(b-a)q^j\right)q^j, \tag{3}$$

where the real numbers a, b and q satisfy that $0 \leq a < b$ and $0 < q < 1$.

For $f \in C(I)$, $I = [0, 1+p]$, $p \in \mathbb{N}_0$, $0 \leq \alpha \leq \beta$, $q \in (0,1)$ and $n \in \mathbb{N}$, we introduce the Kantorovich-type q-Bernstein-Stancu-Schurer operators as follows:

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) = ([n+1]_q + \beta) \sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{-k} \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t) d_q^R t, \tag{4}$$

where $b_{n+p,k}(q;x)$ is defined by (2).

2 Auxiliary Results

In order to obtain the approximation properties, We need the following lemmas:

Lemma 2.1. *Using the definition* (3)*, we easily get*

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} d_q^R t = \frac{q^k}{[n+1]_q + \beta}, \tag{5}$$

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t d_q^R t = \frac{([k]_q+\alpha) q^k}{([n+1]_q+\beta)^2} + \frac{q^{2k}}{[2]_q([n+1]_q+\beta)^2}, \tag{6}$$

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t^2 d_q^R t = \frac{q^k([k]_q+\alpha)^2}{([n+1]_q+\beta)^3} + \frac{2q^{2k}([k]_q+\alpha)}{[2]_q([n+1]_q+\beta)^3} + \frac{q^{3k}}{[3]_q([n+1]_q+\beta)^3}. \tag{7}$$

Lemma 2.2. *(See [2], Lemma 2.1) The following equalities hold*

$$\sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^k = 1 - (1-q)[n+p]_q x, \tag{8}$$

$$\sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{2k} = 1 - \left(1-q^2\right)[n+p]_q x + q(1-q)^2 [n+p]_q [n+p-1]_q x^2. \tag{9}$$

Lemma 2.3. *For the Kantorovich-type q-Bernstein-Stancu-Schurer operators* (4)*, we have*

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(1;x) = 1, \tag{10}$$

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t;x) = \frac{2q[n+p]_q x + 1 + [2]_q \alpha}{[2]_q([n+1]_q+\beta)}, \tag{11}$$

$$\begin{aligned}\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right) &= \frac{\left(q^2[3]_q + 3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q+\beta)^2} x^2 + \frac{[2]_q[3]_q\alpha^2 + 2[3]_q\alpha + [2]_q}{[2]_q[3]_q([n+1]_q+\beta)^2} \\ &\quad + \frac{\left(4q[3]_q\alpha + 3q + 5q^2 + 4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q+\beta)^2} x. \end{aligned} \tag{12}$$

Proof. (10) is easily obtained from (4) and (5). Using (4), (6) and (8), we have

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t;x)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n+p} b_{n+p,q}(k;x)\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}+\frac{q^k}{[2]_q([n+1]_q+\beta)}\right)\\
&= \frac{[n+p]_q}{[n+1]_q+\beta}\sum_{k=0}^{n+p} b_{n+p,q}(k;x)\frac{[k]_q}{[n+p]_q}+\frac{\alpha}{[n+1]_q+\beta}+\frac{1-(1-q)[n+p]_qx}{[2]_q([n+1]_q+\beta)}\\
&= \frac{[n+p]_q}{[n+1]_q+\beta}\sum_{k=0}^{n+p-1}\left[\begin{array}{c} n+p-1\\ k\end{array}\right]_q x^{k+1}(1-x)_q^{n+p-k-1}+\frac{1-(1-q)[n+p]_qx}{[2]_q([n+1]_q+\beta)}\\
&\quad +\frac{\alpha}{[n+1]_q+\beta}\\
&= \frac{[n+p]_q}{[n+1]_q+\beta}x-\frac{(1-q)[n+p]_q}{[2]([n+1]_q+\beta)}x+\frac{1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}.
\end{aligned}
$$

Thus, (11) is proved. Finally, from (4) and (7), we have

$$
\begin{aligned}
&\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right)\\
&= \sum_{k=0}^{n+p} b_{n+p,q}(k;x)\left(\frac{[k]_q^2+2\alpha[k]_q+\alpha^2}{([n+1]_q+\beta)^2}+\frac{2q^k([k]_q+\alpha)}{[2]_q([n+1]_q+\beta)^2}+\frac{q^{2k}}{[3]_q([n+1]_q+\beta)^2}\right),
\end{aligned}
$$

since $[k]_q^2=[k]_q[k-1]_q+q^{k-1}[k]_q$, and from lemma 2.2, we have

$$
\begin{aligned}
&\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right)\\
&= \sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{[k]_q[k-1]_q}{([n+1]_q+\beta)^2}+\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{2\alpha[k]_q}{([n+1]_q+\beta)^2}\\
&\quad +\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{q^{k-1}[k]_q}{([n+1]_q+\beta)^2}+\frac{\alpha^2}{([n+1]_q+\beta)^2}+\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{2q^k[k]_q}{[2]_q([n+1]_q+\beta)^2}\\
&\quad +\frac{2\alpha}{[2]_q([n+1]_q+\beta)^2}\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^k+\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{q^{2k}}{[3]([n+1]_q+\beta)^2}\\
&= \frac{[n+p]_q[n+p-1]_qx^2}{([n+1]_q+\beta)^2}+\frac{2\alpha[n+p]_qx}{([n+1]_q+\beta)^2}+\frac{[n+p]_qx}{([n+1]_q+\beta)^2}-\frac{(1-q)[n+p]_q[n+p-1]_qx^2}{([n+1]_q+\beta)^2}\\
&\quad +\frac{\alpha^2}{([n+1]_q+\beta)^2}+\frac{2q[n+p]_qx}{[2]_q([n+1]_q+\beta)^2}-\frac{2q(1-q)[n+p]_q[n+p-1]_qx^2}{[2]_q([n+1]_q+\beta)^2}\\
&\quad +\frac{2\alpha\left(1-(1-q)[n+p]_qx\right)}{[2]_q([n+1]_q+\beta)^2}+\frac{1-\left(1-q^2\right)[n+p]_qx+q(1-q)^2[n+p]_q[n+p-1]_qx^2}{[3]_q([n+1]_q+\beta)^2}\\
&= \frac{[n+p]_q[n+p-1]_q}{([n+1]_q+\beta)^2}x^2+\frac{(2[2]_q\alpha+[2]_q+2q)[n+p]_q}{[2]_q([n+1]_q+\beta)^2}x+\frac{[2]_q[3]_q\alpha^2+2[3]_q\alpha+[2]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}\\
&\quad -\frac{(1-q)\left(1-q+4q[3]_q\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x^2-\frac{(1-q)\left(2\alpha[3]_q+[2]_q^2\right)[n+p]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x\\
&= \frac{\left(q^2[3]_q+3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x^2+\frac{\left(4q[3]_q\alpha+3q+5q^2+4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x
\end{aligned}
$$

$$+\frac{[2]_q[3]_q\alpha^2+2[3]_q\alpha+[2]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}.$$

Thus, (12) is proved. □

Remark 2.4. *From lemma 2.3, it is observed that for $\alpha=\beta=0$, we get the moments for the operators defined in* (1)*, which are the corresponding results of lemma 2.1 in [2].*

Lemma 2.5. *Using lemma 2.3 and easily computations, we have*

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t-x;x)=\left[\frac{2q[n+p]_q}{[2]_q([n+1]_q+\beta)}-1\right]x+\frac{1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}\doteq A_{n,p,q}^{\alpha,\beta}(x), \tag{13}$$

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t-x)^2;x\right)\leq\left[\frac{\left(q^2[3]_q+3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}+1-\frac{4q[n+p]_q}{[2]_q([n+1]_q+\beta)}\right]x^2$$
$$+\frac{[2]_q[3]_q\alpha^2+2[3]_q\alpha+[2]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}+\frac{\left(4q[3]_q\alpha+3q+5q^2+4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x\doteq B_{n,p,q}^{\alpha,\beta}(x). \tag{14}$$

3 Statistical approximation properties

In this section, we present the statistical approximation properties of the operator $\widetilde{K_{n,p,q}^{\alpha,\beta}}$ by using the Korovkin-type statistical approximation theorem proved in [4].

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by $\delta(K):=\lim_n\frac{1}{n}\sum_{k=1}^{n}\chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x:=\{x_n\}$ is called statistically convergent to a number L if, for every $\varepsilon>0$, $\delta\{n\in\mathbb{N}:|x_n-L|\geq\varepsilon\}=0$. Let $A:=(a_{jn}), j,n=1,2,...$ be an infinite summability matrix. For a given sequence $x:=\{x_n\}$, the $A-$transform of x, denoted by $Ax:=((Ax)_j)$, is given by $(Ax)_j=\sum_{k=1}^{\infty}a_{jn}x_n$ provided the series converges for each j. We say that A is regular if $\lim_n(Ax)_j=L$ whenever $\lim x=L$. Assume that A is a non-negative regular summability matrix. A sequence $x=\{x_n\}$ is called A-statistically convergent to L provided that for every $\varepsilon>0$, $\lim_j\sum_{n:|x_n-L|\geq\varepsilon}a_{jn}=0$. We denote this limit by $st_A-\lim_n x_n=L$. For $A=C_1$, the Cesàro matrix of order one, A-statistical convergence reduces to statistical convergence. It is easy to see that every convergent sequence is statistically convergent but not conversely.

We consider a sequence $q:=\{q_n\}$ for $0<q_n<1$ satisfying

$$st_A-\lim_n q_n=1, \tag{15}$$

If $e_i=t^i,\ t\in\mathbb{R}^+,\ i=0,1,2,...$ stands for the ith monomial, then we have

Theorem 3.1. *Let $A=(a_{nk})$ be a non-negative regular summability matrix and $q:=\{q_n\}$ be a sequence satisfying* (15)*, then for all $f\in C(I)$, $x\in[0,1]$, we have*

$$st_A-\lim_n\left\|\widetilde{K_{n,p,q}^{\alpha,\beta}}f-f\right\|_{C(I)}=0. \tag{16}$$

Proof. Obviously

$$st_A - \lim_n \left\|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_0) - e_0\right\|_{C(I)} = 0. \tag{17}$$

By (13), we have

$$\left|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_1;x) - e_1(x)\right| \le \left|\frac{2q_n[n+p]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)} - 1\right| + \frac{1+[2]_{q_n}\alpha}{[2]_{q_n}([n+1]_{q_n}+\beta)}.$$

Now for a given $\varepsilon > 0$, let us define the following sets:

$$U := \left\{k : \left\|\widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_1) - e_1\right\|_{C(I)} \ge \varepsilon\right\},\ U_1 := \left\{k : \left|\frac{2q_k[n+p]_{q_k}}{[2]_{q_k}([n+1]_{q_k}+\beta)} - 1\right| \ge \frac{\varepsilon}{2}\right\},$$
$$U_2 := \left\{k : \frac{1+[2]_{q_k}\alpha}{[2]_{q_k}([n+1]_{q_k}+\beta)} \ge \frac{\varepsilon}{2}\right\}.$$

Then one can see that $U \subseteq U_1 \cup U_2$, so we have

$$\begin{aligned}\delta\left\{k \le n : \left\|\widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_1) - e_1\right\|_{C(I)}\right\} \quad &\le \quad \delta\left\{k \le n : \left|\frac{2q_k[n+p]_{q_k}}{[2]_{q_k}([n+1]_{q_k}+\beta)} - 1\right| \ge \frac{\varepsilon}{2}\right\} \\ &\quad +\delta\left\{k \le n : \frac{1+[2]_{q_k}\alpha}{[2]_{q_k}([n+1]_{q_k}+\beta)} \ge \frac{\varepsilon}{2}\right\},\end{aligned}$$

since $st_A - \lim_n q_n = 1$, we have

$$st_A - \lim_n \left|\frac{[n+p]_{q_n}}{[n+1]_{q_n}+\beta} - 1\right| = 0, \quad st_A - \lim_n \frac{1+[2]_{q_n}\alpha}{[2]_{q_n}([n+1]_{q_n}+\beta)} = 0,$$

which implies that the right-hand side of the above inequality is zero, thus we have

$$st_A - \lim_n \left\|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_1) - e_1\right\|_{C(I)} = 0. \tag{18}$$

Finally, by (10) and (12), we get

$$\begin{aligned}&\left|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_2;x) - e_2(x)\right| \\ \le\ &\left|\frac{\left(q_n^2[3]_{q_n} + 3q_n^4\right)[n+p]_{q_n}[n+p-1]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n}+\beta)^2} - 1\right| + \frac{\left(4q_n[3]_{q_n}\alpha + 3q_n + 5q_n^2 + 4q_n^3\right)[n+p]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n}+\beta)^2} \\ &+\frac{[2]_{q_n}[3]_{q_n}\alpha^2 + 2[3]_{q_n}\alpha + [2]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n}+\beta)^2} \doteq \alpha_n + \beta_n + \gamma_n.\end{aligned}$$

Since $st_A - \lim_n q_n = 1$, one can see that

$$st_A - \lim_n \alpha_n = st_A - \lim_n \beta_n = st_A - \lim_n \gamma_n = 0. \tag{19}$$

For $\varepsilon > 0$, we define the following four sets

$$V := \left\{k : \left\|\widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_2) - e_2\right\|_{C(I)} \ge \varepsilon\right\},\quad V_1 := \left\{k : \alpha_k \ge \frac{\varepsilon}{3}\right\},\quad V_2 := \left\{k : \beta_k \ge \frac{\varepsilon}{3}\right\},$$

$$V_3 := \left\{ k : \gamma_k \geq \frac{\varepsilon}{3} \right\}.$$

Hence, from (19) we obtain the right-hand side of the above inequality is zero, so we have

$$\delta \left\{ k \leq n : \left\| \widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_2) - e_2 \right\|_{C(I)} \geq \varepsilon \right\} = 0,$$

thus

$$st_A - \lim_n \left\| \widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_2) - e_2 \right\|_{C(I)} = 0. \tag{20}$$

Combining (17), (18) and (20), theorem 3.1 follows from the Korovkin-type statistical approximation theorem established in [4], the proof is completed. □

4 Local approximation properties

Let $f \in C(I)$, endowed with the norm $||f|| = \sup_{x \in I} |f(x)|$. The Peetre's $K-$functional is defined by

$$K_2(f;\delta) = \inf_{g \in C^2} \left\{ ||f - g|| + \delta ||g''|| \right\},$$

where $\delta > 0$ and $C^2 = \{g \in C(I) : g', g'' \in C(I)\}$. By [3, p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{21}$$

where

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta} \sup_{x,x+h,x+2h \in I} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C(I)$. We denote the usual modulus of continuity of $f \in C(I)$ by

$$\omega(f;\delta) = \sup_{0<h\leq\delta} \sup_{x,x+h \in I} |f(x+h) - f(x)|.$$

Now we give a direct local approximation theorem for the operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f,x)$.

Theorem 4.1. *For $q \in (0,1)$, $x \in [0,1]$ and $f \in C(I)$, we have*

$$\left| \widetilde{K_{n,p,q}^{\alpha,\beta}}(f,x) - f(x) \right| \leq C\omega_2 \left(f; \sqrt{\left(A_{n,p,q}^{\alpha,\beta}(x) \right)^2 + B_{n,p,q}^{\alpha,\beta}(x)/2} \right) + \omega \left(f; \left| A_{n,p,q}^{\alpha,\beta}(x) \right| \right), \tag{22}$$

where C is a positive constant, $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14).

Proof. We define the auxiliary operators

$$\overline{K_{n,p,q}^{\alpha,\beta}}(f;x) = \widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f\left(\frac{2q[n+p]_q x + 1 + [2]_q \alpha}{[2]_q([n+1]_q + \beta)} \right) + f(x), \tag{23}$$

$x \in [0,1]$. The operators $\overline{K_{n,p,q}^{\alpha,\beta}}(f;x)$ are linear and preserve the linear functions:

$$\overline{K_{n,p,q}^{\alpha,\beta}}(t-x;x)=0 \tag{24}$$

(see Lemma 2.3).
Let $g \in C^2$. By Taylor's expansion

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-u)g''(u)du,$$

and (24), we get

$$\overline{K_{n,p,q}^{\alpha,\beta}}(g;x)=g(x)+\overline{K_{n,p,q}^{\alpha,\beta}}\left(\int_x^t (t-u)g''(u)du;x\right).$$

Hence, by (23), (13) and (14), we have

$$\begin{aligned}
\left|\overline{K_{n,p,q}^{\alpha,\beta}}(g;x)-g(x)\right| &\leq \left|\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\
&\quad +\left|\int_x^{\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}}\left(\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}-u\right)g''(u)du\right| \\
&\leq \widetilde{K_{n,p,q}^{\alpha,\beta}}\left(\left|\int_x^t (t-u)|g''(u)|du\right|;x\right) \\
&\quad +\int_x^{\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}}\left|\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}-u\right||g''(u)|du \\
&\leq \left\{\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t-x)^2;x\right)+\left[\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}-x\right]^2\right\}||g''|| \\
&\leq \left[\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2+B_{n,p,q}^{\alpha,\beta}(x)\right]||g''||,
\end{aligned}$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). On the other hand, by (23), (4) and lemma 2.3, we have

$$\left|\overline{K_{n,p,q}^{\alpha,\beta}}(f;x)\right| \leq \left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)\right|+2||f|| \leq ||f||\widetilde{K_{n,p,q}^{\alpha,\beta}}(1;x)+2||f|| \leq 3||f||. \tag{25}$$

Now (23) and (25) imply

$$\begin{aligned}
\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)\right| &\leq \left|\overline{K_{n,p,q}^{\alpha,\beta}}(f-g;x)-(f-g)(x)\right|+\left|\overline{K_{n,p,q}^{\alpha,\beta}}(g;x)-g(x)\right| \\
&\quad +\left|f\left(\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}\right)-f(x)\right| \\
&\leq 4||f-g||+\left[\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2+B_{n,p,q}^{\alpha,\beta}(x)\right]||g''||+\omega\left(f;\left|A_{n,p,q}^{\alpha,\beta}(x)\right|\right).
\end{aligned}$$

Hence taking infimum on the right hand side over all $g \in C^2$, we get

$$\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f(x)\right| \leq 4K_2\left(f;\left[\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2 + B_{n,p,q}^{\alpha,\beta}(x)\right]/4\right) + \omega\left(f;\left|A_{n,p,q}^{\alpha,\beta}(x)\right|\right).$$

By (21), for every $q \in (0,1)$, we have

$$\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f(x)\right| \leq C\omega_2\left(f;\sqrt{\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2 + B_{n,p,q}^{\alpha,\beta}(x)/2}\right) + \omega\left(f;\left|A_{n,p,q}^{\alpha,\beta}(x)\right|\right),$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). This completes the proof of Theorem 4.1. □

Remark 4.2. *For any fixed $x \in [0,1]$, $0 \leq \alpha \leq \beta$, $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$ and $\lim_n q_n = 1$, we have*

$$\lim_n A_{n,p,q}^{\alpha,\beta}(x) = 0 \quad \text{and} \quad \lim_n B_{n,p,q}^{\alpha,\beta}(x) = 0,$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). These give us a rate of pointwise convergence of the operators $\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(f;x)$ to $f(x)$.

Next we study the rate of convergence of the operators $K_{n,q}(f;x)$ with the help of functions of Lipschitz class $Lip_M(\xi)$, where $M > 0$ and $0 < \xi \leq 1$. A function f belongs to $Lip_M(\xi)$ if

$$|f(y) - f(x)| \leq M|y - x|^\xi \quad (y, x \in \mathbb{R}). \tag{26}$$

We have the following theorem.

Theorem 4.3. *Let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$, $\lim\limits_n q_n = 1$ and $f \in Lip_M(\xi)$, $0 < \xi \leq 1$. Then we have*

$$\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f(x)\right| \leq M\left(B_{n,p,q}^{\alpha,\beta}(x)\right)^{\frac{\xi}{2}}, \tag{27}$$

where $B_{n,p,q}^{\alpha,\beta}(x)$ is defined in (14).

Proof. Since $\widetilde{K_{n,p,q}^{\alpha,\beta}}$ is a linear positive operator and $f \in Lip_M(\xi)$ $(0 < \xi \leq 1)$, we have

$$\begin{aligned}
&\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f(x)\right| \\
\leq\ & \widetilde{K_{n,p,q}^{\alpha,\beta}}\left(|f(t) - f(x)|;x\right) \\
=\ & ([n+1]_q + \beta)\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} |f(t) - f(x)|d_q^R t \\
\leq\ & M([n+1]_q + \beta)\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} |t - x|^\xi d_q^R t
\end{aligned}$$

$$\begin{aligned}
&\leq M([n+1]_q+\beta)\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^{-k}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}\left[(t-x)^{\xi}\right]^{\frac{2}{\xi}}d_q^R t\right)^{\frac{\xi}{2}}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}d_q^R t\right)^{\frac{2-\xi}{2}}\\
&= M([n+1]_q+\beta)\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^{-k}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2 d_q^R t\right)^{\frac{\xi}{2}}\left(\frac{q^k}{[n+1]_q+\beta}\right)^{\frac{2-\xi}{2}}\\
&= M\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2 d_q^R t\right)^{\frac{\xi}{2}}\left(\frac{[n+1]_q+\beta}{q^k}\right)^{\frac{\xi}{2}}\\
&= M\sum_{k=0}^{n+p}\left[b_{n+p,k}(q;x)\right]^{\frac{2-\xi}{2}}\left(([n+1]_q+\beta)b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2 d_q^R t\right)^{\frac{\xi}{2}}.
\end{aligned}$$

Applying Hölder's inequality for sums, we obtain

$$\begin{aligned}
&|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)|\\
&\leq M\left(\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\right)^{\frac{2-\xi}{2}}\left(\sum_{k=0}^{n+p}([n+1]_q+\beta)b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2 d_q^R t\right)^{\frac{\xi}{2}}\\
&= M\left(\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t-x)^2;x\right)\right)^{\frac{\xi}{2}} = M\left(B_{n,p,q}^{\alpha,\beta}(x)\right)^{\frac{\xi}{2}}.
\end{aligned}$$

Thus, theorem 4.3 is proved. □

5 Graphical analysis

In this section, we will illustrate two examples to state the convergence of operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ to $f(x)$ by means of Graphs.

Example 1: From figure 1, we can observe that as q increases, $n = 50$ be fixed, Kantorovich-type q-Bernstein-Stancu-Schurer operators given by (4) converge to the function $f(x) = \sin(2\pi x)$.

In comparison to figure 1, let $q = 0.99$ be fixed, as n increases, operators given by (4) converge to the function as shown in figure 2.

Example2: Similarly for different values of parameters q and n, let $p = 1$, $\alpha = 2$ and $\beta = 3$, convergence of operators to the function $f(x) = 1 - \cos(4e^x)$ is shown in figure 3 and 4, respectively.

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APPROXIMATION PROPERTIES OF KANTOROVICH-TYPE q-BERNSTEIN-STANCU-SCHURER OPERATORS

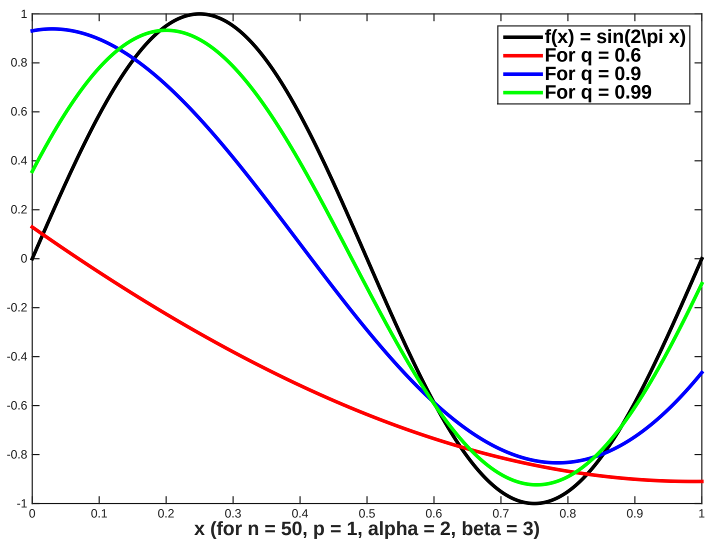

Figure 1: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $n=50$, $p=1$, $\alpha=2$, $\beta=3$ and different values of q.

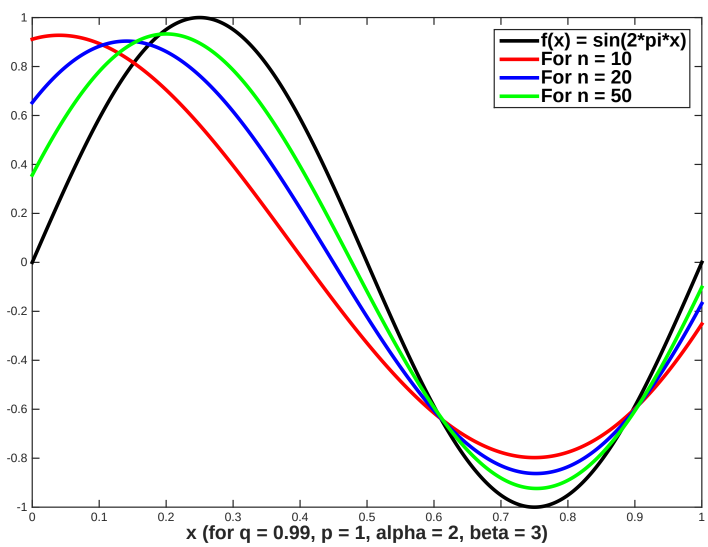

Figure 2: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $q=0.99$, $p=1$, $\alpha=2$, $\beta=3$ and different values of n.

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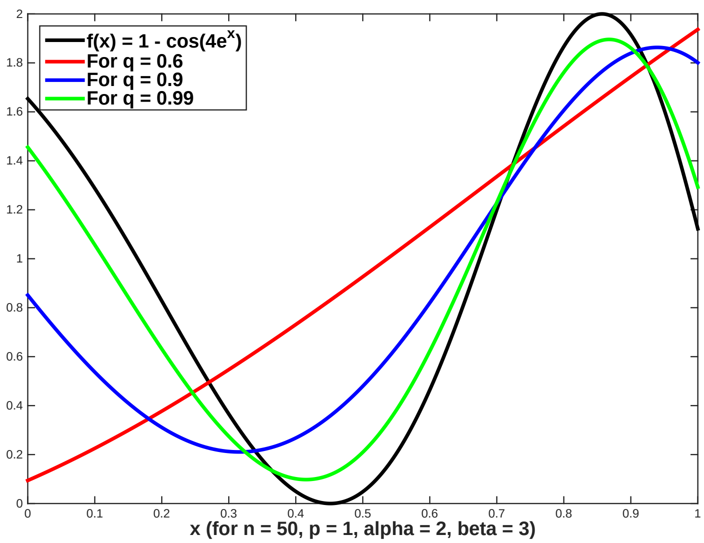

Figure 3: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $n = 50$, $p = 1$, $\alpha = 2$, $\beta = 3$ and different values of q.

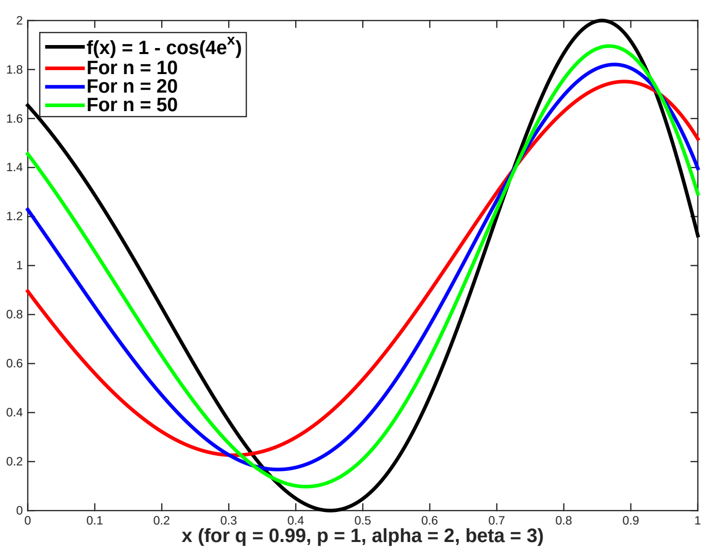

Figure 4: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $q = 0.99$, $p = 1$, $\alpha = 2$, $\beta = 3$ and different values of n.

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On the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$

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Abstract: In this paper, we study the exact values of the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for X being $l_\infty - l_1$ and $l_q - l_1$ spaces. Moreover, we shown that some new conditions for uniformly normal structure of a Banach space X.
Keywords: generalized von Neumann-Jordan constant; $l_\infty - l_1$ and $l_q - l_1$ space; uniformly normal structure
2000 Mathematics subject classification : 46B20.

1. Introduction

In order to study the geometric structure of a Banach space, many geometric constant have been investigated. In particular, the von Neuman-Jordan constant $C_{NJ}(X)$ is widely treated. In[1], as a generalization of the von Neuman-Jordan constant, a new geometric constant called the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ was introduced. It is proved that the $C_{NJ}^{(p)}(X)$ is strongly connected with geometric structure, such as uniformly non-square, uniformly normal structure. Hence it's necessary to compute the $C_{NJ}^{(p)}(X)$ for some concrete spaces.

Throughout this paper, let $X = (X, \|\cdot\|)$ be a real Banach spaces. We will use B_X, S_X and $ex(B_X)$ to denote unit ball, unit sphere of X and the set of extreme points of B_X, respectively.

Recall that the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson[3], as the smallest constant C for which,

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C,$$

holds for all $x, y \in X$.

An equivalent definition of the constant is

$$C_{NJ}(X) = \sup\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\}.$$

The properties of $C_{NJ}(X)$ have been investigated in many papers(see for instances [2],[4],[8],[9],[10]). Recently, a generalized form of this constant was introduced as following

Definition 1.[1] The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ is defined by

$$C_{NJ}^{(p)}(X) := \sup\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} : x, y \in X, (x,y) \neq (0,0)\},$$

where $1 \leq p < \infty$.

It's equivalent to

$$C_{NJ}^{(p)}(X) = \sup\{\frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} : x, y \in S_X, 0 \leq t \leq 1\},$$

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where $1 \leq p < \infty$.

Now let us collect some properties of this constant (see [1]):
(i) $1 \leq C_{NJ}^{(p)}(X) \leq 2$;
(ii) X is uniformly non-square if and only if $C_{NJ}^{(p)}(X) < 2$;
(iii) Let $r \in (1, 2]$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Then for $X = L_r[0, 1]$,
(1) if $1 < p \leq r$ then $C_{NJ}^{(p)}(X) = 2^{2-p}$ and if $r < p \leq r'$ then $C_{NJ}^{(p)}(X) = 2^{\frac{p}{r}-p+1}$,
(2) if $r' < p < \infty$ then $C_{NJ}^{(p)}(X) = 1$.

In this paper, we study the exact values of the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for X being $l_\infty - l_1$ and $l_q - l_1$ space. Moreover, we shown that some new conditions for uniformly normal structure of a Banach space X.

2. Main Results

Firstly, we consider $l_\infty - l_1$ space. As $C_{NJ}^{(1)}(X) = 2$ for any Banach space X, we only consider the case $p > 1$.
Theorem 2.1. ($l_\infty - l_1$ spaces). Let $p > 1$ and $X = l_\infty - l_1$ which is $\mathbb{R}^2$ endowed with the norm

$$\|x\| = \begin{cases} \|x\|_\infty, & if\ x_1 x_2 \geq 0, \\ \|x\|_1, & if\ x_1 x_2 \leq 0. \end{cases}$$

Then

$$C_{NJ}^{(p)}(l_\infty - l_1) = \frac{(1+t_0)^p + 1}{2^{p-1}(1+t_0^p)} = \frac{1}{2^{p-1}(1-t_0^{p-1})}, \tag{2.1}$$

where $t_0 \in (0, 1)$ is the unique solution of the equation

$$(1+t)^{p-1} - t^{p-1} - t^{p-1}(1+t)^{p-1} = 0. \tag{2.2}$$

Proof. Firstly we shall show that $\|x + ty\|^p + \|x - ty\|^p \leq 1 + (1+t)^p$ for any $x, y \in S_X$ and every $t \in [0, 1]$.
By Minkowski inequality, for any $\alpha, \beta \in [0, 1]$ and any $x_1, x_2, y_1, y_2 \in B_X$ with $x = \alpha X_1 + (1-\alpha)x_2, y = \beta y_2 + (1-\beta)y_2$, we have

$$\begin{aligned}
&\|x + ty\|^p + \|x - ty\|^p \\
&= \|\alpha(x_1 + ty) + (1-\alpha)(x_2 + ty)\|^p + \|\alpha(x_1 - ty) + (1-\alpha)(x_2 - ty\|^p \\
&\leq \alpha\|x_1 + ty\|^p + (1-\alpha)\|x_2 + ty\|^p + \alpha\|x_1 - ty\|^p + (1-\alpha)\|x_2 - ty\|^p \\
&= \alpha[\|\beta(x_1 + ty_1) + (1-\beta)(x_1 + ty_2)\|^p + \|\beta(x_1 - ty_1) + (1-\beta)(x_1 - ty_2)\|^p] \\
&+(1-\alpha)[\|\beta(x_2 + ty_1) + (1-\beta)(x_2 + ty_2)\|^p + \|\beta(x_2 - ty_1) + (1-\beta)(x_2 - ty_2)\|^p] \\
&\leq \alpha\beta[\|x_1 + ty_1\|^p + \|x_1 - ty_1\|^p] + \alpha(1-\beta)[\|x_1 + ty_2\|^p + \|x_1 - ty_2\|^p] \\
&+(1-\alpha)\beta[\|x_2 + ty_1\|^p + \|x_2 - ty_1\|^p] + (1-\alpha)(1-\beta)[\|x_2 + ty_2\|^p + \|x_2 - ty_2\|^p]
\end{aligned}$$

Hence, we only need to prove $\|x + ty\|^p + \|x - ty\|^p \leq 1 + (1+t)^p$ for any $x, y \in ex(B_X)$ and every $t \in [0, 1]$.
Since $ex(B_X) = \{(1, 0), (0, 1), (1, 1), (-1, 0), (-1, -1), (0, -1)\}$ and we can change x into $-x$ or y into $-y$. So we may assume that $x, y = (0, 1), (1, 0)$ or $(1, 1)$. Obviously, for these x, y we easily have $\|x + ty\|^p + \|x - ty\|^p \leq 1 + (1+t)^p$ for every $t \in [0, 1]$. Therefore,

$$C_{NJ}^{(p)}(l_\infty - l_1) \leq \sup_{t\in[0,1]}\{\frac{(1+t)^p+1}{2^{p-1}(1+t^p)}\}.$$

Let $f(t) = \frac{(1+t)^p+1}{1+t^p}$, then

$$f^{'}(t) = \frac{p(1+t)^{p-1}}{(1+t^p)^2}[1 - t^{p-1} - (\frac{t}{1+t})^{p-1}].$$

Defining $h(t) = 1 - t^{p-1} - (\frac{t}{1+t})^{p-1}$, we have $h(t)$ is decreasing from 1 to $-\frac{1}{2^{p-1}}$ on $[0,1]$. Whence there exists an unique $t_0 \in (0,1)$ such that $h(t_0) = 0$. Therefore,

$$C_{NJ}^{(p)}(l_\infty - l_1) \leq \frac{(1+t_0)^p+1}{2^{p-1}(1+t_0^p)}.$$

On the other hand, by taking $x_0 = (1,0), y_0 = (t_0, t_0)$, we have

$$C_{NJ}^{(p)}(l_\infty - l_1) \geq \frac{(1+t_0)^p+1}{2^{p-1}(1+t_0^p)}.$$

Hence,

$$C_{NJ}^{(p)}(l_\infty - l_1) = \frac{(1+t_0)^p+1}{2^{p-1}(1+t_0^p)},$$

where $t_0 \in (0,1)$ is the unique solution of $1 - t^{p-1} = (\frac{t}{1+t})^{p-1}$.
From (2.2), we also have

$$(1+t_0)^p + 1 = (1+t_0)\frac{t_0^{p-1}}{1-t_0^{p-1}} + 1 = \frac{1+t_0^p}{1-t_0^{p-1}}.$$

Therefore (2.1) is obtained.
Corollary 2.2. For $X = l_\infty - l_1$, we have

$$C_{NJ}^{(\frac{3}{2})}(X) = \frac{1}{\sqrt{2} - \sqrt{2\sqrt{2}+1-\sqrt{5+4\sqrt{2}}}} \approx 1.5077. \tag{2.3}$$

and

$$C_{NJ}^{(3)}(X) = \frac{3+2\sqrt{2}+\sqrt{5+4\sqrt{2}}}{8} \approx 1.1366. \tag{2.4}$$

Proof. (1) For $p = \frac{3}{2}$, (2.2) is equivalent to $t^4 + 1 - 2t^3 - 2t - 5t^2 = 0$.
that is

$$t^2 + \frac{1}{t^2} - 2(t + \frac{1}{t}) = 5.$$

Hence, we can get $t = \frac{2\sqrt{2}+1-\sqrt{5+4\sqrt{2}}}{2}$ and (2.3) is valid by(2.1).
(2) For $p = 3$, (2.2) is equivalent to $t^2 = (1+t)^2(1-t^2)$. Letting $t = u - 1$, we have

$$u^4 + 1 - 2u^3 - 2u + u^2 = 0.$$

that is

$$u^2 + \frac{1}{u^2} - 2(u + \frac{1}{u}) = -1.$$

Hence, $u = \frac{\sqrt{2}+1+\sqrt{2\sqrt{2}-1}}{2}$ and $t = \frac{\sqrt{2}-1+\sqrt{2\sqrt{2}-1}}{2}$. Therefore

$$C_{NJ}^{(3)}(X) = \frac{1}{4(1-t^2)} = \frac{1}{2-2(\sqrt{2}-1)\sqrt{2\sqrt{2}-1}} = \frac{3+2\sqrt{2}+\sqrt{5+4\sqrt{2}}}{8} \approx 1.1366.$$

Theorem 2.3. ($l_q - l_1$ spaces). If $p \geq q > 1$. Let $X = \mathbb{R}^2$ endowed with the norm

$$\|x\| = \begin{cases} \|x\|_q, if \ x_1x_2 \geq 0 \\ \|x\|_1, if \ x_1x_2 \leq 0 \end{cases},$$

then

$$C_{NJ}^{(p)}(l_q - l_1) = 1 + 2^{\frac{p}{q}-p}.$$

In order to prove this theorem, firstly we give the following lemma.

Lemma2.4. Let $a, b, c, d \geq 0$ and $p \geq q > 1$ such that $a^q + b^q = 1$ and $c^q + d^q = 1$. If $0 \leq t \leq 1$, $a \geq ct$ and $b \leq dt$, then

$$[(a+ct)^q + (b+dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p \leq (1+t)^p + (1+t^q)^{\frac{p}{q}}.$$

Proof. Clearly, $0 \leq a - ct + dt - b \leq 1 + t$. So we will consider the following two cases.

Case I. if $0 \leq a - ct + dt - b \leq (1+t^q)^{\frac{1}{q}}$, then

$$\begin{aligned} &[(a+ct)^q + (b+dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p \\ &\leq [(a^q + b^q)^{\frac{1}{q}} + t(c^q + d^q)^{\frac{1}{q}}]^p + (1+t^q)^{\frac{p}{q}} \\ &= (1+t)^p + (1+t^q)^{\frac{p}{q}}. \end{aligned}$$

Case II. if $(1+t^q)^{\frac{1}{q}} \leq a - ct + dt - b \leq 1 + t$, then

$$\begin{aligned} &[(a+ct)^q + (b+dt)^q]^{\frac{1}{q}} + (a - ct + dt - b) \\ &\leq (a^q + d^q t^q)^{\frac{1}{q}} + (c^q t^q + b^q)^{\frac{1}{q}} + a - ct + dt - b \\ &\leq (1+t^q)^{\frac{1}{q}} + ct + b + a - ct + dt - b \\ &\leq (1+t^q)^{\frac{1}{q}} + 1 + t. \end{aligned}$$

So,

$$[(a+ct)^q + (b+dt)^q]^{\frac{1}{q}} \leq (1+t^q)^{\frac{1}{q}} + 1 + t - (a - ct + dt - b).$$

Thus,

$$\begin{aligned} &[(a+ct)^q + (b+dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p \\ &\leq [(1+t^q)^{\frac{1}{q}} + 1 + t - (a - ct + dt - b)]^p + (a - ct + dt - b)^p \\ &\leq \max_{u \in [(1+t^q)^{\frac{1}{q}}, 1+t]} [(1+t^q)^{\frac{1}{q}} + 1 + t - u]^p + u^p \\ &= (1+t)^p + (1+t^q)^{\frac{p}{q}}. \end{aligned}$$

Proof of Theorem 2.3

Note that $ex(B_X) = \{(x_1, x_2) : x_1^q + x_2^q = 1, x_1x_2 \geq 0\}$.

Now we prove that

$$\|x + ty\|^p + \|x - ty\|^p \leq (1+t)^p + (1+t^q)^{\frac{p}{q}},$$

holds for any $x, y \in ex(B_X)$ and any $t \in [0, 1]$.

Case I. If $(a-ct)(b-dt)\geq 0$. By Minkowski inequality, we have

$$\begin{aligned}&\|x+ty\|^p+\|x-ty\|^p\\ &=\|x+ty\|_q^p+\|x-ty\|_q^p\\ &=[(a+ct)^q+(b+dt)^q]^{\frac{p}{q}}+[|a-ct|^q+|b-dt|^q]^{\frac{p}{q}}\\ &\leq[(a^q+b^q)^{\frac{1}{q}}+(c^qt^q+d^qt^q)^{\frac{1}{q}}]^p+1\\ &\leq(1+t)^p+1\\ &\leq(1+t)^p+(1+t^q)^{\frac{p}{q}}.\end{aligned}$$

Case II. If $(a-ct)(b-dt)\leq 0$. By Lemma2.4, we have that

$$\begin{aligned}&\|x+ty\|^p+\|x-ty\|^p\\ &=\|x+ty\|_q^p+\|x-ty\|_1^p\\ &=[(a+ct)^q+(b+dt)^q]^{\frac{p}{q}}+(a-ct+dt-b)^p\\ &\leq(1+t)^p+(1+t^q)^{\frac{p}{q}}.\end{aligned}$$

Therefore, $\|x+ty\|^p+\|x-ty\|^p\leq(1+t)^p+(1+t^q)^{\frac{p}{q}}$ is also valid for any $x,y\in S_X$. Hence ,

$$C_{NJ}^{(p)}(l_q-l_1)\leq\frac{(1+t)^p+(1+t^q)^{\frac{p}{q}}}{2^{p-1}(1+t^p)}.$$

On the other hand, for every $t\in[0,1]$, taking $x_0=(1,0),y_0=(0,1)$, we have

$$\begin{aligned}&C_{NJ}^{(p)}(l_q-l_1)\\ &\geq\tfrac{\|x_0+ty_0\|^p+\|x_0-ty_0\|^p}{2^{p-1}(1+t^p)}\\ &=\tfrac{(1+t)^p+(1+t^q)^{\frac{p}{q}}}{2^{p-1}(1+t^p)}.\end{aligned}$$

Hence,

$$C_{NJ}^{(p)}(l_q-l_1)=\max_{t\in[0,1]}\frac{(1+t)^p+(1+t^q)^{\frac{p}{q}}}{2^{p-1}(1+t^p)}.$$

We let $f(t)=\frac{(1+t)^p+(1+t^q)^{\frac{p}{q}}}{1+t^p}$, so

$$f^{'}(t)=\frac{p\{(1+t^q)^{\frac{p}{q}-1}(t^{q-1}-t^{p-1})+(1+t)^{p-1}(1-t^{p-1})\}}{(1+t^p)^2}\geq 0.$$

That imply $f(t)$ is not decreasing. Hence,

$$\begin{aligned}&C_{NJ}^{(p)}(l_q-l_1)\\ &=2^{1-p}\max\nolimits_{t\in[0,1]}f(t)\\ &=2^{1-p}f(1)=1+2^{\frac{p}{q}-p}.\end{aligned}$$

Lemma2.6. Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$C_{NJ}^{(p)}(X)=2^{1-\frac{p}{q}}C_{NJ}^{(q)}(X^*)^{\frac{p}{q}}$$

and

$$C_{NJ}^{(p)}(X)=C_{NJ}^{(p)}(X^{**}),$$

where X^* is the dual of X.

Proof. Let $l_p(X)=\{(x_1,x_2):\|(x_1,x_2)\|=(\|x_1\|^p+\|x_2\|^p)^{\frac{1}{p}}\}$ and define the operator $A:l_p(X)\to l_p(X)$ by $(x_1,x_2)\mapsto(x_1+x_2,x_1-x_2)$. Then we easily have $C_{NJ}^{(p)}(X)=\frac{\|A\|^p}{2^{p-1}}$. Similarly, $C_{NJ}^{(q)}(X^*)=$

$\frac{\|A^*\|^q}{2^{q-1}}$. So $C_{NJ}^{(p)}(X) = 2^{1-\frac{p}{q}} C_{NJ}^{(q)}(X^*)^{\frac{p}{q}}$ by $\|A\| = \|A^*\|$, and hence $C_{NJ}^{(q)}(X^*) = 2^{1-\frac{q}{p}} C_{NJ}^{(p)}(X^{**})^{\frac{q}{p}}$. Therefore, we have $C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(X^{**})$.

The relationship between the constant $C_{NJ}^{(p)}(X)$ and the uniformly normal structure of X as follows:

Theorem 2.7. The Banach space X has uniformly normal structure if any one of the following conditions is valid

(i)$C_{NJ}^{(p)}(X) < \frac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}}$ for some $p \in \left(1, \frac{3-log_2 3}{2-log_2 3}\right)$;

(ii)$C_{NJ}^{(q)}(X^*) < \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2}$ for some $q > 1$,

where $p^{-1} + q^{-1} = 1$.

Proof. According to $C_{NJ}^{(p)}(X) < 2$, we have X is uniformly non-square, so we only need to prove X has weak normal structure.

Assume that X has no weak normal structure. Then it is well known (see[5])that for any $\varepsilon > 0$ there exists $z_1, z_2, z_3 \in S_X$ and $g_1, g_2, g_3 \in S_{X^*}$ satisfying the following statements:

(i) for all $i \neq j$, we have $|\|z_i - z_j\| - 1| < \varepsilon, |g_i(z_j)| < \varepsilon$,

(ii) $g_i(z_j) = 1$ for $i = 1, 2, 3$,

(iii) $\|z_3 - (z_2 + z_1)\| \geq \|z_2 + z_1\| - \varepsilon$.

Let us fix $\varepsilon > 0$ as small as needed. Then, we can find $z_1, z_2, z_3 \in S_X$ and $g_1, g_2, g_3 \in S_{X^*}$ satisfying the above properties.

(1)Taking $\alpha = \frac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}}$. We will consider the following two cases:

Case I. If $\|z_2 + z_1\| \leq \alpha$. Then,

$$\begin{aligned}
&\frac{\|g_1+g_2\|^q+\|g_2-g_1\|^q}{2^{q-1}(\|g_2\|^q+\|g_1\|^q)}\\
\geq\ &\frac{[(g_1+g_2)(\frac{z_2+z_1}{\alpha})]^q+[(g_2-g_1)(\frac{z_2-z_1}{\|z_2-z_1\|})]^q}{2^q}\\
\geq\ &\frac{(\frac{2-2\varepsilon}{\alpha})^q+(\frac{2-2\varepsilon}{1+\varepsilon})^q}{2^q}\\
=\ &(\frac{1-\varepsilon}{\alpha})^q + (\frac{1-\varepsilon}{1+\varepsilon})^q.
\end{aligned}$$

Case II. If $\|z_2 + z_1\| > \alpha$. Then, the contains two sub-cases:

(i) If $\|z_3 - z_2 + z_1\| \leq \alpha$. Then,

$$\begin{aligned}
&\frac{\|g_1+g_3\|^q+\|g_3-g_1\|^q}{2^{q-1}(\|g_3\|^q+\|g_1\|^q)}\\
\geq\ &\frac{[(g_1+g_3)(\frac{z_3-z_2+z_1}{\alpha})]^q+[(g_3-g_1)(\frac{z_3-z_1}{\|z_3-z_1\|})]^q}{2^q}\\
\geq\ &\frac{(\frac{2-4\varepsilon}{\alpha})^q+(\frac{2-2\varepsilon}{1+\varepsilon})^q}{2^q}\\
=\ &(\frac{1-2\varepsilon}{\alpha})^q + (\frac{1-\varepsilon}{1+\varepsilon})^q.
\end{aligned}$$

(ii) If $\|z_3 - z_2 + z_1\| > \alpha$. Then,

$$\begin{aligned}
&\frac{\|z_3-z_2+z_1\|^p+\|z_3-z_2-z_1\|^p}{2^{p-1}(\|z_3-z_2\|^p+\|z_1\|^p)}\\
\geq\ &\frac{\alpha^p+(\|z_2+z_1\|-\varepsilon)^p}{2^{p-1}[(1+\varepsilon)^p+1]}\\
\geq\ &\frac{\alpha^p+(\alpha-\varepsilon)^p}{2^{p-1}[(1+\varepsilon)^p+1]}.
\end{aligned}$$

Letting $\varepsilon \to 0$, and by lemma2.6 we have

$$C_{NJ}^{(p)}(X) \geq \min\{2^{1-\frac{p}{q}}(\frac{1}{\alpha^q}+1)^{\frac{p}{q}}, \frac{\alpha^p}{2^{p-1}}\} = \frac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}},$$

which contradicts to the hypothesis(i).

(2)Taking $\alpha = \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2}$. By the proof of (1), we have

$$C_{NJ}^{(q)}(X^*) \geq \min\{\frac{1}{\alpha^q}+1, 2^{1-\frac{q}{p}}(\frac{\alpha^p}{2^{p-1}})^{\frac{q}{p}}\} = \min\{\frac{1}{\alpha^q}+1, \frac{\alpha^q}{2^{q-1}}\} = \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2},$$

which contradicts to the hypothesis(ii).

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Discrete dynamical systems in soft topological spaces *

Wenqing Fu, Hu Zhao

Abstract In this paper the iteration of soft continuous functions is investigated and their discrete dynamical systems in soft topological spaces are defined. Some basic concepts related to discrete dynamical systems (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) are introduced into soft topological spaces. Soft topological mixing and soft topological transitivity are also studied. At last, soft topological entropy is defined and several properties of it are discussed.

Keywords Soft point, Soft ω-limit set, Soft nonwandering point, Soft topological mixing, Soft topological transitivity, Soft topological entropy

1 Introduction and preliminaries

The real world is too complex for our immediate and direct understanding, so we create *models* which are simplifications of the real word. In 1999, Molodtsov [1] introduced the concept of soft set which gives a new approach to modeling uncertainties. And he also discussed the application of soft set theory in many fields, such as: operations analysis, game theory, the smoothness of function, and so on[2]. Maji et al.[3] and Ali et al.[4] defined some operators of soft sets. Beyond these theoretical works of soft set, research works on its applications in various fields are progressing rapidly, and great progress has been achieved, including soft set theory in abstract algebras[5−10], decision making, data analysis, information system, and so on[11−14]. The application of soft set theory in algebraic structures was introduced by Aktaş and Çağman[5], they defined the notion of soft groups and progressed some basic properties. Jun[6,7] investigated soft BCK/BCI-algebras and its application in ideal theory. Dudek et al.[8] discussed soft ideals in BCC-algebras. Zhang[9] studied intuitionistic fuzzy soft rings. Feng et al.[10] worked on soft semirings, soft ideals and idealistic soft semirings. Maji et al.[11] first applied soft sets to solve the decision making problem that is based on the concept of knowledge reduction in the theory of rough sets[12]. Based on the analysis of the rough set model on a tolerance relation and the fuzzy rough set, two types of fuzzy rough sets models on tolerance relations are constructed and researched by Xu et al.[13]. Chen et al.[14] presented a

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new definition of soft set parametrization reduction so as to improve the soft set based decision making in [11]. Yang[15] combined the multi-fuzzy set and soft set, from which they obtained a new soft set model named multi-fuzzy soft set, and applied it to decision making. Soft set theory is also be used in topology. Shabir and Naz's work[16] on soft topological spaces defined over an initial universe with a fixed set of parameters. The notions of soft open set, soft closed set, soft closure, soft interior point, soft neighborhood of a point, and soft separation axioms (such as soft T_i-space for $i = 1, 2, 3, 4$, soft normal space, and soft regular space) were also introduced and their basic properties were investigated. Min[17] pointed out some mistakes of [16] and investigated some properties of the soft separation axioms defined in [15]. Zorlutuna etc.[18] introduced some new concepts in soft topological spaces (such as soft point, interior point, interior, neighborhood, continuity, and compactness).

Motivated by Chen etc.[19] and Liu[20], this paper will investigate iteration of soft continuous functions and their discrete dynamical systems in soft topological spaces. Some basic concepts on dynamical systems (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) are introduced in soft topological spaces, soft topological mixing, soft topological transitivity, soft topological entropy and its several properties are studied. As a result, some conclusions of discrete dynamical systems in ordinary topological spaces are generalized. Now we give some definitions and results to be used in this paper.

Definition 1[1] A soft set on a set X is a triple (M, E, X), where $M : E \longrightarrow 2^X$ (the set of all subsets of X) is a mapping. The set of all soft sets on X is denoted by $\mathbb{S}(X, E)$.

Roughly speaking, a soft set on a set X is just a family $\{M_e\}_{e\in E}$ of subsets of X; it can be looked to be a subset of X if E is a singleton.

Let $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$. If $M(e) \subseteq N(e)$ $(\forall e \in E)$, then (M, E, X) is called a soft subset of (N, E, X), denoted by $(M, E, X)\widetilde{\subseteq}(N, E, X)$. If $(M, E, X)\widetilde{\subseteq}(N, E, X)$ and $(M, E, X)\widetilde{\supseteq}(N, E, X)$, then (M, E, X) and (N, E, X) are said to be soft equal, denoted by $(M, E, X) = (N, E, X)$.

Remark 1[16] (1) Let X be a set, and $A \in 2^X$. Define $\widetilde{A} : E \longrightarrow 2^X$ as $\widetilde{A}(e) = A$ $(\forall e \in E)$, then $(\widetilde{A}, E, X) \in \mathbb{S}(X, E)$; we use $\widetilde{A}$ to denote this soft set (particularly, we use $\widetilde{x}$ to denote the soft set $\widetilde{\{x\}}$).

(2) Let X be a set, and $(M, E, X) \in \mathbb{S}(X, E)$. Then $(M', E, X) \in \mathbb{S}(X, E)$, where $M' : E \longrightarrow 2^X$ is defined as

$$M'(e) = X - M(e) \ (\forall e \in E).$$

Sometimes we use $(M, E, X)'$ to replace (M', E, X).

(3) Let X be a set, $\{(H_j, E, X)\}_{j\in J} \subseteq \mathbb{S}(X, E)$. Then $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$, called the union (denoted as $\widetilde{\bigcup}_{j\in J}(H_j, E, X)$) and intersection (denoted as $\widetilde{\bigcap}_{j\in J}(H_j, E, X)$)

of the family $\{(H_j, E, X)\}_{j\in J}$ respectively, where

$$M(e) = \bigcup_{j\in J} H_j(e)\ (\forall e \in E)$$

and

$$N(e) = \bigcap_{j\in J} H_j(e)\ (\forall e \in E).$$

(4) Let X be a set, $(H, E, X) \in \mathbb{S}(X, E)$, and $x \in X$. Write $x \in (H, E, X)$ if $x \in H(e)$ $(\forall e \in E)$, and $x \notin (H, E, X)$ if $x \notin H(e)$ for some $e \in E$.

(5) Let X be a set. The difference of the two soft sets (M, E, X) and (N, E, X) is a soft set (H, E, X) over X (usually, denoted by $(M, E, X) - (N, E, X)$) which is defined by $H(e) = M(e) - N(e)$ $(\forall e \in E)$.

(6) Let X be a set, and $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$. Then

(i) $((M, E, X)\widetilde{\cup}(N, E, X))' = (M, E, X)'\widetilde{\cap}(N, E, X)'$;

(ii) $((M, E, X)\widetilde{\cap}(N, E, X))' = (M, E, X)'\widetilde{\cup}(N, E, X)'$.

Definition 2[18] (1) A soft set $(M, E, X) \in \mathbb{S}(X, E)$ is called elementary (or a soft point in $\widetilde{X}$, denoted by e_M) if $M(e) \neq \emptyset$ for some $e \in E$ and $M(e') = \emptyset$ for all $e' \in E - \{e\}$.

(2) Let e_M be a soft point in $\widetilde{X}$, and (N, E, X) is a soft set. If $M(e) \subseteq N(e)$, then e_M is said to be in (N, E, X), denoted by $e_M\widetilde{\in}(N, E, X)$.

Definition 3[17] Let X and Y be two sets, E and F be two nonempty parameter sets, and $f : E \longrightarrow F$ and $g : X \longrightarrow Y$ are mappings. For each $(M, E, X) \in \mathbb{S}(X, E)$, define

$$(f, g)(M, E, X) = (g^{\rightarrow}(M), f(E), Y),$$

where

$$g^{\rightarrow}(M)(\alpha) = \bigcup_{f(e)=\alpha} g(M(e))\ (\forall \alpha \in F).$$

Then we obtain a mapping

$$(f, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, F).$$

For each $(N, F, Y) \in \mathbb{S}(Y, F)$, define

$$(f, g)^{-1}(N, F, Y) = (g^{-1} \circ N \circ f, f^{-1}(F), X),$$

where

$$(g^{-1} \circ N \circ f)(e) = g^{-1}(N(f(e)))\ \ (\forall e \in f^{-1}(F)).$$

Then we obtain another mapping

$$(f, g)^{-1} : \mathbb{S}(Y, F) \longrightarrow \mathbb{S}(X, E).$$

Definition 4[16] (1) Let X be a set, and $\mathscr{T} \subseteq \mathbb{S}(X, E)$ satisfies

(i) $\emptyset$ and $X \in \mathscr{T}$;

(ii) $\mathscr{T}$ is closed under arbitrary unions;

(ii) $\mathscr{T}$ is closed under finite intersections.

Then $\mathscr{T}$ is called a soft topology on X, and $(X, \mathscr{T}, E)$ is called a soft topological space. The members of $\mathscr{T}$ are called soft open sets, members of $\mathscr{T}' = \{(M', E, X) \mid (M, E, X) \in \mathscr{T}\}$ are called soft closed sets.

(2) Let $(X, \mathscr{T}, E)$ be a soft topological space, and Y be a non-empty subset of X. Then

$$\mathscr{T}_Y = \{(M_Y, E, X) \mid (M, E, X) \in \mathscr{T}\}$$

is a soft topology on Y, it is called the soft relative topology on Y, and $(Y, \mathscr{T}_Y, E)$ is called a soft subspace of $(X, \mathscr{T}, E)$, where

$$(M_Y, E, X) = \widetilde{Y} \widetilde{\cap} (M, E, X) \ (\forall (M, E, X) \in \mathscr{T}).$$

Example 1 (1) Let $X = \{x_1, x_2, x_3\}$ be a 3-element set, $E = \{e_1, e_2\}$ be a 2-element set, and

$$\mathscr{T} = \{(M_i, E, X) \mid i = 1, 2, \cdots, 6\} \cup \{\widetilde{\emptyset}, \widetilde{X}\},$$

where (M_i, E, X) $(i = 1, 2, \cdots, 6)$ are defined as follows:

$$M_1(e) = \begin{cases} \{x_2\}, & \text{if } e = e_1; \\ \{x_1\}, & \text{if } e = e_2. \end{cases}$$

$$M_2(e) = \begin{cases} \{x_1\}, & \text{if } e = e_1; \\ \{x_3\}, & \text{if } e = e_2. \end{cases}$$

$$M_3(e) = \begin{cases} \{x_3\}, & \text{if } e = e_1; \\ \{x_2\}, & \text{if } e = e_2. \end{cases}$$

$$M_4(e) = \begin{cases} \{x_2, x_3\}, & \text{if } e = e_1; \\ \{x_1, x_2\}, & \text{if } e = e_2. \end{cases}$$

$$M_5(e) = \begin{cases} \{x_1, x_2\}, & \text{if } e = e_1; \\ \{x_1, x_3\}, & \text{if } e = e_2. \end{cases}$$

$$M_6(e) = \begin{cases} \{x_1, x_3\}, & \text{if } e = e_1; \\ \{x_2, x_3\}, & \text{if } e = e_2. \end{cases}$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

(2) Let $X = R$ (the set of all real numbers), $E = \{e_1, e_2\}$ be a 2-element set,

$$\mathcal{J} = \{A \subseteq X \mid X - A \text{ is a finite subset of } X\} \cup \{\emptyset, X\}$$

(i.e. the finite complement topology on X), and

$$\mathscr{T} = \{(M, E, X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

(3) Let $X = R$, $E = \{e_1, e_2\}$ be a 2-element set, $\mathcal{J}$ be the ordinary topology on X (i.e. $\mathcal{J}$ is the topology on X generated by the basis $\mathcal{B} = \{(a,b) \mid a,b \in R, a < b\}$), and

$$\mathscr{T} = \{(M,E,X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

(4) Let $X = [0,1]$, $E = \{e_1, e_2\}$ be a 2-element set, $\mathcal{J}$ be the ordinary topology on X (i.e. $\mathcal{J}$ is the topology on $[0,1]$ generated by the basis

$$\mathcal{B} = \{(a,b) \mid a \in [0,1), b \in (0,1], a < b\}),$$

and

$$\mathscr{T} = \{(M,E,X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

Remark 2 (1)[16] Let $(X, \mathscr{T}, E)$ be a soft topological space, e_M is a soft point in $\widetilde{X}$, $(N,E,X) \in \mathbb{S}(X,E)$. If there exists a $(A,E,X) \in \mathscr{T}$ such that

$$e_M \widetilde{\in} (A,E,X) \widetilde{\subseteq} (N,E,X),$$

then (N,E,X) is called a neighborhood of e_M.

(2) It can be easily seen that $\widetilde{\emptyset}, \widetilde{X} \in \mathscr{T}'$, and $\mathscr{T}'$ is closed under the operations of arbitrary intersections and finite unions. It can be also seen that $(N,E,X) \in \mathscr{T}'$ if and only if

$$((A,E,X) - e_M) \widetilde{\cap} (N,E,X) \neq \widetilde{\emptyset}$$

for any$e_M \in \widetilde{X}$ and any neighborhood (A,E,X) of e_M.

(3)[16] Let$(X, \mathscr{T}, E)$ be a soft topological space, and $(M,E,X) \in \mathbb{S}(X,E)$. Then

$$\overline{(M,E,X)} = \widetilde{\bigcap}\{(N,E,X) \mid (M,E,X) \widetilde{\subseteq} (N,E,X),$$
$$(N,E,X) \in \mathscr{T}'_X\}$$

is called the closure of (M,E,X). Clearly, $(M,E,X) \in \mathbb{S}(X,E)$ is a soft closed set of $(X, \mathscr{T}, E)$ if and only if $\overline{(M,E,X)} = (M,E,X)$.

(4)[16] Let $(X, \mathscr{T}, E)$ be a soft topological space over X, then $\mathscr{T}^e = \{M(e) \mid (M,E,X) \in \mathscr{T}\}$ is a topology on X $(e \in E)$.

(5) If E is a single point set, then a soft topological space $(X, \mathscr{T}, E)$ can be seen as a common topological space.

Definition 5 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces. A soft function

$$(f,g) : \mathbb{S}(X,E) \longrightarrow \mathbb{S}(Y,E)$$

is said to be a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$ if

$$(f,g)^{-1}(N,E,Y) \in \mathscr{T}_X \quad (\forall (N,E,Y) \in \mathscr{T}_Y).$$

Remark 3 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces, and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$$

be a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$. Then $g : X \longrightarrow Y$ is a continuous function from $(X, \mathscr{T}_X^e)$ to $(Y, \mathscr{T}_Y^e)$ $(\forall e \in E)$.

Definition 6[18] (1) Let $(X, \mathscr{T}, E)$ be a soft topological space, $(P, E, X) \in \mathbb{S}(X, E)$, and $\mathscr{A} \subseteq \mathscr{T}$. If

$$\widetilde{\bigcup}\mathscr{A} = (P, E, X),$$

then $\mathscr{A}$ is called an soft open cover of (P, E, X).

(2) Let $(X, \mathscr{T}, E)$ be a soft topological space, and $(P, E, X) \in \mathbb{S}(X, E)$. (P, E, X) is said to be soft compact if every open soft cover of it has a finite subcover. If $\widetilde{X}$ is compact, then $(X, \mathscr{T}, E)$ is called a soft compact topological space.

Theorem 1[18] Let $(X, \mathscr{T}, E)$ be a soft compact topological space, then each soft closed subset (P, E, X) is a soft compact subset of $\widetilde{X}$.

Theorem 2 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces, and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$$

is a soft function. Then the following conditions are equivalent:

(1) (id_E, g) is a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$.

(2) $(id_E, g)^{-1}(N, E, Y) \in \mathscr{T}'_X \quad (\forall\ (N, E, Y) \in \mathscr{T}'_Y)$.

(3) $(id_E, g)\overline{(M, E, X)} \widetilde{\subseteq} \overline{(id_E, g)(M, E, X)} \quad (\forall (M, E, X) \in \mathbb{S}(X, E))$.

(4) $(id_E, g)^{-1}\overline{(P, E, Y)} \widetilde{\supseteq} \overline{(id_E, g)^{-1}(P, E, Y)} \quad (\forall (P, E, Y) \in \mathbb{S}(Y, E))$.

Proof Straightforward. □

2 Discrete dynamical systems in soft topological spaces

Let X be a topological space, and $g : X \longrightarrow X$ a continuous mapping, then the family $\{g^n\}_{n \in N}$ defines a (discrete) semi-dynamical system in topological space X, where N stands for the set of all nonnegative integers. In addition, if g is a homeomorphism (i.e. g is a one-to-one correspondence and both g and its inverse mapping g^{-1} are continuous), then we can define g^{-n} by $g^{-n} = (g^{-1})^n$ $(\forall n \in N)$, then $\{g^n\}_{n \in Z}$ defines a discrete dynamical system in topological space X, where Z stands for the set of all integers.

Let $(X, \mathscr{T}, E)$ be a soft topological space and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(X, E)$$

be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$. It can be seen from definition 3 that

$$(g^n)^{\rightarrow} = (g^{\rightarrow})^n,$$

so we can define the n-th iterate of (id_E, g) $(n \in N)$ as follows:

$$\begin{aligned}(id_E, g)^n &= (id_E, g) \circ (id_E, g)^{n-1} = (id_E \circ id_E, g \circ g^{n-1}) \\ &= (id_E, g^n),\end{aligned}$$

$(id_E, g)^0 = (id_E, g^0) = (id_E, id_X)$,

where id_E (resp. id_X) denotes the identity mapping of E (resp., X) onto itself. Then the family $\{(id_E, g)^n\}_{n\in N}$ defines a (discrete) semi-dynamical system in soft topological space $(X, \mathscr{T}, E)$, where N stands for the set of all nonnegative integers. If g is a one-to-one correspondence and both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous, it can be seen from definition 3 that

$$(g^{\leftarrow})^n = (g^n)^{\leftarrow} \ (\forall n \in N - \{0\})$$

and

$$((g^n)^{\leftarrow})^m = (g^{\leftarrow})^{nm} \quad (\forall n \in N - \{0\}, \forall m \in N).$$

Let

$$(id_E, g)^{-n} = (id_E, g^{-n}) = (id_E, (g^n)^{-1}) \quad (\forall n \in N),$$

then $\{(id_E, g)^n\}_{n\in Z}$ defines a discrete dynamical system in soft topological space, and it is denoted by $(X, (id_E, g))$. If $(X, \mathscr{T}, E)$ is a soft compact topological space, then $(X, (id_E, g))$ is called a soft compact discrete topological dynamical system. It is easy to show that $(id_E, g)^n(e_M)$
$(\forall n \in Z)$ is a soft point when e_M is a soft point.

Example 2 Let us consider the soft topological space in Example 1(1). Define $g : X \longrightarrow X$ as follows:

$$g(x_1) = x_2, \quad g(x_2) = x_3, \quad g(x_3) = x_1.$$

We will verify that both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous. In fact,

$$(id_E, g)^{-1}(M_1, E, X) = (g^{-1} \circ M_1 \circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1} \circ M_1 \circ id_E(e) &= g^{-1}((M_1)(e)) \\ &= \begin{cases} g^{-1}(\{x_2\}), & \text{if } e = e_1; \\ g^{-1}(\{x_1\}), & \text{if } e = e_2. \end{cases} \\ &= \begin{cases} \{x_1\}, & \text{if } e = e_1; \\ \{x_3\}, & \text{if } e = e_2. \end{cases} \\ &= M_2(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_1, E, X) = (M_2, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_2, E, X) = (g^{-1} \circ M_2 \circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_2\circ id_E(e) &= g^{-1}((M_2)(e)) \\ &= \begin{cases} g^{-1}(\{x_1\}), & \text{if } e=e_1; \\ g^{-1}(\{x_3\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_3\}, & \text{if } e=e_1; \\ \{x_2\}, & \text{if } e=e_2. \end{cases} \\ &= M_3(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_2, E, X) = (M_3, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_3, E, X) = (g^{-1}\circ M_3\circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_3\circ id_E(e) &= g^{-1}((M_3)(e)) \\ &= \begin{cases} g^{-1}(\{x_3\}), & \text{if } e=e_1; \\ g^{-1}(\{x_2\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_2\}, & \text{if } e=e_1; \\ \{x_1\}, & \text{if } e=e_2. \end{cases} \\ &= M_1(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_3, E, X) = (M_1, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_4, E, X) = (g^{-1}\circ M_4\circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_4\circ id_E(e) &= g^{-1}((M_4)(e)) \\ &= \begin{cases} g^{-1}(\{x_2, x_3\}), & \text{if } e=e_1; \\ g^{-1}(\{x_1, x_2\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_1, x_2\}, & \text{if } e=e_1; \\ \{x_3, x_1\}, & \text{if } e=e_2. \end{cases} \\ &= M_5(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_4, E, X) = (M_5, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_5, E, X) = (g^{-1}\circ M_5\circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_5\circ id_E(e) &= g^{-1}((M_5)(e)) \\ &= \begin{cases} g^{-1}(\{x_1, x_2\}), & \text{if } e=e_1; \\ g^{-1}(\{x_3, x_1\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_3, x_1\}, & \text{if } e=e_1; \\ \{x_2, x_3\}, & \text{if } e=e_2. \end{cases} \\ &= M_6(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_5, E, X) = (M_6, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_6, E, X) = (g^{-1}\circ M_6\circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_6\circ id_E(e) &= g^{-1}((M_6)(e)) \\ &= \begin{cases} g^{-1}(\{x_1, x_3\}), & \text{if } e=e_1; \\ g^{-1}(\{x_2, x_3\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_3, x_2\}, & \text{if } e=e_1; \\ \{x_1, x_2\}, & \text{if } e=e_2. \end{cases} \\ &= M_4(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_6, E, X) = (M_4, E, X) \in \mathscr{T}$. It is easy to see that

$$(id_E, g)^{-1}(\widetilde{\emptyset}) = \widetilde{\emptyset} \in \mathscr{T}$$

and

$$(id_E, g)^{-1}(\widetilde{X}) = \widetilde{X} \in \mathscr{T}.$$

Therefore, (id_E, g) is continuous.

From the above, it is easy to see that

$$(id_E, g)^{-1} = (id_E, g^{-1}),$$

since for any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g^{-1})(M, E, X) = ((g^{-1})^{\rightarrow}(M), E, X),$$

where

$$(g^{-1})^{\rightarrow}(M)(e) = g^{-1}(M)(e) = g^{-1} \circ M \circ id_E(e).$$

Thus for any $(M, E, X) \in \mathscr{T}$,

$$\begin{aligned} & ((id_E, g)^{-1})^{-1}(M, E, X) \\ = \ & (id_E, g^{-1})^{-1}(M, E, X) \\ = \ & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\ = \ & (g \circ M \circ id_E, E, X) \end{aligned}$$

Hence

$$((id_E, g)^{-1})^{-1}(M_1, E, X) = (g \circ M_1 \circ id_E, E, X),$$

where

$$\begin{aligned} g \circ M_1 \circ id_E(e) &= g((M_1)(e)) \\ &= \begin{cases} g(\{x_2\}), & \text{if } e = e_1; \\ g(\{x_1\}), & \text{if } e = e_2. \end{cases} \\ &= \begin{cases} \{x_3\}, & \text{if } e = e_1; \\ \{x_2\}, & \text{if } e = e_2. \end{cases} \\ &= M_3(e) \end{aligned}$$

Thus $((id_E, g)^{-1})^{-1}(M_1, E, X) = (M_3, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_2, E, X) = (g \circ M_2 \circ id_E, E, X),$$

where

$$\begin{aligned} g \circ M_2 \circ id_E(e) &= g((M_2)(e)) \\ &= \begin{cases} g(\{x_1\}), & \text{if } e = e_1; \\ g(\{x_3\}), & \text{if } e = e_2. \end{cases} \\ &= \begin{cases} \{x_2\}, & \text{if } e = e_1; \\ \{x_1\}, & \text{if } e = e_2. \end{cases} \\ &= M_1(e) \end{aligned}$$

Thus $((id_E, g)^{-1})^{-1}(M_2, E, X) = (M_1, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_3, E, X) = (g \circ M_3 \circ id_E, E, X),$$

where

$$\begin{aligned} g\circ M_3\circ id_E(e) &= g((M_3)(e)) \\ &= \begin{cases} g(\{x_3\}), & \text{if } e=e_1; \\ g(\{x_2\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_1\}, & \text{if } e=e_1; \\ \{x_3\}, & \text{if } e=e_2. \end{cases} \\ &= M_2(e) \end{aligned}$$

Thus $((id_E,g)^{-1})^{-1}(M_3,E,X)=(M_2,E,X)\in\mathscr{T}$.

$$((id_E,g)^{-1})^{-1}(M_4,E,X)=(g\circ M_4\circ id_E,E,X),$$

where

$$\begin{aligned} g\circ M_4\circ id_E(e) &= g((M_4)(e)) \\ &= \begin{cases} g(\{x_2,x_3\}), & \text{if } e=e_1; \\ g(\{x_1,x_2\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_1,x_3\}, & \text{if } e=e_1; \\ \{x_2,x_3\}, & \text{if } e=e_2. \end{cases} \\ &= M_6(e) \end{aligned}$$

Thus $((id_E,g)^{-1})^{-1}(M_4,E,X)=(M_6,E,X)\in\mathscr{T}$.

$$((id_E,g)^{-1})^{-1}(M_5,E,X)=(g\circ M_5\circ id_E,E,X),$$

where

$$\begin{aligned} g\circ M_5\circ id_E(e) &= g((M_5)(e)) \\ &= \begin{cases} g(\{x_1,x_2\}), & \text{if } e=e_1; \\ g(\{x_3,x_1\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_2,x_3\}, & \text{if } e=e_1; \\ \{x_1,x_2\}, & \text{if } e=e_2. \end{cases} \\ &= M_4(e) \end{aligned}$$

Thus $((id_E,g)^{-1})^{-1}(M_5,E,X)=(M_4,E,X)\in\mathscr{T}$.

$$((id_E,g)^{-1})^{-1}(M_6,E,X)=(g\circ M_6\circ id_E,E,X),$$

where

$$\begin{aligned} g\circ M_6\circ id_E(e) &= g((M_6)(e)) \\ &= \begin{cases} g(\{x_1,x_3\}), & \text{if } e=e_1; \\ g(\{x_2,x_3\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_2,x_1\}, & \text{if } e=e_1; \\ \{x_3,x_1\}, & \text{if } e=e_2. \end{cases} \\ &= M_5(e) \end{aligned}$$

Thus $((id_E,g)^{-1})^{-1}(M_6,E,X)=(M_5,E,X)\in\mathscr{T}$. It is easy to see that

$$((id_E,g)^{-1})^{-1}(\widetilde{\emptyset})=\widetilde{\emptyset}\in\mathscr{T}$$

and

$$((id_E,g)^{-1})^{-1}(\widetilde{X})=\widetilde{X}\in\mathscr{T}.$$

Therefore, $(id_E,g)^{-1}$ is continuous. Hence, $(X,(id_E,g))$ is a soft topological dynamical system.

Example 3 Let us consider the soft topological space in Example 1(2). Let $g : X \longrightarrow X$ be an arbitrary one-to-one respondence on X. Then for any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

the complement $X - g^{-1} \circ M \circ id_E(e)$ is still a finite subset of X since g is an one-to-one respondence, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous.

On the other hand, for any $(M, E, X) \in \mathscr{T}$,

$$\begin{aligned} & ((id_E, g)^{-1})^{-1}(M, E, X) \\ = \ & (id_E, g^{-1})^{-1}(M, E, X) \\ = \ & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\ = \ & (g \circ M \circ id_E, E, X) \end{aligned}$$

where

$$g \circ M \circ id_E(e) = g(M(e)) \quad (\forall e \in E),$$

the complement $X - g \circ M \circ id_E(e)$ is still a finite subset of X since g is an one-to-one respondence, thus

$$(id_E, g)(M, E, X) \in \mathscr{T}.$$

Therefore, $(id_E, g)^{-1}$ is continuous. Hence, $(X, (id_E, g))$ is a soft topological dynamical system.

Example 4 Let us consider the soft topological space in Example 1(3). Define $g : X \longrightarrow X$ as follows:

$$g(x) = x + 1 \quad (\forall x \in X).$$

Then for every $(a, b) \in \mathcal{B}$, $g(a, b) = (a + 1, b + 1)$, and $g^{-1}(a, b) = (a - 1, b - 1)$, thus $g(\mathcal{B}) = g^{-1}(\mathcal{B}) = \mathcal{B}$. Denote the topology on X generated by $g(\mathcal{B})$ and $g^{-1}(\mathcal{B})$ by $g(\mathcal{J})$ and $g^{-1}(\mathcal{J})$. Then $g(\mathcal{J}) = g^{-1}(\mathcal{J}) = \mathcal{J}$.

For any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g^{-1}(M(e)) \in g^{-1}(\mathcal{J}) = \mathcal{J}$, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous.

On the other hand, for any $(M, E, X) \in \mathscr{T}$,

$$\begin{aligned} & ((id_E, g)^{-1})^{-1}(M, E, X) \\ = \ & (id_E, g^{-1})^{-1}(M, E, X) \\ = \ & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\ = \ & (g \circ M \circ id_E, E, X) \end{aligned}$$

where

$$g \circ M \circ id_E(e) = g(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g(M(e)) \in g(\mathcal{J}) = \mathcal{J}$, thus $(id_E, g)(M, E, X) \in \mathscr{T}$. Therefore, $(id_E, g)^{-1}$ is continuous. Hence, $(X, (id_E, g))$ is a soft topological dynamical system.

Example 5 Let us consider the soft topological space in Example 1(4). Define $g : X \longrightarrow X$ as follows:

$$g(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}]; \\ 2 - 2x, & x \in (\frac{1}{2}, 1]. \end{cases}$$

For every $(a, b) \in \mathcal{B}$,

$$g^{-1}(a, b) = \begin{cases} (\frac{a}{2}, \frac{b}{2}), & b \leq \frac{1}{2}; \\ (\frac{2-a}{2}, \frac{2-b}{2}), & a \geq \frac{1}{2}; \\ (\frac{a}{2}, \frac{2-b}{2}), & a < \frac{1}{2} < b. \end{cases}$$

Thus $g^{-1}(\mathcal{B}) \subseteq \mathcal{B}$. Let $g^{-1}(\mathcal{J})$ be the topology on X generated by $g^{-1}(\mathcal{B})$, then $g^{-1}(\mathcal{J}) \subseteq \mathcal{J}$.

For any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g^{-1}(M(e)) \in g^{-1}(\mathcal{J}) \subseteq \mathcal{J}$, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous. Hence, $(X, (id_E, g))$ is a semi-soft topological dynamical system.

Definition 7 Let $(X, (id_E, g))$ be a soft discrete topological dynamical system and $e_M \in \widetilde{X}$ is a soft point. Define several soft sets as follows:

$$Orb_{(id_E, g)}(e_M) = \{(id_E, g)^n(e_M) \mid n \in Z\},$$

$$Orb^+_{(id_E, g)}(e_M) = \{(id_E, g)^n(e_M) \mid n \in N - \{0\}\}.$$

$$Orb^-_{(id_E, g)}(e_M) = \{(id_E, g)^{-n}(e_M) \mid n \in N - \{0\}\}.$$

Then we call $Orb_{(id_E, g)}(e_M)$ (resp., $Orb^+_{(id_E, g)}(e_M)$, $Orb^-_{(id_E, g)}(e_M)$) the soft orbit (resp., soft positive semi-orbit, soft negative semi-orbit) of the soft dynamical system of (id_E, g).

Let $e_M \in \widetilde{X}$, if $(id_E, g)^n(e_M) = e_M$ for some $n \in N - \{0\}$, then e_M is called a soft periodic point of (id_E, g), the smallest one of such integers is referred to as the soft period of e_M. In particular, if $(id_E, g)(e_M) = e_M$, then e_M is called a soft fixed point of (id_E, g). Let $Per(id_E, g)$ (resp. $Fix(id_E, g)$) be the set of all soft periodic points (resp. all soft fixed points) of (id_E, g). Then $Fix(id_E, g) \subseteq Per(id_E, g)$.

Definition 8 Let $e_M \in \widetilde{X}$ be a soft point, then the soft set

$$\omega(e_M) = \widetilde{\bigcap}_{n \in N - \{0\}} \overline{\widetilde{\bigcup} \{(id_E, g)^k(e_M) \mid k \geq n\}},$$

is called a soft ω-limit set.

Obviously $\omega(e_M)$ is a soft closed set of $(X, \mathscr{T}, E)$. If the soft topological space $(X, \mathscr{T}, E)$ is soft compact, then $\omega(e_M) \neq \widetilde{\emptyset}$ by Theorem 7.4 in [20].

Definition 9 Let $(X, (id_E, g))$ be a soft discrete topological dynamical system, and $e_M \in \widetilde{X}$ a soft point.

(1) If for each soft open neighborhood (N, E, X) of e_M, there exists an $n \in N - \{0\}$ such that $(id_E, g)^n(e_M)$
$\widetilde{\in}(N, E, X)$, then e_M is called a soft recurrent points of (id_E, g). The set of all soft recurrent points of (id_E, g) is denoted by $Rec(id_E, g)$. Clearly, $Per(id_E, g) \subseteq Rec(id_E, g)$.

(2) If for each soft open neighborhood (N, E, X) of e_M, there exists an $n \in N - \{0\}$ such that

$$(id_E, g)^{-n}(N, E, X)\widetilde{\cap}(N, E, X) \neq \widetilde{\emptyset}.$$

Then e_M is called a soft nonwandering point of (id_E, g). The set of all soft nonwandering points of (id_E, g) is denoted by $\Omega(id_E, g)$, i.e.,

$\Omega(id_E, g) = \{e_M \in \widetilde{X} \mid e_M$ be a soft nonwandering point of $(id_E, g)\}$.
Each soft point of $\widetilde{X} - \Omega(id_E, g)$ is called a soft wandering point.

Definition 10 Let (id_E, g) be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$.

(1) (id_E, g) is called soft topological mixing if, for any pair (M, E, X) and $(N, E, X) \in \mathscr{T}$ of nonempty soft open sets of $(X, \mathscr{T}, E)$, there exists an $n \in N - \{0\}$ such that
$(id_E, g)^n(M, E, X)\widetilde{\cap}(N, E, X) \neq \widetilde{\emptyset}$.

(2) (id_E, g) is called soft topological transitivity if there exists a soft point $e_M \in \widetilde{X}$ such that $Orb_{(id_E,g)}(e_M)$ is dense in $\widetilde{X}$ (i.e. $\overline{Orb_{(id_E,g)}(e_M)} = \widetilde{X}$).

(3) A soft set (N, E, X) is said to be soft invariant of (id_E, g) if $(id_E, g)(N, E, X)\widetilde{\subseteq}(N, E, X)$ (i.e. $g(N(e)) \subseteq N(e)$ for each $e \in E$).

Theorem 3 Let $(X, \mathscr{T}, E)$ be a soft topological space, and $(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(\mathbb{X}, \mathbb{E})$ be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$. Then

(1) $\Omega(id_E, g)$ is a soft closed set of $\widetilde{X}$, and $Rec(id_E, g)$
$\widetilde{\subseteq}\Omega(id_E, g)$.

(2) $Orb_{(id_E,g)}(e_M)$, $\omega(e_M)$, $Per(id_E, g)$, $Fix(id_E, g)$ and $\Omega(id_E, g)$ are invariant of (id_E, g).

(3) $\Omega((id_E, g)^m)$ is an invariant and closed soft set, and

$$\Omega((id_E, g)^m)\widetilde{\subseteq}\Omega(id_E, g) \quad (m \in N - \{0\}).$$

(4) Each soft point $e_M \in \widetilde{X}$ is a soft nonwandering point if one of the following conditions is satisfied:

(i) (id_E, g) is soft topological mixing, g is a one-to-one correspondence, and both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous

(ii) $Per(id_E, g) = \widetilde{X}$.

Proof (1) Suppose that a soft point e_M is not a soft wandering point of (id_E, g), then there exists some soft open neighborhood (N, E, X) and some $n \in N - \{0\}$ such that

$$(id_E, g)^{-n}(N, E, X)\widetilde{\cap}(N, E, X) = \widetilde{\emptyset}.$$

So all the soft points in (N, E, X) are not soft wandering points of (id_E, g), it follows that $\Omega(id_E, g)$ be a soft closed set of $\widetilde{X}$.

Now let soft point $e_M \in Rec(id_E, g)$, then for each soft open neighborhood (N, E, X) of e_M, there exists some $n \in N - \{0\}$ such that $(id_E, g)^n(e_M)\widetilde{\subseteq}(N, E, X)$, so for any $e \in E$, $g^n(M(e)) \subseteq N(e)$, thus $M(e) \subseteq g^{-n}(N(e))$, it implies that

$$e_M\widetilde{\in}(id_E, g^{-n})(N, E, X) = (id_E, g)^{-n}(N, E, X),$$

then

$$e_M\widetilde{\in}(id_E, g)^{-n}(N, E, X)\widetilde{\cap}(N, E, X),$$

hence

$$Rec(id_E, g)\widetilde{\subseteq}\Omega(id_E, g).$$

(2) We only show that $\omega(e_M)$ and $\Omega(id_E, g)$ are invariant sets of (id_E, g). Firstly, we have

$$\begin{aligned}
& (id_E, g)(\omega(e_M)) \\
= \;& (id_E, g)(\widetilde{\bigcap}_{n\in N-\{0\}}\overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}}) \\
\widetilde{\subseteq} \;& \widetilde{\bigcap}_{n\in N-\{0\}}(id_E, g)\overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}}) \\
\widetilde{\subseteq} \;& \widetilde{\bigcap}_{n\in N-\{0\}}\overline{\widetilde{\bigcup}\{(id_E, g)^{k+1}(e_M) \mid k \geq n\}}) \\
\widetilde{\subseteq} \;& \widetilde{\bigcap}_{n\in N-\{0\}}\overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}} = \omega(e_M)
\end{aligned}$$

Now let soft point $e_M\widetilde{\in}\Omega(id_E, g)$ and (N, E, X) a soft open neighborhood of soft point $(id_E, g)(e_M)$, we can obtain that $(id_E, g)^{-1}(N, E, X)$ is a soft open neighborhood of soft point e_M since (id_E, g) is a soft continuous function, then there exists some $n \in N - \{0\}$ such that

$$\begin{aligned}
& (id_E, g)^{-1}((id_E, g)^{-n}(N, E, X))\widetilde{\cap}(N, E, X)) \\
= \;& (id_E, g)^{-n}((id_E, g)^{-1}(N, E, X))\widetilde{\cap}(id_E, g)^{-1}(N, E, X) \\
\neq \;& \widetilde{\emptyset}
\end{aligned}$$

So

$$(id_E, g)^{-n}(N, E, X))\widetilde{\cap}(N, E, X) \neq \widetilde{\emptyset}.$$

Therefore

$$(id_E, g)(e_M)\widetilde{\in}\Omega(id_E, g),$$

Hence

$$(id_E, g)(\Omega(id_E, g))\widetilde{\subseteq}\Omega(id_E, g).$$

(3) Straightforward.

(4) Let (i) hold, $e_M\widetilde{\in}\widetilde{X}$ be a soft point and $(N,E,X)\in\mathscr{T}$ be a soft open neighborhood of e_M. Because (id_E,g) is soft topological mixing, there exists some $n\in N-\{0\}$ such that

$$(id_E,g)^n(M,E,X)\widetilde{\cap}(M,E,X)\neq\widetilde{\emptyset}.$$

Then

$$(id_E,g)^{-n}(M,E,X)\widetilde{\cap}(M,E,X)\neq\widetilde{\emptyset}$$

since g is a one-to-one correspondence and both (id_E,g) and its inverse mapping $(id_E,g)^{-1}$ are continuous. Thus $e_M\in\Omega(id_E,g)$.

Let (ii) hold. Then

$$\widetilde{X}=\overline{Per(id_E,g)}\widetilde{\subseteq}\overline{Rec(id_E,g)}\widetilde{\subseteq}\overline{\Omega(id_E,g)}=\Omega(id_E,g)\widetilde{\subseteq}\widetilde{X}.$$

Therefore $\Omega(id_E,g)=\widetilde{X}$. $\square$

Remark 4 If g is a one-to-one correspondence, both

$$(id_E,g):\mathbb{S}(X,E)\longrightarrow\mathbb{S}(X,E)$$

and its inverse mapping

$$(id_E,g)^{-1}:\mathbb{S}(X,E)\longrightarrow\mathbb{S}(X,E)$$

are continuous, and $(M,E,X)\in\mathbb{S}(X,E)$. Then

$$(id_E,g)^n(M,E,X)\widetilde{\cap}(M,E,X)\neq\widetilde{\emptyset}$$

if and only if

$$(id_E,g)^{-n}(M,E,X)\widetilde{\cap}(M,E,X)\neq\widetilde{\emptyset}\ (\forall n\in N-\{0\}).$$

So $\Omega(id_E,g)=\Omega(id_E,g)^{-1}$.

Definition 11 Let $(X,\mathscr{T}_X,E)$ and $(Y,\mathscr{T}_Y,E)$ be soft topological spaces,

$$(id_E,g):\mathbb{S}(X,E)\longrightarrow\mathbb{S}(X,E)$$

be a soft continuous function from $(X,\mathscr{T}_X,E)$ to $(X,\mathscr{T}_X,E)$,

$$(id_E,f):\mathbb{S}(Y,E)\longrightarrow\mathbb{S}(Y,E))$$

be a soft continuous function from $(Y,\mathscr{T}_Y,E)$ to $(Y,\mathscr{T}_Y,E))$. If there exists a soft continuous function $(id_E,h):\mathbb{S}(X,E)\longrightarrow\mathbb{S}(Y,E)$ from $(X,\mathscr{T}_X,E)$ to $(Y,\mathscr{T}_Y,E)$ such that

$$(id_E,h)\circ(id_E,f)=(id_E,g)\circ(id_E,h)$$

(i.e. $(id_E,h\circ f)=(id_E,g\circ h)$), then (id_E,h) is said to be soft topology semi-conjugate from (id_E,g) to (id_E,f). If g is a one-to-one correspondence and both (id_E,g) and its inverse

mapping $(id_E, g)^{-1}$ are continuous, then (id_E, h) is said to soft topological conjugate from (id_E, g) to (id_E, f). Here, we denote $(id_E, g) \cong (id_E, f)$.

$$\begin{array}{ccc} \mathbb{S}(X,E) & \xrightarrow{(id_E, g)} & \mathbb{S}(X,E) \\ \downarrow{\scriptstyle (id_E, h)} & & \downarrow{\scriptstyle (id_E, h)} \\ \mathbb{S}(Y,E) & \xrightarrow[(id_E, f)]{} & \mathbb{S}(Y,E) \end{array}$$

fig.1

Remark 5 (1) $\cong$ is an equivalence relation.

(2) If (id_E, h) is a soft topological conjugate mapping from (id_E, g) to (id_E, f), then for each soft point $e_M \in \widetilde{X}$ and $n \in N - \{0\}$, we have

$$(id_E, h)((id_E, f)^n(e_M)) = (id_E, g^n)((id_E, h)(e_M)),$$

it follows that

$$(id_E, h)(Orb_{(id_E, g)}(e_M)) = Orb_{(id_E, g)}((id_E, h)(e_M)),$$

and it is easy to show that

$$(id_E, h)(\omega(e_M)) = \omega((id_E, h)(e_M));$$

$$(id_E, h)(Per(id_E, g)) = Per(id_E, f);$$

$$(id_E, h)(Fix(id_E, g)) = Fix(id_E, f);$$

$$(id_E, h)(Rec(id_E, g)) = Rec(id_E, f);$$

$$(id_E, h)(\Omega(id_E, g)) = \Omega(id_E, f).$$

3 Soft topological entropy

In this section, the definition of soft topological entropy will be given and some fundamental properties of the soft topological entropy will be studied.

Definition 12 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, and α be a soft open cover of $\widetilde{X}$. Denote the smallest cardinality of all subcovers (for $\widetilde{X}$) of α by $N_{\widetilde{X}}(\alpha)$, i.e.,

$$N_{\widetilde{X}}(\alpha) = \min\left\{|\beta| \mid \beta \subseteq \alpha \text{ and } \widetilde{X} = \widetilde{\bigcup}\beta\right\}.$$

Since $\widetilde{X}$ is compact soft set, $N_{\widetilde{X}}(\alpha)$ is a positive integer. Let $H_{\widetilde{X}}(\alpha) = \log N_{\widetilde{X}}(\alpha)$.

Let α and β be two soft open covers of $\widetilde{X}$. Define their join by

$$\alpha \widehat{\cup} \beta = \{(P, E, X) \widetilde{\cap} (Q, E, X) \mid (P, E, X) \in \alpha, (Q, E, X) \in \beta\}.$$

Clearly, the join $\alpha\widehat{\bigcup}\beta$ is a soft open cover of $\widetilde{X}$. It is well known that β is called a refinement of α (denoted by $\alpha \prec \beta$) if for each $(Q, E, X) \in \beta$, there exists a $(P, E, X) \in \alpha$ such that $(Q, E, X)\widetilde{\subseteq}(P, E, X)$.

Theorem 4 Let$(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α and β be two soft open covers of $\widetilde{X}$. Then the following hold.

(1) $H_{\widetilde{X}}(\alpha) \geq 0$.

(2) if $\beta \prec \alpha$,then $H_{\widetilde{X}}(\alpha) \leq H_{\widetilde{X}}(\beta)$.

(3) $H_{\widetilde{X}}(\alpha\widehat{\bigcup}\beta) \leq H_{\widetilde{X}}(\alpha) + H_{\widetilde{X}}(\beta)$.

(4) $H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = H_{\widetilde{X}}(\alpha)$.

Proof we only prove (4). Let $N_{\widetilde{X}}(\alpha) = n$, then any subcover of α containing less than n elements of α would not cover $\widetilde{X}$. Let

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_n, E, X)\}$$

be a subcover (for $\widetilde{X}$) of α with a cardinality n, since (id_E, g) is continuous,

$$\{(id_E, g)^{-1}(P_1, E, X), (id_E, g)^{-1}(P_2, E, X),$$
$$\cdots, (id_E, g)^{-1}(P_n, E, X)\}$$

is a subcover (for $(id_E, g)^{-1}(\widetilde{X})$) of $(id_E, g)^{-1}(\alpha)$. By $(id_E, g)(\widetilde{X}) = \widetilde{X}$ we can know $\widetilde{X} = (id_E, g)^{-1}(\widetilde{X})$, so

$$\{(id_E, g)^{-1}(P_1, E, X), (id_E, g)^{-1}(P_2, E, X),$$
$$\cdots, (id_E, g)^{-1}(P_n, E, X)\}$$

is a finite open subcover (for $\widetilde{X}$) of $(id_E, g)^{-1}(\alpha)$. Therefore,

$$N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \leq n = N_{\widetilde{X}}(\alpha)$$

which implies $H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \leq H_{\widetilde{X}}(\alpha)$.

Now, suppose that $N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = m$. Let

$$\{(id_E, g)^{-1}(Q_1, E, X), (id_E, g)^{-1}(Q_2, E, X),$$
$$\cdots, (id_E, g)^{-1}(Q_m, E, X)\}$$

be a finite open subcover (for $\widetilde{X}$) of $(id_E, g)^{-1}(\alpha)$. Therefore,

$$\widetilde{X} = \widetilde{\bigcup}_{i=1}^{m}\{(id_E, g)^{-1}(Q_i, E, X)\}.$$

Since $(id_E, g)(\widetilde{X}) = \widetilde{X}$, then

$$\begin{aligned} \widetilde{X} &= (id_E, g)(\widetilde{X}) = \widetilde{\bigcup}_{i=1}^{m}\{(id_E, g)^{-1}(Q_i, E, X)\} \\ &= \widetilde{\bigcup}_{i=1}^{m}\{(id_E, g)(id_E, g)^{-1}((Q_i, E, X))\} \\ &= \widetilde{\bigcup}_{i=1}^{n}\{(Q_i, E, X)\}. \end{aligned}$$

So

$$\{(Q_i, E, X) \mid i = 1, 2, \cdots, m\}$$

is a finite open subcover (for $\widetilde{X}$) of α, Hence, $m \geq N_{\widetilde{X}}(\alpha)$, i.e.,

$$N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \geq N_{\widetilde{X}}(\alpha)$$

which implies

$$H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \geq H_{\widetilde{X}}(\alpha).$$

By the above, we can get that

$$H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = H_{\widetilde{X}}(\alpha). \quad \square$$

Theorem 5 Let$(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α be a soft open cover of $\widetilde{X}$. Then the limit

$$\lim_{n\to\infty} \frac{1}{n} H_{\widetilde{X}}(\widehat{\bigcup}_{k=1}^{n-1} \{(id_E, g)^{-k}(\alpha)\})$$

exists.

Proof. Let

$$a_n = H_{\widetilde{X}}(\widehat{\bigcup}_{k=1}^{n-1} \{(id_E, g)^{-k}(\alpha)\}).$$

We only need to show that

$$a_{n+p} \leq a_n + a_p \ (\forall n, p \in N - \{0\}).$$

From theorem 2.7(3) and (4), we have

$$\begin{aligned}
a_{n+p} &= H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{n+p-1} \left\{(id_E, g)^{-k}(\alpha)\right\}\right) \\
&= H_{\widetilde{X}}(\left(\widehat{\bigcup}_{k=0}^{n-1} \left\{(id_E, g)^{-k}(\alpha)\right\}\right) \\
&\qquad \widehat{\bigcup} \left(\widehat{\bigcup}_{k=n}^{n+p-1} \left\{(id_E, g)^{-k}(\alpha)\right\}\right)) \\
&= H_{\widetilde{X}}(\left(\widehat{\bigcup}_{k=0}^{n-1} \left\{(id_E, g)^{-k}(\alpha)\right\}\right) \\
&\qquad \widehat{\bigcup} \left((id_E, g)^{-n} \widehat{\bigcup}_{k=0}^{p-1} \left\{(id_E, g)^{-k}(\alpha)\right\}\right)) \\
&\leq H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{n-1} \left\{(id_E, g)^{-k}(\alpha)\right\}\right) \\
&\qquad + H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{p-1} \left\{(id_E, g)^{-k}(\alpha)\right\}\right).
\end{aligned}$$

Thus $a_{n+p} \leq a_n + a_p$. $\square$

Definition 13 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, let α be a soft open cover of $\widetilde{X}$. Then

$$Ent((id_E, g), \alpha, \widetilde{X}) = \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=1}^{n-1} \{(id_E, g)^{-k}(\alpha)\}\right)$$

is called the soft topological entropy of (id_E, g) on $\widetilde{X}$ relative to α, and

$$Ent(id_E, g) = \sup_{\alpha}\{Ent((id_E, g), \alpha, \widetilde{X}) \mid \alpha \text{ is a soft open cover of } \widetilde{X}\}$$

is called the soft topological entropy of (id_E, g).

By Theorem 1, each soft closed subset of $\widetilde{X}$ is a soft compact subset of $\widetilde{X}$, then the following theorem holds.

Theorem 6 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α be a soft open cover of $\widetilde{X}$, (A_1, E, X) and (A_2, E, X) be two closed soft sets, and $(A_1, E, X)\widetilde{\subseteq}(A_2, E, X)$, Then

(1)

$$Ent((id_E, g), \alpha, (A_1, E, X)) \leq Ent((id_E, g), \alpha, (A_2, E, X)).$$

(2)

$$Ent((id_E, g), (A_1, E, X)) \leq Ent((id_E, g), (A_2, E, X)).$$

Proof. (1) Let

$$N_{(A_2,E,X)}(\widetilde{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}) = s.$$

Then there exists a soft open subcover

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_s, E, X)\}$$

of

$$\widetilde{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}$$

for (A_2, E, X). Since $(A_1, E, X) \widetilde{\subseteq}(A_2, E, X)$, we have

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_s, E, X)\}$$

is also a subcover of

$$\widetilde{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}$$

for (A_1, E, X), and hence

$$N_{(A_1,E,X)}(\widetilde{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}) \leq s$$

$$= N_{(A_2,E,X)}(\widetilde{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}).$$

So

$$H_{(A_1,E,X)}(\widetilde{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\})$$

$$\leq H_{(A_2,E,X)}(\widetilde{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}).$$

Then

$$Ent((id_E,g),\alpha,(A_1,E,X))$$
$$\leq Ent((id_E,g),\alpha,(A_2,E,X)).$$

(2)

$$\begin{aligned} & Ent((id_E,g),(A_1,E,X)) \\ = \;& \sup_{\alpha}\{Ent((id_E,g),\alpha,(A_1,E,X)) \mid \alpha \text{ is a soft open} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{cover of } \widetilde{X}\} \\ \leq \;& \sup_{\alpha}\{Ent((id_E,g),\alpha,(A_2,E,X)) \mid \alpha \text{ is a soft open} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{cover of } \widetilde{X}\} \\ = \;& Ent((id_E,g),(A_2,E,X)). \ \square \end{aligned}$$

Theorem 7 Let $(X,(id_E,g))$ be a soft compact discrete topological dynamical system, and α be a soft open cover of $\widetilde{X}$. Then $Ent(id_E,id_X)=0$.

Proof Straightforward.

Theorem 8 $Ent(id_E,g^m)\geq m\cdot Ent(id_E,g)$ ($\forall m\in N-\{0\}$).

Proof As

$$((g^n)^{\leftarrow})^m=(g^{\leftarrow})^{nm} \quad (\forall n\in N-\{0\},\forall m\in N),$$

we have

$$\begin{aligned} & \widehat{\bigcup}_{t=0}^{n-1}\{(id_E,g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E,g)^{-t}(\alpha)\}\} \\ = \;& \widehat{\bigcup}_{s=0}^{mn-1}\{(id_E,g)^{-s}(\alpha)\} \end{aligned}$$

Hence

$$H_{\widetilde{X}}(\widehat{\bigcup}_{t=0}^{n-1}\{(id_E,g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E,g)^{-t}(\alpha))\}\}$$
$$=H_{\widetilde{X}}(\widehat{\bigcup}_{s=0}^{mn-1}\{(id_E,g)^{-s}(\alpha)\}).$$

Denote

$$\beta=\widehat{\bigcup}_{s=0}^{m-1}\{(id_E,g)^{-s}(\alpha)\}.$$

Then

$$\begin{aligned} & Ent(id_E,g^m)=Ent(id_E,g)^m\geq Ent((id_E,g)^m,\beta,\widetilde{X}) \\ = \;& \lim_{n\to\infty}\frac{1}{n}H_{\widetilde{X}}\left(\widehat{\bigcup}_{t=0}^{n-1}\{(id_E,g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E,g)^{-t}(\alpha))\}\}\right) \\ = \;& \lim_{n\to\infty} m\cdot\frac{1}{mn}H_{\widetilde{X}}(\widehat{\bigcup}_{s=0}^{mn-1}\{(id_E,g)^{-s}(\alpha)\}) \\ = \;& m\cdot Ent((id_E,g),\alpha,\widetilde{X}). \end{aligned}$$

Hence,

$$Ent(id_E,g^m)\geq m\cdot\sup_{\alpha}Ent((id_E,g),\alpha,\widetilde{X})$$
$$=m\cdot Ent(id_E,g). \ \square$$

4 Conclusion

In this paper, the discrete dynamical systems in soft topological spaces are defined, and simple examples are also given. Some basic concepts (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) of the discrete dynamical system are introduced into soft topological spaces. Soft topological mixing and soft topological transitivity are also studied. At last, soft topological entropy is defined and several properties of it are discussed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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FUNCTIONAL INEQUALITIES IN VECTOR BANACH SPACE

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Abstract. In this paper, we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(ax+by+cz)+f(bx+ay+bz)+f(cx+cy+az)\| \le \|(a+b+c)f(x+y+z)\|$$

in vector Banach space, where $a \neq b \neq c \in \mathbb{R}$ are fixed points with $3 > |a+b+c|$.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [24] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[8],[10], [12]–[16], [22]–[25],[26]-[31],[34]).

We recall some basic facts concerning generalized norm.

Definition 1.1 (see [15]). Let E be a real vector space. A generalized norm for E is a mapping $\|\cdot\|_G : E \to \mathbb{R}_+^k$ denoted by

$$\|x\|_G = (\alpha_1(x), \alpha_2(x), \alpha_3(x), \cdots, \alpha_k(x))$$

such that

(a) $\|x\|_G \ge 0$, that is, $\alpha_i(x) \ge 0$ for all $i = 1, 2, \cdots, k$;
(b) $\|x\|_G = 0$ if and only if $x = 0$, that is, $\alpha_i(x) = 0$ for all i, if and only if $x = 0$;
(c) $\|\lambda x\|_G = |\lambda|\|x\|_G$,that is,$\alpha_i(\lambda x) = |\lambda|\alpha_i(x)$;
(d) $\|x+y\|_G \le \|x\|_G + \|y\|_G$, which means, $\alpha_i(x+y) \le \alpha_i(x) + \alpha_i(y)$;

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Example 1.2. *In* $\mathbb{R}^2$, $\|x\|_G = (|x_1|, |x_2|)$.

Definition 1.3. Let $(X, \|\cdot\|_G)$ be a general normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} \alpha_i(x_n - x) = 0$ for all $i = 1, 2\cdots, k$. In that case, x is called the limit of the sequence x_n and we denote it by $G\text{-}lim_{n\to\infty}\, x_n = x$.

Definition 1.4. A sequence x_n in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $\|x_{n+p} - x_n\|_G \leq \epsilon$, that is, $\alpha_i(x_{n+p} - x_n) \leq \epsilon$.

It is known that every convergent sequence in the general normed space is Cauchy. If each Cauchy sequence is convergent, then the the general normed space is said to be complete and the general normed space is called a *vector Banach space*.

2. HYers-Ulam Stability In vector Banach Space

From now on , Let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a vector Banach space.

This paper,we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(ax+by+cz) + f(bx+cy+bz) + f(cx+ay+az)\|_G \leq \|(a+b+c)f(x+y+z)\|_G$$

in the vector Banach space, where $a \neq b \neq c \in \mathbb{R}$ are fixed points with $3 > |a+b+c|$.

Lemma 2.1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping. If it satisfies*

$$\begin{aligned} &\|f(ax+by+cz) + f(bx+cy+bz) + f(cx+ay+az)\|_G \\ &\leq \|(a+b+c)f(x+y+z)\|_G \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$ *and* a, b, c *are fixed real numbers with* $3 > |a+b+c|$. *Then* f *is additive.*

Proof. Letting $x = y = z = 0$ in (2.1) for all $x, y, z \in \mathcal{X}$, we get

$$\|3f(0)\|_G \leq \|(a+b+c)f(0)\|_G \tag{2.2}$$

for $a, b, c \in \mathbb{R}$.

For any $i = 1, 2, \cdots, k$,

$$\alpha_i(3f(0)) \leq \alpha_i((a+b+c)f(0))$$

we get

$$3\alpha_i(f(0)) \leq |a+b+c|\alpha_i(f(0)),$$

Thus $f(0) = 0$.

Letting $x = 0$ and Replacing z by $-y$ in (2.1), we get

$$\|f((b-c)y) + f((c-b)y)\|_G \leq \|(a+b+c)f(0)\|_G = |a+b+c|\alpha_i(f(0)) = 0$$

and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$.

Replacing x by $-y-z$ in (2.1), we have

$$\|f((b-a)y+(c-a)z)+f((a-b)y)+f((a-c)z)\|_G \leq 0$$

for all $y, z \in \mathcal{X}$. Then we can obtain

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$. □

Theorem 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\begin{aligned}&\|f(ax+by+cz)+f(bx+cy+bz)+f(cx+ay+az)\|_G \\ &\leq \|(a+b+c)f(x+y+z)\|_G + (\underbrace{\varphi(x,y,z), \varphi(x,y,z), \cdots, \varphi(x,y,z)}_{k})\end{aligned} \tag{2.3}$$

and

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left((-2)^j x, (-2)^j x, (-2)^j x\right) < \infty \tag{2.4}$$

for all $x, y, z \in \mathcal{X}$ and $a \neq b \neq c \in \mathbb{R}$ are fixed points with $3 > |a+b+c|$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned}&\|f(x) - A(x)\|_G \\ &\leq \left(\underbrace{\widetilde{\varphi}\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right), \cdots, \widetilde{\varphi}\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right)}_{k}\right)\end{aligned} \tag{2.5}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x = -y-z$ in (2.3), we get

$$\begin{aligned}&\|f((b-a)y+(c-a)z)+f((a-b)y)+f((a-c)z)\|_G \\ &\leq \underbrace{(\varphi(-y-z,y,z), \cdots, \varphi(-y-z,y,z))}_{k}\end{aligned} \tag{2.6}$$

for all $y, z \in \mathcal{X}$.

Letting $y = \frac{x}{b-a}, z = \frac{y}{c-a}$ in (2.6), we get

$$\begin{aligned}&\|f(x+y)+f(-x)+f(-y)\|_G \\ &\leq \underbrace{\left(\varphi\left(\frac{x}{a-b}+\frac{y}{a-c}, \frac{x}{b-a}, \frac{y}{c-a}\right), \cdots, \varphi\left(\frac{x}{a-b}+\frac{y}{a-c}, \frac{x}{b-a}, \frac{y}{c-a}\right)\right)}_{k}\end{aligned} \tag{2.7}$$

for all $x, z \in \mathcal{X}$.

Letting $x = y$ in (2.7) we get

$$\begin{aligned}
&\|2f(-x) + f(2x)\|_G \\
&\leq \left(\varphi\left(\frac{2a-b-c}{(a-b)(a-c)}x, \frac{1}{b-a}x, \frac{1}{c-a}x\right), \cdots,\right. \\
&\left.\varphi\left(\frac{2a-b-c}{(a-b)(a-c)}x, \frac{1}{b-a}x, \frac{1}{c-a}x\right)\right)
\end{aligned}$$

for all $x \in \mathcal{X}$. Thus

$$\begin{aligned}
&\left\|f(x) - \frac{f(-2x)}{-2}\right\|_G \\
&\leq \frac{1}{2}\left(\varphi\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right), \cdots,\right. \\
&\left.\varphi\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right)\right)
\end{aligned}$$

for all $x \in \mathcal{X}$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\begin{aligned}
&\left\|\frac{1}{(-2)^l} f\left((-2)^l x\right) - \frac{1}{(-2)^m} f\left((-2)^m x\right)\right\|_G \\
&\leq \sum_{i=l}^{m-1} \frac{1}{2^i}\left(\varphi\left(\frac{(-2)^i(b+c-2a)}{(a-b)(a-c)}x, \frac{(-2)^i}{a-b}x, \frac{(-2)^i}{a-c}x\right), \cdots,\right. \\
&\left.\varphi\left(\frac{(-2)^i(b+c-2a)}{(a-b)(a-c)}x, \frac{(-2)^i}{a-b}x, \frac{(-2)^i}{a-c}x\right)\right)
\end{aligned}$$

for all $x \in \mathcal{X}$. That is,

$$\begin{aligned}
&\alpha_i\left(\frac{1}{(-2)^l} f\left((-2)^l x\right) - \frac{1}{(-2)^m} f\left((-2)^m x\right)\right) \\
&\leq \sum_{i=l}^{m-1} \frac{1}{2^i}\varphi\left(\frac{(-2)^i(b+c-2a)}{(a-b)(a-c)}x, \frac{(-2)^i}{a-b}x, \frac{(-2)^i}{a-c}x\right)
\end{aligned} \tag{2.8}$$

for all $x \in mathcalX$. It follows from (2.8) that the sequence $\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a generalized norm space, the sequence $\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}$ converges. So one may define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := G - \lim_{k\to\infty}\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}, \quad \forall x \in \mathcal{X}.$$

Taking $m = 0$ and letting l tend to ∞ in (2.8), we have the inequality (2.5).

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It follows from (2.3) that

$$\begin{aligned}
&\|A(ax+by+cz)+A(bx+ay+bz)+A(cx+cy+az)\|_G \\
&=\lim_{k\to\infty}\left|\frac{1}{(-2)^k}\right|\big\|f((-2)^k(ax+by+cz))+f((-2)^k(bx+ay+bz)) \\
&+f((-2)^k(cx+cy+az))\big\|_G \\
&\le \lim_{k\to\infty}\left|\frac{1}{(-2)^k}\right|\big\|(a+b+c)f((-2)^k(x+y+z))\big\|_G \\
&+\lim_{k\to\infty}\left|\frac{1}{(-2)^k}\right|\left(\underbrace{\varphi((-2)^kx,(-2)^ky,(-2)^kz),\cdots,\varphi((-2)^kx,(-2)^ky,(-2)^kz)}_{k}\right) \\
&\le \|(a+b+c)A(x+y+z)\|_G
\end{aligned} \tag{2.9}$$

for all $x, y, z \in \mathcal{X}$. One see that A satisfies the inequality (2.1) and so it is additive by Lemma (2.1).

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (2.5). Then one has

$$\begin{aligned}
&\|A(x)-T(x)\|_G=\left\|\frac{1}{(-2)^k}A\left((-2)^kx\right)-\frac{1}{(-2)^k}T\left((-2)^kx\right)\right\|_G \\
&\le \frac{1}{2^k}\left(\left\|A\left((-2)^kx\right)-f\left((-2)^kx\right)\right\|_G\right. \\
&\left.+\left\|T\left((-2)^kx\right)-f\left((-2)^kx\right)\right\|_G\right) \\
&\le 2\frac{1}{2^k}\left(\underbrace{\widetilde{\varphi}\left(\frac{(b+c-2a)(-2)^k}{(a-b)(a-c)}x,\frac{(-2)^k}{a-b}x,\frac{(-2)^k}{a-c}x\right),\cdots}_{k}\right)
\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. □

Theorem 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.3) such that*

$$\widetilde{\varphi}(x,y,z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{(-2)^j},\frac{y}{(-2)^j},\frac{z}{(-2)^j}\right) < \infty \tag{2.10}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x)-A(x)\|_G \le \widetilde{\varphi}(x,x,-2x) \tag{2.11}$$

for all $x \in \mathcal{X}$.

Proof. The proof is similar with Theorem (2.2), we can get

$$\left\|f(x)-(-2)f\left(\frac{x}{-2}\right)\right\|_G$$
$$\leq \left(\underbrace{\varphi\left(\frac{(2a-b-c)x}{2(a-b)(a-c)},\frac{x}{2(b-a)},\frac{x}{2(c-a)}\right)\cdots\varphi\left(\frac{(2a-b-c)x}{2(a-b)(a-c)},\frac{x}{2(b-a)},\frac{x}{2(c-a)}\right)}_{k}\right)$$

for all $x\in\mathcal{X}$.

Next, we can prove that the sequence $\{(-2)^n f\left(\frac{x}{(-2)^n}\right)$ is a Cauchy sequence for all $x\in\mathcal{X}$,and define a mapping $A:\mathcal{X}\to\mathcal{Y}$ by

$$A(x):=\lim_{n\to\infty}(-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x\in\mathcal{X}$ that is similar to the corresponding part of the proof of Theorem (2.2). □

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Coupled fixed point theorems for generalized $(\psi, \phi)-$ weak contraction in partially ordered G-metric spaces

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In this manuscript, we give coupled fixed point results for generalized $(\psi, \phi)-$ weak contraction, satisfying rational type expression in the context of partially ordered G-metric spaces. The derived results generalize the result of K. Chakrabarti (K. Chakrabarti, Coupled fixed point theorems with rational type contractive condition in a partially ordered G-metric space, Journal of Mathematics, Volume 2014, Article ID 785357, 7 pages). To demonstrate our result and also to demonstrate the authenticity of our result from the previous one, we give suitable example.

Key Words: Coupled fixed point, Mixed monotone property, Partially ordered G-metric space, $(\psi, \phi)-$weak contraction.

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1 Introduction and preliminaries

Fixed point theory provide one of the most important and useful technique for the existence of fixed point, coincidence point, common fixed point and coupled fixed pint for self map under different condition. It is used for the existence and uniqueness of the solution of mathematical model which may be in the form of differential equations, matrix equations, integral equations, functional equations, linear inequalities or mixed see ([5], [17], [19], [30]). In this area the first well known result proved by Banach [8] known as Banach contraction principle. Many authors generalized this principle in various spaces by using different contractive conditions ([6], [13], [15], etc.).

In recent years, metric fixed point theory has been developed rapidly in partially ordered metric space. Ran and Reurings [30] extended the Banach contraction principle in partially ordered sets and also discuss some applications to linear and nonlinear matrix equations. Nieto and Rodriguez-Lopez [23]

extended the result of Ran and Reurings and used their established result to obtain a unique solution for first order ODEs. Jaggi [15] construct rational type contraction in complete metric space. Harjani et al [13] extend the result of Jaggi to partial ordered complete metric space. For more details (see[13], [33]).

Alber and Gurre [6] gave the concept of weak contraction as a generalization of contraction and established the existence of fixed points for a self map in a Hilbert space. Rhoades [31] extended this concept to metric spaces and defined ϕ-weak contraction. Dutta and Choudhury [12] generalized ϕ-weak contraction to the concept of (ψ, ϕ) weak contraction and studies some fixed point results. Zhang and Song [34] extend weak contraction for the study of two self map. Furthermore Djorić [11] generalized the result of Zhang and Song and studied common fixed point for generalized (ψ, ϕ) weak contraction. For some other similar results see [22], [25], [29], [32].

The concept of mixed monotone mappings introduced by Bhaskar and Lakshmikantham [9] and derived some coupled fixed point results. Furthermore, they applied their results on a first order differential equation with periodic boundary conditions [14]. Lakshmikanthem and Ćirić [17] generalized the concept of mixed monotone mapping and established a coupled fixed point theorem for nonlinear contractions in partially ordered metric spaces. Recently Chakrabarti [10] investigated coupled fixed point theorems for map satisfying nonlinear rational type contraction and mixed monotone property in partially ordered G-metric space.

In this work, using the concept of generalized rational type $(\psi, \phi)-$weak contraction condition, coupled fixed point results in the framework of complete partially ordered generalized metric spaces are investigated. Through out the paper $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ will denote the set of all non-negative real numbers, the set of positive integer and the set of non-negative integer respectively.

Definition 1. *[20] Let $(X, \preceq)$ be a partially ordered set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:*

1. *$G(u, v, w) = 0$ if $u = v = w$;*
2. *$0 < G(u, u, v)$ for all $u, v \in X$ with $u \neq v$;*
3. *$G(u, u, v) \leq G(u, v, w)$ for all $u, v, w \in X$ with $v \neq w$;*
4. *$G(u, v, w) = G(u, w, v) = G(v, w, u) = \ldots$ (symmetry in all three variables);*
5. *$G(u, v, w) \leq G(u, p, p) + G(p, v, w)$ for all $u, v, w, p \in X$ (rectangle inequality).*

Then it is called a G-metric on X and the triple $(X, G, \preceq)$ is called partially ordered G-metric space.

Definition 2. *[20] The pair (X, G) is said to be symmetric G-metric space if $G(u, v, v) = G(u, u, v)$ for all $u, v \in X$.*

Example 1. *(1) Let $X = \mathbb{R}^+$ and $G : X \times X \times X \to \mathbb{R}^+$ be the function defined is follows $G(u, v, w) = \max\{|u - v|, |v - w|, |w - u|\}$, for all $u, v, w \in X$. Then G is symmetric G-metric on X.*

(2) Let $X = \{a, b\}$ *. Define* $G(a, a, a) = G(b, b, b) = 0, G(a, a, b) = 1, G(a, b, b) = 2,$ *and extend* G *to* X^3 *by using the symmetry in the variables. Then it is clear that* (X, G) *is an asymmetric* $G-$*metric space.*

(3) Also see examples of asymmetric $G-$*metric spaces in ([2], Example 2.6; [3], Example 2.2; [18], Example 2.2; [22], Example 3.4.).*

Definition 3. *[20] Let* (X, G) *be a G-metric space and let* α_n *be a sequence in* X*. A point* $\alpha \in X$ *is said to be the limit of the sequence* α_n *if*

$$\lim_{n,m\to\infty} G(\alpha_n, \alpha_m, \alpha) = 0$$

and the sequence α_n *is said to be G-convergent in* X*.*

Definition 4. *[20] A sequence* α_n *is called a G-Cauchy sequence if for every* $\varepsilon > 0$*, there is a positive integer* $\mathbb{N}$ *such that* $G(\alpha_n, \alpha_m, \alpha_l) < \varepsilon$ *for all* $n, m, l > \mathbb{N}$*.*

Definition 5. *[20] A metric space* (X, G) *is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in* (X, G) *is G-convergent in* X*.*

Definition 6. *[9] Let* $(X, \preceq)$ *be a partially ordered set,* $T : X \times X \to X$*. Then* T *is said to have mixed-monotone property if* $T(x, y)$ *is monotone non-decreasing in* x *and monotone non-increasing in* y*. That is., for all* $x, y \in X$

Definition 7. *[17] Let* $(X, \preceq)$ *be a partially ordered set,* $T : X \times X \to X$ *and* $g : X \to X$*. We say* T *is the g-mixed monotone property if* T *is monotone g-nondecreasing in its first argument and monotone g-non-increasing in its second argument. That is., for all* $x, y \in X$

$$\begin{aligned}
&x_1, x_2 \in X,\ gx_1 \preceq gx_2 \Rightarrow T(x_1, y) \preceq T(x_2, y),\\
&y_1, y_2 \in X,\ gy_1 \preceq gy_2 \Rightarrow T(x, y_1) \succeq T(x, y_2).
\end{aligned}$$

Definition 8. *[9] Let* $T : X \times X \to X$ *be a map such that* $T(x, y) = x$ *and* $T(y, x) = y$ *then the pair* $(x, y) \in X \times X$ *is called a coupled fixed point of* T*. It is clear that* (x, y)*is a coupled fixed point if and only if* (y, x) *is such.*

Definition 9. *[17] Let* $T : X \times X \to X$ *and* $g : X \to X$ *be two map such that* $T(x, y) = gx$ *and* $T(y, x) = gy$ *then the pair* $(x, y) \in X \times X$ *is called a coupled coincidence point of* T *and* g*.*

Definition 10. *[17] Two maps* $T : X \times X \to X$ *and* $g : X \to X$ *are said to be commutative if* $g(T(x, y)) = T(gx, gy)$*.*

Chakrababati [10] proved the following results.

Theorem 1. *[10]Let* $(X, \preceq)$ *be a partially ordered set and let* (X, G) *be a* $G-$*complete G-metric space. Suppose* $T : X \times X \longrightarrow X$ *be a continuous mapping on* X *having the mixed monotone property. Suppose for all* $(x, y), (u, v), (w, z) \in X \times X$ *with* $(x, y) \preceq (u, v) \preceq (w, z)$ *holds*

$$\begin{aligned}
&G(T(x, y), T(u, v), T(w, z))\\
&\leq \alpha \frac{G(x, T(x, y), T(x, y))\, G(u, T(u, v), T(u, v))\, G(w, T(w, z), T(w, z))}{G^2(x, u, w)}\\
&\quad + \beta G(x, u, w),
\end{aligned}$$

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where $8\alpha + \beta < 1$. *If there exist* $x_0, y_0 \in X$ *such that* $x_0 \preceq T(x_0, y_0)$ *and* $y_0 \succeq T(y_0, x_0)$, *then* T *has a coupled fixed point* $(x_*, y_*) \in X$.

Theorem 2. *[10] Let* $(X, \preceq)$ *be a partially ordered set and let* (X, G) *be a* $G-$*complete G-metric space. Suppose* $T : X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *be a continues mappings on* X *such that* T *has the mixed g-monotone property. Suppose that* $T(X \times X) \subseteq g(X)$, g *commute with* T *and for* $(x, y), (u, v), (w, z) \in X \times X$ *with* $(x, y) \preceq (u, v) \preceq (w, z)$ *and* $gx \preceq gu \preceq gw$ *or* $gy \succeq gu \succeq gz$ *holds*

$$\begin{aligned} &G(T(x,y), T(u,v), T(w,z)) \\ &\leq \alpha \frac{G\left(gx, T(x,y), T(x,y)\right) G\left(gu, T(u,v), T(u,v)\right) G\left(gw, T(w,z), T(w,z)\right)}{G^2(gx, gu, gw)} \\ &\quad + \beta G(gx, gu, gw), \end{aligned}$$

where $8\alpha + \beta < 1$. *If there exist* $x_0, y_0 \in X$ *such that* $gx_0 \preceq T(x_0, y_0)$ *and* $gy_0 \succeq T(y_0, x_0)$ *then* T *and* g *have a coupled coincidence point* $(x_*, y_*) \in X \times X$, *that is.,* (x_*, y_*) *satisfies* $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

2 Main Results

In our main results we used the following two classes.

$\psi \in \Psi$ if and only if $\psi : [0, \infty) \to [0, \infty)$, ψ is continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

$\phi \in \Phi$ if and only if $\phi : [0, \infty) \to [0, \infty)$, ψ is a lower semi continuous and non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Also, for more details of G-metric spaces see ([1]-[4], [7], [16], [18], [21], [26]-[28]).

Remark 1. *It is worth to noticing that both results in [10] without the conditions* $G^2(x, u, w) \neq 0$ *that is.,* $G(gx, gu, gw) \neq 0$ *are not correct.*

Now, we announce the first our result.

Theorem 3. *Let* $(X, \preceq)$ *be a partially ordered set and let* (X, G) *be a* $G-$*complete symmetric G-metric space. Suppose* $T : X \times X \longrightarrow X$ *be a continuous mapping having the mixed monotone property and satisfying*

$$\psi(G(T(x,y), T(u,v), T(w,z)) \leq \psi(M(x,u,w,y,v,z)) - \phi(M(x,u,w,y,v,z)), \tag{2.1}$$

for all $x, y, z, u, v, w \in X$ *with* $G(x, u, w) \neq 0$ *and* $(x, y) \preceq (u, v) \preceq (w, z)$ *or* $(x, y) \succeq (u, v) \succeq (w, z)$, *where*

$$\begin{aligned} &M(x,u,w,y,v,z) \\ &= \max\left\{ \frac{[G(x, T(x,y), T(x,y)G(u, T(u,v), T(u,v)G(w, T(w,z), T(w,z)]}{G^2(x,u,w)}, \right. \\ &\qquad \left. G(x,u,w) \right\}, \end{aligned}$$

$\psi \in \Psi$ *and* $\phi \in \Phi$. *If there exist* $x_0, y_0 \in X$ *such that* $x_0 \preceq T(x_0, y_0)$ *and* $y_0 \succeq T(y_0, x_0)$. *Then* T *has a coupled fixed point* $(x_*, y_*) \in X$.

Proof. Suppose that there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$. Further, define $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$. Using the mixed monotone property and the mathematical induction we obtain that $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$ for all $n \in \mathbb{N}$ (very known method).

Consider now

$$\psi(G(x_{n+1}, x_n, x_n)) = \psi(G(T(x_n, y_n), T(x_{n-1}, y_{n-1}), T(x_{n-1}, y_{n-1}))).$$

Using (2.1) we have that

$$\begin{aligned}\psi(G(x_{n+1}, x_n, x_n)) \leq& \psi(M(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1})) \\ &- \phi(M(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}))\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}&M(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}) \\ =&\max\left\{\frac{G(x_n, T(x_n, y_n), T(x_n, y_n))\, G^2(x_{n-1}, T(x_{n-1}, y_{n-1})\, T(x_{n-1}, y_{n-1}))}{G^2(x_n, x_{n-1}, x_{n-1})},\right. \\ &\left. G(x_n, x_{n-1}, x_{n-1})\right\}.\end{aligned}$$

Let $G_n = G(x_n, x_{n-1}, x_{n-1})$ then,

$$M(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}) = \max\{G_{n+1}, G_n\}.$$

Further we show that G_n is non-incresing. Suppose their exist n_0 such that $G_{n_0+1} > G_{n_0}$ then from (2.2)

$$\psi(G_{n_0+1}) \leq \psi(G_{n_0+1}) - \phi(G_{n_0+1}).$$

Which implies that $\phi(G_{n_0+1}) \leq 0$. A contradiction. Hence $G_n \geq G_{n+1}$ for all $n \geq 1$. Since $\{G_n\}$ is a non-increasing sequence of positive real numbers there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G_n = r. \tag{2.3}$$

We shall show that $r = 0$. Suppose $r > 0$ then applying limit in (2.2) and using (2.3), we have

$$\psi(r) \leq \psi(r) - \phi(r) < \psi(r).$$

We obtain a contradiction. Therefore $r = 0$ that is.,

$$\lim_{n\to\infty} G_n = 0. \tag{2.4}$$

Now, we show that $\{x_n\}$ is a G-Cauchy sequence. Suppose that, $\{x_n\}$ is not G-Cauchy. Then, there exist $\epsilon > 0$ and subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that,

$$G(x_{m(k)}, x_{m(k)}, x_{n(k)}) \geq \epsilon, \quad \forall\, k \in \mathbb{N}. \tag{2.5}$$

Furthermore, corresponding to $m(k)$ one can choose $n(k)$ such that, it is the smallest integer with $n(k) > m(k)$ satisfying (2.5) then,

$$G(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) < \epsilon, \quad \forall\, k \in \mathbb{N} \tag{2.6}$$

Now

$$\begin{aligned}\epsilon &\leq G(x_{m(k)}, x_{m(k)}, x_{n(k)}) = G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \\ &\leq G(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}),\end{aligned}$$

Taking limit $k \to \infty$ and using (2.4) we get

$$\lim_{k\to\infty} G(x_{m(k)}, x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.7}$$

Now

$$\begin{aligned}G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) &= G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \\ &\leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}) \\ &\leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \\ &\quad + G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}),\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \\ &\leq G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) \\ &\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \\ &\quad + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}),\end{aligned} \tag{2.9}$$

Using limit $k \to \infty$ in (2.8) and (2.9) and using (2.4) and (2.7) we get

$$\lim_{k\to\infty} G(x_{m(k)-1}, x_{m(k)-1} x_{n(k)-1}) = \epsilon. \tag{2.10}$$

Consider

$$\begin{aligned}&\psi\Big(G(x_{m(k)}, x_{m(k)}, x_{n(k)}\Big) \\ &\leq \psi\Big(M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1})\Big) \\ &\quad - \phi\Big(M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1})\Big),\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}&M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1}) \\ &= \max\Bigg\{\frac{[[G(x_{m(k)-1}, x_{m(k)}, x_{m(k)})]^2 G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}]}{G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})^2}, \qquad (2.12) \\ &\qquad G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})\Bigg\}. \qquad (2.13)\end{aligned}$$

Applying limit $k \to \infty$ in (2.13), using (2.7), (2.10) and (2.4) we get

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1}) = \epsilon. \tag{2.14}$$

Taking limit of (2.11) using (2.7), (2.14) and lower semi continuity of ϕ we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) < \psi(\epsilon),$$

which is contradiction. So $\epsilon = 0$. Therefore x_n is a G-Cauchy sequence. Similarly by the same argument we can show that y_n is a G-Cauchy sequence. By completeness of X, there is $x_*,\ y_* \in X$ such that $x_n \to x_*$ and $y_n \to y_*$ as $n \to \infty$.
Now we have to show that (x_*, y_*) is a coupled fixed point of T. Since T is continuous on X and G is also continuous in each of its variable, so

$$G(T(x_*, y_*), x_*, x_*) = G(\lim_{n\to\infty} T(x_n, y_n), x_*, x_*) = G(x_*, x_*, x_*) = 0.$$

Hence, we proved that $T(x_*, y_*) = x_*$ Similarly by the same argument we obtain that $T(y_*, x_*) = y_*$. So (x_*, y_*) is a coupled fixed point of T. □

Theorem 4. *Suppose that the conditions of Theorem 3 are valid. In addition suppose that for each $(x, y), (u, v) \in X \times X$ exists $(w, z) \in X \times X$ which is comparable to (x, y) and (u, v). Then coupled fixed point of T is unique.*

Proof. Suppose that $(x_*, y_*), (x^{'}, y^{'}) \in X \times X$ are two coupled fixed points.

Case 1

If (x_*, y_*), $(x\prime, y\prime)$ are comparable then from (2.1)

$$\begin{aligned}\psi(G(T(x_*, y_*), T(x^{'}, y^{'}), T(x^{'}, y^{'}) \leq &\psi(M(x_*, x^{'}, x^{'}, y_*, y^{'}, y^{'})\\ &- \phi(M(x_*, x^{'}, x^{'}, y_*, y^{'}, y^{'}),\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}&M(x_*, x^{'}, x^{'}, y_*, y^{'}, y^{'})\\ &= \max\left\{\frac{G(x_*, T(x_*, y_*), T(x_*, y_*))[G(x^{'}, T(x^{'}, y^{'}), T(x^{'}, y^{'})]^2]}{G(x_*, x^{'}, x^{'})}, G(x_*, x^{'}, x^{'})\right\}\\ &= \max\left\{\frac{G(x_*, x_*, x_*)[G(x^{'}, x^{'}, x^{'}]^2}{G(x_*, x^{'}, x^{'})}, G(x_*, x^{'}, x^{'})\right\}.\end{aligned}$$

Which implies that

$$M(x_*, x^{'}, x^{'}, y_*, y^{'}, y^{'}) = G(x_*, x^{'}, x^{'}).$$

From (2.15) we have

$$\psi(G(x_*, x^{'}, x^{'}) = \psi(G(T(x_*, y_*), T(x^{'}, y^{'}), T(x^{'}, y^{'}) < \phi(G(x_*, x^{'}, x^{'}),$$

which is contradiction. Hence we must have $x_* = x^{'}$. Similarly we can easily show that $y_* = y^{'}$ so couple fixed point is unique.

Case 2

If (x_*, y_*), $(x^{'}, y^{'})$ are not comparable by Theorem 3 there is a $(u, v) \in X \times X$ comparable to (x_*, y_*) and $(x^{'}, y^{'})$ if there is $m_0 \in \mathbb{N}$ such that $T^{m_0}(u, v) = (x_*, y_*)$, then
$T^{m_0+1}(u, v) = T(x_*, y_*) = x_*$, in last we get $T^m(u, v) = x_*$ for $m \geq m_0$ this mean $T^m(u, v) \to x_*$ for $m \to \infty$
if there is no such m_0 then for any $m \geq 1$

$$\begin{aligned}\psi(G(T^m(u, v), x_*, x_*) &= \psi(G(T^m(u, v), T^m(x_*, y_*), T^m(x_*, y_*)\\ &\leq \psi(M(u, x_*, x_*, v, y_*, y_*) - \phi(M(u, x_*, x_*, v, y_*, y_*),\end{aligned} \tag{2.16}$$

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where

$$\begin{aligned}
&M(u, x_*, x_*, v, y_*, y_*)\\
&= \max\left\{\frac{G(T^{m-1}(u,v), T^m(u,v), T^m(u,v))[G(T^{m-1}(x_*,y_*), T^m(x_*,y_*), T^m(x_*,y_*))]^2}{G(T^{m-1}(u,v), T^{m-1}(x_*,y_*), T^{m-1}(x_*,y_*))},\right.\\
&\qquad \left. G(T^{m-1}(u,v), T^{m-1}(x_*,y_*), T^{m-1}(x_*,y_*))\right\}\\
&= \max\left\{\frac{G(T^{m-1}(u,v), x_*, x_*)[G(x_*,x_*,x_*)]^2}{G(T^{m-1}(u,v), x_*, x_*)}, G(T^{m-1}(u,v), x_*, x_*)\right\}.
\end{aligned}$$

Which implies that

$$M(u, x_*, x_*, v, y_*, y_*) = G(T^{m-1}(u,v), x_*, x_*).$$

Putting M in (2.16), we have

$$\begin{aligned}
\psi(G(T^m(u,v), x_*, x_*) \leq \psi(G(T^{m-1}(u,v), x_*, x_*))\\
\phi(G(T^{m-1}(u,v), x_*, x_*)). \qquad (2.17)
\end{aligned}$$

This implies that

$$\psi(G(T^m(u,v), x_*, x_*) < \psi(G(T^{m-1}(u,v), x_*, x_*),$$

since ψ is non-decreasing therefore,

$$G(T^m(u,v), x_*, x_*) < G(T^{m-1}(u,v), x_*, x_*)$$

that is, $\{G(T^m(u,v), x_*, x_*)\}$ is a decreasing sequence of positive real numbers. Therefore, there is an α_1 such that $\{G(T^m(u,v), x_*, x_*)\} \to \alpha_1$. We shall show that $\alpha_1 = 0$. Suppose, to the contrary, that $\alpha_1 > 0$. Taking the limit in equation (2.17) we get contradiction. So α_1 =0. Implies $G(T^m(u,v), x_*, x_*)$=0, that is., $T^m(u,v) = x_*$. Similarly we can show that $T^m(u,v) = y_*$, $(T^m(u,v) = x^{'}$ and $(T^m(u,v) = y^{'}$. Hence the coupled fixed point is unique. □

The next result is the generalization of Theorem 3. Because the proof is similar, then it is omitted.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and let (X, G) be a $G-$complete symmetric G-metric space. Suppose that $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are a continues mappings such that T has the $g-$mixed monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commute with T and satisfying*

$$\psi(G(T(x,y), T(u,v), T(w,z)) \leq \psi(M(x,u,w,y,v,z) - \phi(M(x,u,w,y,v,z), \qquad (2.18)$$

for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$, where

$$\begin{aligned}
&M(x,u,w,y,v,z)\\
&= \max\left\{\frac{G(gx, T(x,y), T(x,y))\, G(gu, T(u,v), T(u,v))\, G(gw, T(w,z), T(w,z))}{G^2(gx, gu, gw)},\right.\\
&\qquad \left. G(gx, gu, gw)\right\},
\end{aligned}$$

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$\psi \in \Psi$ and $\phi \in \Phi$. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x_*, y_*) \in X \times X$, that is., (x_*, y_*) satisfies $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

Corollary 1. *Let (X, G) be a partially ordered set and let (X, G) be a $G-$complete symmetric G-metric space. Suppose that $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are a continues mappings such that T has the $g-$mixed monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commute with T and for $0 < k < 1$ satisfying*

$$G(T(x, y), T(u, v), T(w, z)) \leq k(M(x, u, w, y, v, z),$$

for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$, where

$$\begin{aligned} &M(x, u, w, y, v, z) \\ = &\max\left\{ \frac{G\left(gx, T(x, y), T(x, y)\right) G\left(gu, T(u, v), T(u, v)\right) G\left(gw, T(w, z), T(w, z)\right)}{G^2(gx, gu, gw)}, \right. \\ &\left. \qquad G(gx, gu, gw) \right\}. \end{aligned}$$

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x_*, y_*) \in X \times X$, that is., (x_*, y_*) satisfies $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

Proof. The proof follows by taking $\psi(t) = t$, $\phi(t) = (1 - k)t$ where $0 < k < 1$ in Theorem 5. □

Remark 2. *For $0 < \alpha < \frac{1}{8}$, $0 < \beta < \frac{1}{16}$ and for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ we have*

$$\begin{aligned} &G(T(x, y), T(u, v), T(w, z)) \\ \leq &\alpha \frac{[G(gx, T(x, y), T(x, y)G(gu, T(u, v), T(u, v)G(gw, T(w, z), T(w, z)]}{G(gx, gu, gw)^2} \\ &+ \beta G(gx, gu, gw), \\ \leq &(\alpha + \beta) \max\left\{ \frac{G\left(gx, T(x, y), T(x, y)\right) G\left(gu, T(u, v), T(u, v)\right) G\left(gw, T(w, z), T(w, z)\right)}{G^2(gx, gu, gw)}, \right. \\ &\left. \qquad G(gx, gu, gw) \right\}. \end{aligned}$$

where $k = \alpha + \beta < 1$. Clearly, the relation $0 < 8\alpha + \beta < 1$ implies that Corollary 1 is the generalization of Theorem 2. Therefore Theorem 5 is the generalization of Theorem 2.

Now we give example which satisfying Theorem 5 but does not Theorem 2.

Example 2. *Let $X = [0, 1]$ and consider the natural ordered relation in X, defined $G : X \times X \times X \to \mathbb{R}^+$ by*

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}. \end{cases}$$

Then (X,G) is $G-$complete symmetric G-metric space. Let $T : X\times X \to X$, $g : X \to X$, $\phi : [0,\infty) \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ define by,

$$T(x,y) = \begin{cases} \frac{x^3-y^3}{4}, & \text{if } x \geq y, \\ 0, & \text{if } x < y, \end{cases}$$

$$g(x) = x^2, \quad \phi(t) = \frac{t}{2}, \quad \psi(t) = \frac{t}{4}.$$

We discuss the following cases.

Case 1. $(x,y) = (0,0), (u,v) = (0,0), (w,z) = (1,0)$ *it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$ *and*

$$\psi(G(T(0,0),T(0,0),T(1,0)) \leq \psi(M(0,0,1,0,0,0) - \phi(M(0,0,1,0,0,0),$$

where $G(T(0,0),T(0,1),T(0,1)) = 1$ *and* $M(0,1,1,1,1,1) = 1$.

Case 2. $(x,y) = (0,1), (u,v) = (1,1), (w,z) = (1,1)$ *it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$*or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$ *and*

$$\psi(G(T(0,1),T(1,1),T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1),$$

where $G(T(0,1),T(1,1),T(1,1)) = 0$ *and* $M(0,1,1,1,1,1) = 1$.

Case 3. $(x,y) = (0,0), (u,v) = (1,0), (w,z) = (1,0)$ *it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$ *and*

$$\psi(G(T(0,0),T(1,0),T(1,0)) \leq \psi(M(0,1,1,0,0,0) - \phi(M(0,1,1,0,0,0),$$

where $G(T(0,0),T(1,0),T(1,0)) = \frac{1}{4}$ *and* $M(0,1,1,0,0,0) = 1$.

Case 4. $(x,y) = (0,1), (u,v) = (1,1), (w,z) = (1,1)$ *again it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$*and*

$$\psi(G(T(0,1),T(1,1),T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1),$$

where $G(T(0,1),T(1,1),T(1,1)) = 0$ *and* $M(0,1,1,1,1,1) = 1$.

Case 5. $(x,y) = (u,v) = (0,1), (w,z) = (1,1)$ *also it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$ *and*

$$\psi(G(T(0,1),T(0,1),T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1),$$

where $G(T(0,1),T(0,1),T(1,1)) = 0$ *and* $M(0,0,1,1,1,1) = 1$.

Clearly for $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$*all the conditions of Theorem 5 hold. So* $(0,0)$ *is the unique common coupled fixed point of* T *and* g*. On the other side if we taking in the Case 3* $\alpha = \beta = \frac{1}{6}$ *then Theorem 2 fail to satisfy.*

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TRIANGULAR NORMS BASED ON INTUITIONISTIC FUZZY BCK-SUBMODULES

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Abstract: *We introduce the concept of intuitionistic fuzzy BCK-submodules of a BCK-module with respect to a t-norm and a s-norm and present some basic properties.*

Keywords : Intuitionistic fuzzy BCK-submodules, Triangular Norms, (Imaginable) Intuitionistic (T, S)-fuzzy BCK-submodules.

1. Introduction

The theory of fuzzy sets proposed by Zadeh [11] in 1965, and later on several researchers worked in this field. As a natural advancement of these research works we get one of the interesting generalizations of the theory of fuzzy sets that is the theory of intuitionstic fuzzy sets propounded by Atanassov [1, 2]. In 1966 Imai and Iseki [5] proposed the concept of BCK-algebra. Xi [10] applied the concept of fuzzy set to BCK-algebras. Also Bakhshi [3] in 2011 introduced the concept of fuzzy BCK-submodule of BCK-module and gave some related results. Recently, Badhurays and Bashammakh [4] considered the intuitionistic fuzzification of the concept of BCK-submodules in a BCK-module and investigated some properties of such BCK-modules. In this paper, we are going to introduce the notion of intuitionistic (T,S)-fuzzy BCK-submodules by using triangular norms, say T and S, and investigate several properties. We obtain some results on level sets of an intuitionistic (T,S)-fuzzy BCK-submodule by using the concept of level sets and triangular norms.

For the notations and terminology not given in this paper, the reader is referred to Atanassov [1, 2] (1986, 1994), Jun [8] (2001), Janiŝ [6] (2010), and Zadeh [11] (1965).

2. Preliminaries

First we present the fundamental definitions.

Definition 2.1. (Imai and Iseki [5]) a BCK-algebra is a set X with a binary operation $*$ and a constant 0 satisfying the following axioms :

(BCK1) $((x * y) \star (x * z)) \star (z * y) = 0$

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(BCK2) $(x * (x * y)) * y = 0$,
(BCK3) $x * x = 0$,
(BCK4) $0 * x = 0$,
(BCK5) $x * y = 0$ and $y * x = 0$ imply that $x = y$,
for all $x, y, z \in X$.
A partial ordering "$\leq$" is defined on X by $x \leq y$ iff $x * y = 0$.

Definition 2.2. (Zadeh [11]) By a fuzzy set μ in a nonempty set X we mean a function $\mu\colon X \longmapsto [0,1]$, and the complement of μ denoted by $\bar{\mu}$ is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Definition 2.3. (Atanassov [1]) An intuitionistic fuzzy set *(IFS)* in a universe X is an object of the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\},$$

where the functions $\mu\colon X \longmapsto [0,1]$ and $\lambda\colon X \longmapsto [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x)) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A(x), \lambda_A(x))$ for the *IFS*

$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\}$$

Definition 2.4. (Atanassov [1]) Let X be a non-empty set and $A = (\mu_A(x), \lambda_A(x))$, $B = (\mu_B(x), \lambda_B(x))$ be *IFS*'s of X. Then

(1) $A \subset B$ iff $\mu_A(x) < \mu_B(x)$ and $\lambda_A(x) > \lambda_B(x)$ for all $x \in X$.
(2) $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\lambda_A(x) = \lambda_B(x)$ for all $x \in X$
(3) $A^C = (\lambda_A, \mu_A)$.
(4) $A \cap B = \{x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\} : x \in X\}$.
(5) $A \cup B = \{x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\} : x \in X\}$.
(6) $\Box A = \{(x, \mu_A(x), \bar{\mu_A}(x)) | x \in X\}$.
(7) $\Diamond A = \{(x, \bar{\lambda}_A(x), \lambda_A(x)) | x \in X\}$.

Definition 2.5. (Atanassov [1]) Let $A = (\mu_A(x), \lambda_A(x))$ be an intuitionistic fuzzy set in M and let $\alpha \in [0,1]$. Then the sets

$$U(\mu_A, \alpha) = \{x \in M : \mu_A(x) \geq \alpha\},$$
$$L(\lambda_A, \alpha) = \{x \in M : \lambda_A(x) \leq \alpha\}$$

are called a μ-level α-cut and a λ-level α-cut of A, respectively.

Theorem 2.1. (Bakhshi [3]) *Let X be a bounded implicative BCK-algebra. Then $(X, +, 0)$ is an X-module where "$+$" is defined as $x + y = (x \star y) \vee (y \star x)$ and $xy = x \wedge y$.*

Theorem 2.2. (Bakhshi [3]) *A subset A of a BCK-module M is a BCK-submodule of M iff $a - b, xa \in A$, for every $a, b \in A$ and $x \in X$.*

Definition 2.6. (Bakhshi [3]) A fuzzy subset A of M is said to be a fuzzy *BCK*-submodule if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold :

(1) $A(m_1 + m_2) \geq \min\{A(m_1), A(m_2)\}$

(2) $A(m) = A(-m)$
(3) $A(xm) \geq A(m)$

Definition 2.7. (Badhurays and Bashammakh [4]) An intuitionistic fuzzy subset $A = (\mu_A(x), \lambda_A(x))$ of M is said to be an intuitionistic fuzzy BCK-submodule of M if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold :

(1) $\mu_A(m_1 + m_2) \geq \min\{\mu_A(m_1), \mu_A(m_2)\}$,
$\lambda_A(m_1 + m_2) \leq \max\{\lambda_A(m_1), \lambda_A(m_2)\}$.
(2) $\mu_A(m) = \mu_A(-m), \lambda_A(m) = \lambda_A(-m)$,
(3) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

Definition 2.8 (Klir and Yuan [9]) a triangular norm (or t-norm) T is a mapping $T\colon [0,1] \times [0,1] \longmapsto [0,1]$, which satisfies the following axioms for every $x, y, z, \in [0,1]$:

(T1) $T(x, 1) = x$ (boundary condition);
(T2) $y \leq z$ implies $T(x, y) \leq T(x, z)$ (monotonicity);
(T3) $T(x, y) = T(y, x)$ (commutativity);
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity).

Definition 2.9. (Klir and Yuan [9]) a triangular conorm (or t-conorm) S is a mapping $S\colon [0,1] \times [0,1] \longmapsto [0,1]$, which satisfies the following axioms for every $x, y, z, \in [0,1]$:

(S1) $S(x, 0) = x$ (boundary condition);
(S2) $y \leq z$ implies $S(x, y) \leq S(x, z)$ (monotonicity);
(S3) $S(x, y) = S(y, x)$ (commutativity);
(S4) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity).

Both t-norm and s-norm are called triangular norms. For all $\alpha, \beta \in [0,1]$, It is clear that

$$T(\alpha, \beta) \leq \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq S(\alpha, \beta).$$

Definition 2.10. (Jun and Hong [7]) For a t-norm T and a s-norm S, we use the symbols Δ_T and Δ_S as the sets :

$$\Delta_T = \{a \in [0,1] | T(a, a) = a\},$$
$$\Delta_S = \{a \in [0,1] | S(a, a) = a\},$$

respectively.

Definition 2.11. (Jun and Hong [7]) We say that the intuitionistic fuzzy set $A = (\mu_A(x), \lambda_A(x))$ in M satisfies the imaginable property if

$$Im(\mu_A) \subseteq \Delta_T \text{ and } Im(\lambda_A) \subseteq \Delta_S.$$

Definition 2.12. (Klir and Yuan [9]) The norms T and S are called dual if and only if

D1) $\bar{T}(x, y) = S(\bar{x}, \bar{y})$,
D2) $\bar{S}(x, y) = T(\bar{x}, \bar{y})$ for all $x, y \in [0,1]$

A few t-norms which are frequently encountered are T_l, T_m, and T_w defined by $T_l(a,b) = \max\{a+b-1, 0\}$ (Lukasiewicz), $T_m(a,b) = \min\{a,b\}$ (minimum) and

$$T_w(a,b) := \begin{cases} \min\{a,b\} & \text{if } a = 1 \text{ or } b = 1, \\ 0 & \text{otherwise (weak).} \end{cases},$$

A few s-norms which are frequently encountered are S_l,S_m, and S_w defined by $S_l(a,b) = \min\{a+b, 1\}$ (Lukasiewicz), $S_m(a,b) = \max\{a,b\}$ (maximum) andæ

$$S_w(a,b) := \begin{cases} \max\{a,b\} & \text{if } a = 0 \text{ or } b = 0, \\ 1 & \text{otherwise (strong).} \end{cases},$$

3. Intuitionistic (T,S)-fuzzy BCK-submodules

Throughout this paper, M is a BCK-module and T is a t-norm and S is a s-norm unless otherwise specified. we can extend the concept of the intuitionistic fuzzy BCK-submodules of M to the concept of intuitionistic (T,S)-fuzzy BCK-submodules in the following way:

Definition 3.1. Let T be a t-norm and S be a s-norm on $[0,1]$. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M is called an intuitionistic fuzzy BCK-submodule of M with respect to t-norm and s-norm (briefly, intuitionistic (T,S)-fuzzy BCK-submodule of M) if it satisfies the following conditions for all $m, m_1, m_2 \in M$:

(1) $\mu_A(m_1+m_2) \geq T\{\mu_A(m_1), \mu_A(m_2)\}$,
$\lambda_A(m_1+m_2) \leq S\{\lambda_A(m_1), \lambda_A(m_2)\}$.
(2) $\mu_A(m) = \mu_A(-m), \lambda_A(m) = \lambda_A(-m)$,
(3) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

Example 3.2. Let $X = \{0,1,2,3\}$ and consider the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then $(X,*)$ is a BCK-module over itself. Define a fuzzy set $\mu_A : M \longmapsto [0,1]$ by $\mu(0) = 0.5$, $\mu(m) = 0.3$, $m \in M$ and $\lambda_A : M \longmapsto [0,1]$ by $\lambda_A(0) = 0.3$, $\lambda_A(m) = 0.5$, $m \in M$. Let $T_l : [0,1] \times [0,1] \longmapsto [0,1]$ be a function defined by $T_l(a,b) = \max(a+b-1, 0)$ for all $a,b \in [0,1]$ and let $S_l : [0,1] \times [0,1] \longmapsto [0,1]$ be a function defined by $S_l(a,b) = \min(a+b, 1)$ for all $a,b \in [0,1]$. Then T_l is a t-norm and S_l is a s-norm. By routine calculations, we know that $A = (\mu_A(x), \lambda_A(x))$ is an intuitionistic (T_l, S_l)-fuzzy BCK-submodule of M.

Theorem 3.3. *An intuitionistic fuzzy subset A of M is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if*

(1) $\mu_A(m_1-m_2) \geq T\{\mu_A(m_1), \mu_A(m_2)\}$,
$\lambda_A(m_1-m_2) \leq S\{\lambda_A(m_1), \lambda_A(m_2)\}$.
(2) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

proof. Let A be an intuitionistic (T,S)-fuzzy BCK-submodule of M, then

$$\begin{aligned}\mu_A(m_1 - m_2) &= \mu_A(m_1 + (-m_2)) \\ &\geq T(\mu_A(m_1), \mu_A(-m_2)) \\ &= T(\mu_A(m_1), \mu_A(m_2)),\end{aligned}$$

Similarly, $\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2))$. Condition 2 is hold by definition. Conversely suppose A satisfies 1 and 2. Then we have by 2

$$\mu_A(-m) = \mu_A((-1).m) \geq \mu_A(m),$$

and

$$\mu_A(m) = \mu_A((-1).(-1).m) \geq \mu_A(-m).$$

Thus $_A(m) = {}_A(-m)$. Similarly, $\lambda_A(m) = \lambda_A(-m)$.
Also we have

$$\begin{aligned}\mu_A(m_1 + m_2) &= \mu_A(m_1 - (-m_2)) \\ &\geq T(\mu_A(m_1), \mu_A(-m_2)) \\ &\geq T(\mu_A(m_1), \mu_A(m_2))\end{aligned}$$

Similarly,

$$\lambda_A(m_1 + m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

Thus A is an intuitionistic (T,S)-fuzzy BCK-submodule of M.

Proposition 3.4. *Let $T$and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M, then so is $\Box A = (\mu_A, \overline{\mu}_A)$.*

Proof. For all $m_1, m_2 \in M$, we have

$$T(\mu_A(m_1), \mu_A(m_2)) \leq \mu_A(m_1 + m_2)$$

and so

$$T(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m_2)) \leq 1 - \overline{\mu}_A(m_1 + m_2)$$

hence

$$1 - T(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m)) \geq 1 - (1 - \overline{\mu}_A(m_1 + m_2)$$

which implies

$$\overline{T}(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m_2)) \geq \overline{\mu}_A(m_1 + m_2)$$

since T and S are dual, we get

$$S(\overline{\mu}_A(m_1), \overline{\mu}_A(m_2)) \geq \overline{\mu}_A(m_1 + m_2)\ ,$$

Moreover $\mu_A(m) = \mu_A(-m)$ imply that

$$1 - \mu_A(m) = 1 - \mu_A(-m),$$

Thus $\overline{\mu}_A(m) = \overline{\mu}_A(-m)$. Now, let $m \in M$ and $x \in X$, since μ_A is T-fuzzy BCK-submodule of M, we have $\mu_A(x.m) \geq \mu_A(m)$. Hence $1 - \mu_A(x.m) \leq 1 - \mu_A(m)$ which implies $\overline{\mu}_A(xm) \leq \overline{\mu}_A(m)$. Therefore $\Box A = (\mu_A, \overline{\mu}_A)$ is an intuitionistic (T,S) - fuzzy BCK-submodule of M.

Proposition 3.5. *Let T and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK- submodule of M, then so is $\diamond A = (\overline{\lambda}_A, \lambda_A)$.*

Proof. For all $m_1, m_2 \in M$, we have

$$S(\lambda_A(m_1), \lambda_A(m_2)) \geq \lambda_A(m_1 + m_2)$$

and so

$$S(1 - \overline{\lambda}_A(m_1), 1 - \overline{\lambda}_A(m_2)) \geq 1 - \overline{\lambda}_A(m_1 + m_2)$$

hence

$$1 - S(1 - \overline{\lambda}_A(m_1), 1 - \overline{\lambda}_A(m_2)) \leq 1 - (1 - \overline{\lambda}_A(m_1 + m_2))$$

which implies

$$1 - S(\overline{\overline{\lambda}}_A(m_1), \overline{\overline{\lambda}}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2)$$

since T and S are dual

$$1 - \overline{T}(\overline{\lambda}_A(m_1), \overline{\lambda}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2)$$

that is

$$T(\overline{\lambda}_A(m_1), \overline{\lambda}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2).$$

Moreover

$$\overline{\lambda}_A(m) = \overline{\lambda}_A(-m)$$

imply that $1 - \lambda_A(m) = 1 - \lambda_A(-m)$, Thus $\lambda_A(m) = \lambda_A(-m)$. Now, let $m \in M$ and $x \in X$, since λ_A is T-fuzzy BCK-submodule of M we have $\lambda_A(x.m) \leq \lambda_A(m)$. Hence $1 - \lambda_A(x.m) \geq 1 - \lambda_A(m)$ which implies $\overline{\lambda}_A(xm) \geq \overline{\lambda}_A(m)$. Therefore $\diamond A = (\overline{\lambda}_A, \lambda_A)$ is an intuitionistic (T, S) - fuzzy BCK-submodule of M.

Combining the above two Propositions it is not difficult to verify that the following theorem is valid.

Theorem 3.6. *Let T and S be dual norms. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M if and only if $\square A$ and $\diamond A$ are intuitionistic (T, S)-fuzzy BCK-submodule of M.*

Corollary 3.7. *Let T and S be dual norms. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M if and only if μ_A and $\overline{\lambda_A}$ are T-fuzzy BCK-submodule of M.*

From corollary 3.7 we immediately obtain the following result.

Theorem 3.8. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)- fuzzy BCK- submodule of M if and only if the fuzzy sets μ_A and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M.*

Theorem 3.9. *An intuitionistic fuzzy set $A = (\mu_A\ ,\ \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M if and only if $\square A = (\mu_A, \bar{\mu}_A)$ and $\diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. By Theorem 3.8, we get $\mu_A = \overline{\overline{\mu}}_A$and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M.

Therefore $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. Conversely, assume that $A = (\mu_A, \lambda_A)$ and $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCKsubmodule of M. Then the fuzzy sets μ_A and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M. Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M.

Definition 3.10. An intutionistic (T, S)-fuzzy BCK-submodule of M is called an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M if it satisfies the imaginable property.

Proposition 3.11. *Every imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M is an intuitionistic fuzzy BCK-submodule of M.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M. Then

$$\mu_A(m_1 + m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 + m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2))$$

for all $m_1, m_2 \in M$.
Since $A = (\mu_A, \lambda_A)$ is imaginable, we have

$$\begin{aligned}
&\min\{\mu_A(m_1), \mu_A(m_2)\} \\
&= T(\min\{\mu_A(m_1), \mu_A(m_2)\}, \min\{\mu_A(m), \mu_A(m_2)\}) \\
&\leq T(\mu_A(m_1), \mu_A(m_2)) \\
&\leq \min\{\mu_A(m_1), \mu_A(m_2)\},
\end{aligned}$$

and

$$\begin{aligned}
&\max\{\lambda_A(m_1), \lambda_A(m_2)\} \\
&= S(\max\{\lambda_A(m_1), \lambda_A(m_2)\}, \max\{\lambda_A(m), \lambda_A(m_2)\}) \\
&\geq S(\lambda_A(m_1), \lambda_A(m_2)) \\
&\geq \max \lambda_A(m_1), \lambda_A(m_2).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mu_A(m_1 - m_2) \\
&\geq T(\mu_A(m_1), \mu_A(m_2)) \\
&= \min\{\mu_A(m_1), \mu_A(m_2)\},
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_A(m_1 - m_2) \\
&\leq S(\lambda_A(m_1), \lambda_A(m_2)) \\
&= \max\{\lambda_A(m_1), \lambda_A(m_2)\}.
\end{aligned}$$

Now let $x \in X$ and $m \in M$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, we have $\mu_A(xm) \geq \mu_A(m)$, $\lambda_A(xm) \leq \lambda_A(m)$. Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M.

Note that every intuitionistic fuzzy BCK-submodule is an intuitionistic (T,S)-fuzzy BCK-submodule but the converse is not true as seen in the following Example.

Example 3.12. We consider the BCK-module M which is given in Example 3.2. Define an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M

$$\mu_A(m) = \begin{cases} 0.2 & \text{if} \quad m = 1 \\ 0.3 & \text{if} \quad m = 2,3 \\ 0.5 & \text{if} \quad m = 0 \end{cases} ; \quad \lambda_A(m) = \begin{cases} 0.5 & \text{if} \quad m = 1 \\ 0.3 & \text{if} \quad m = 2,3 \\ 0.1 & \text{if} \quad m = 0 \end{cases}$$

Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_w, S_w)-fuzzy BCK-submodule of M, but it is not an intuitionistic fuzzy BCK-submodule of M since

$$\mu_A(2+3) = \mu_A(1) = 0.2 < 0.3 = \min(\mu_A(2), \mu_A(3)).$$

Proposition 3.13. *If an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M is an imaginable intuitionistic (T,S)-fuzzy BCK-submodule of M, then for all $m \in M$, $\mu_A(0) \geq \mu_A(m)$ and $\lambda_A(0) \leq \lambda_A(m)$.*

Proof. From Definition 3.1 (3) it follows that

$$\mu_A(0) = \mu_A(0.m) \geq \mu_A(m)$$

and

$$\lambda_A(0) = \lambda_A(0.m) \leq \lambda_A(m)$$

for all $m \in M$.

Theorem 3.14. *If $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T,S)-fuzzy BCK-submodule of M, then the set $H = \{m \in M | \mu(m) = \mu(0)\}$ and $K = \{m \in M | \lambda_A(m) = \lambda_A(0)\}$ are BCK-submodule of M.*

Proof. Assume that $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T,S)-fuzzy BCK-submodule of M, and let $m_1, m_2 \in M$. Since $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T,S)-fuzzy BCK-submodule of M, we have

$$\begin{aligned} \mu_A(m_1 - m_2) &\geq T(\mu_A(m), \mu_A(m)) \\ &= T(\mu_A(0), \mu_A(0)) \\ &= \mu_A(0) \end{aligned}$$

for all $m_1, m_2 \in M$, Using Lemma Proposition 3.11., we get $\mu_A(m_1 - m_2) = \mu_A(0)$. Hence $m_1 - m_2 \in H$. Now let $x \in X$and $m \in M$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M, we have $\mu_A(x.m) \geq \mu_A(m) = \mu_A(0)$. Using Lemma Proposition 3.11., we get $\mu_A(x.m) = \mu_A(0)$ and so $x.m \in H$. Therefore H is a BCK-submodule of M. By similar method, we get K is a BCK-submodule of M.

Definition 3.15. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in BCK-submodule M and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then the set

$$A_{(\alpha,\beta)} := \{m \in M | \mu_A(m) \geq \alpha, \lambda_A(m) \leq \beta\}$$

is called an (α, β)-level set of $A = (\mu_A, \lambda_A)$.

Theorem 3.16. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in M such that*

$A_{(\alpha,\beta)}$ is a BCK-submodule of M , for all $(\alpha,\beta) \in [0,1]$ with $\alpha+\beta \leq 1$. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M .

Proof. Let $m_1, m_2, m \in M$ and $x \in X$ be such that $A(m_1) = (\alpha_1, \beta_1)$, $A(m_2) = (\alpha_2, \beta_2)$ where $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$. Then $m_1, m_2 \in A_{(\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))}$, and so $m_1 - m_2 \in A_{\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))}$.
Hence

$$\mu_A(m_1 - m_2) \geq \min(\alpha_1, \alpha_2) \geq T(\alpha_1, \alpha_2),$$

and

$$\lambda_A(m_1 - m_2)) \leq \max(\beta_1, \beta_2) \leq S(\beta_1, \beta_2).$$

Also, if we put $s' = {}_A(m), t' = {}_A(m)$ where $s' + t' \leq 1$. Then $m \in A_{(s',t')}$. Since $A_{(s',t')}$ is a BCK- submodule of M, we have $xm \in A_{(s',t')}$. It follows that

$$\mu_A(xm) \geq s' = \mu_A(m)$$

and

$$\lambda_A(xm) \leq t' = \lambda_A(m)$$

Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M.

The following Example shows that the converse of Theorem 3.16 is not true.

Example 3.17. We consider the intuitionistic (T_w, S_w)-fuzzy BCK-submodule A of M which is given in Example 3.2. Then $A_{(0.3,0.5)} = \{2, 3, 0\}$ is not BCK-submodule of M since $2 + 3 = 1 \notin A_{(0.3,0.5)}$

Theorem 3.18. *If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, then $A_{(1,0)}$ is either empty or a BCK-submodule of M .*

Proof. Let $m_1, m_2 \in A_{(1,0)}$. Then $\mu_A(m_1) \geq 1$, $\mu_A(m_2) \geq 1$, $\lambda_A(m_1) \leq 0$ and $\lambda_A(m_2) \leq 0$. It follows from Definitions 2.10 and Theorem 3.3 that

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2)) \geq T(1, 1) = 1$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)) \leq S(0, 0) = 0,$$

so $m_1 - m_2 \in A_{(1,0)}$. Let $m \in A_{(1,0)}$ and $x \in X$. Then

$$\mu_A(xm) \geq \mu_A(m) \geq 1$$

and

$$\lambda_A(xm) \leq \lambda_A(m) \leq 0,$$

so $xm \in A_{(1,0)}$.

As a generalization of Theorem 3.18, we get the following Theorem.

Theorem 3.19. *If $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M, then $A_{(\alpha,\beta)}$ is either empty or a BCK-submodule of M for all $\alpha \in \Delta_T$ and $\beta \in \Delta_S$. with $\alpha + \beta \leq 1$.*

Proof. Let $m_1, m_2 \in A_{(\alpha,\beta)}$ where $\alpha \in \Delta_T$, $\beta \in \Delta_S$ and $\alpha+\beta \leq 1$. Then

$$\begin{aligned}&\mu_A(m_1 - m_2)\\ &\geq T(\mu_A(m_1), \mu_A(m_2))\\ &\geq T(\alpha,\alpha) = \alpha\end{aligned}$$

and

$$\begin{aligned}&\lambda_A(m_1 - m_2)\\ &\leq S(\lambda_A(m_1), \lambda_A(m_2))\\ &\leq S(\beta,\beta) = \beta,\end{aligned}$$

and so $m_1 - m_2 \in A_{(\alpha,\beta)}$. Let $m \in A_{(\alpha,\beta)}$ and $x \in X$. Then

$$\mu_A(xm) \geq \mu_A(m) \geq \alpha$$

and

$$\lambda_A(xm) \leq \lambda_A(m) \leq \beta,$$

so $xm \in A_{(\alpha,\beta)}$.Hence $A_{(\alpha,\beta)}$ is a BCK-submodule of M .

Proposition 3.20. (Bakhshi [3]) *A fuzzy set in M is a fuzzy BCK-submodule of M if and only if the non-empty $U(\mu,\alpha)$, $\alpha \in [0,1]$ is a BCK-submodule of M.*

By the above Proposition , we get the following result.

Corollary 3.21. *If $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic fuzzy set in M. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if the non-empty sets $U(\mu,\alpha)$ and $L(\lambda,\alpha)$ are BCK-submodules of M, for every $(\alpha,\beta) \in [0,1]$.*

From corollary 3.21 we immediately obtain the following Theorem.

Theorem 3.22. *Let T be the minimum t-norm and let S the maximum s-norm dual of T . Then an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ of M is is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if*

$$A_{(\alpha,\beta)} := \{m \in M | \mu_A(m) \geq \alpha, \lambda_A(m) \leq \beta\}$$

is a BCK-submodule of M, where $(\alpha,\beta) \in [0,1]$.

Proposition 3.23. *Let S be a non-empty subset of a BCK-module M. Then an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ defined by*

$$\mu_A(m) = \begin{cases} 1 & \text{if} \quad m \in S, \\ \alpha & \quad \text{otherwise.} \end{cases}, \lambda_A(m) = \begin{cases} 0 & \text{if} \quad m \in S, \\ \beta & \quad \text{otherwise.} \end{cases}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ *and $\alpha + \beta \leq 1$ is an intuitionistic (T,S)-fuzzy BCK -submodule of M if and only if S is a BCK-submodule of M.*

Proof. Let S be a BCK-submodule of M. Let $m_1, m \in M$. If $m_1, m_2 \in S$,

then $m_1 - m_2 \in S$, and so

$$\begin{aligned}\mu_A(m_1 - m_2) = 1 \geq 1 \\ &= T(1,1) \\ &= T(\mu_A(m_1), \mu_A(m_2))\end{aligned}$$

and

$$\begin{aligned}\lambda_A(m_1 - m_2) &= 0 \\ &= S(0,0) \\ &= S(\lambda_A(m_1), \lambda_A(m_2))\end{aligned}$$

For $m_1 \in S$, $m_2 \notin S$, we have

$$\begin{aligned}\mu_A(m_1 - m_2) = \alpha \geq \alpha \\ &= T(1,\alpha) \\ &= T(\mu_A(m_1), \mu_A(m_2))\end{aligned}$$

and

$$\begin{aligned}\lambda_A(m_1 - m_2) = \beta \leq \beta \\ &= S(0,\beta) \\ &= S(\lambda_A(m_1), \lambda_A(m_2))\end{aligned}$$

Similarly, for the case $m_1 \notin S$, $m_2 \in S$, we have

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

For $m_1 \notin S$, $m_2 \notin S$,

$$\begin{aligned}\mu_A(m_1 - m_2) &\geq \alpha \\ &= T(1,\alpha) \\ &\geq T(\alpha,\alpha) \\ &= T(\mu_A(m_1), \mu_A(m_2)),\end{aligned}$$

and

$$\begin{aligned}\lambda_A(m_1 - m_2) &\leq \beta \\ &= S(0,\beta) \\ &\leq S(\beta,\beta) \\ &= S(\lambda_A(m_1), \lambda_A(m_2)).\end{aligned}$$

Thus for all cases,

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

Next, let $m \in M$ and $x \in X$, Then, if $m \in S$ then $xm \in S$ and so,

$$\mu_A(xm) = 1 \geq 1 = \mu_A(m)$$

and

$$\lambda_A(xm) = 0 \leq 0 = \lambda_A(m).$$

If $m \notin S$, then

$$\mu_A(xm) \geq \alpha = {}_A(m)$$

and

$$\lambda_A(xm) \leq \beta = \lambda_A(m).$$

Therefore $\mu_A(xm) \geq \mu_A(m)$ and $\lambda_A(xm) \leq \lambda_A(m)$. Thus $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M .

Conversely, we assume $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in S$, $x \in X$. Then,

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2)) = T(1, 1) = 1,$$

hence $\mu_A(m_1 - m_2) = 1$. Thus $m_1 - m_2 \in S$. Also, $\mu_A(xm) \geq \mu_A(m) = 1$ implies $\mu_A(xm) = 1$ implies $xm \in S$. Hence, S is a BCK-submodule of M .

Corollary 3.24. *Let S be a non-empy subset of a BCK-module M and let χ_s be the characteristic function of S. Then $A = (\chi_s, \chi_s^c)$ is an intutionistic (T, S)-fuzzy BCK-submodule of M if and only if S is a BCK-submodule of M.*

Definition 3.25. (Janiŝ [6]) Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X and let T be a t-norm. Then $A_{T,\alpha}$ is a subset of X defined by

$$A_{T,\alpha} = \{x \in X | T(\mu_A(x), 1 - \lambda_A(x)) \geq \alpha\},$$

for every $\alpha \in [0, 1]$

Theorem 3.26. *Let T and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M. Then*

$$A_{T,1} = \{m \in M | T(\mu_A(m), 1 - \lambda_A(m)) = 1\}$$

is a BCK-submodule of M.

Proof. Let $m_1, m_2 \in A_{T,1}$. Then,

$$\begin{aligned}
&T(\mu_A(m_1 - m_2), 1 - {}_A(m_1 - m_2)) \\
&\geq T(T(\mu_A(m_1), {}_A(m_2)), 1 - S({}_A(m_1), {}_A(m_2))) \\
&= T(T(\mu_A(m_2), (\mu_A(m_1)), T(1 - \lambda_A(m_1), 1 - \lambda_A(m_2))) \\
&= T(\mu_A(m_2), T(\mu_A(m_1), T(1 - \lambda_A(m_1), 1 - \lambda_A(m_2)))) \\
&= T(\mu_A(m_2), T(T(\mu_A(m_1), 1 - \lambda_A(m_1)), 1 - \lambda_A(m_2))) \\
&= T(\mu_A(m_2), T(1 - \lambda_A(m_2), T(\mu_A(m_1), 1 - \lambda_A(m_1)))) \\
&= T(T(\mu_A(m_2), 1 - \lambda_A(m_2)), T(\mu_A(m_1), 1 - \lambda_A(m_1))) \\
&= T(1, 1) = 1
\end{aligned}$$

Thus, we have $T(\mu_A(m_1 - m_2), 1 - \lambda_A(m_1 - m_2)) = 1$ Therefore $m_1 - m_2 \in A_{T,1}$. Also, let $x \in X$ and $m \in A_{T,1}$. Then $T(\mu_A(m), 1 - \lambda_A(m)) = 1$. Further, $T(\mu_A(xm), 1 - \lambda_A(xm)) \geq T(\mu_A(m), 1 - \lambda_A(m)) = 1$. Therefore $xm \in A_{T,1}$. Hence, $A_{T,1}$ is a is a BCK-submodule of M.

For any tringular norm T, the level set $A_{T,\alpha}$ of an intuitionistic (T,S)-fuzzy BCK-submodule of M is not necessarily to be a BCK-submodule of M. However, if T is the minimum tringular norm, then all level sets $A_{T,\alpha}$ of an intuitionistic (T,S)-fuzzy BCK-submodule of M are BCK-submodules of M.

Theorem 3.27. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M such that T_m, S_m are dual. Then for every $\alpha \in [0,1]$,*

$$A_{T_m,\alpha} = \{m \in M | T(\mu_A(m), 1 - \lambda_A(m)) \geq \alpha\}$$

is a BCK-submodule of M.

Proof. Let $A = (\mu_A(x), \lambda_A(x))$ is an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in A_{T,\cdot}$. Then,

$$\begin{aligned}
&T_m(\mu_A(m_1 - m_2), 1 -{}_A(m_1 - m_2))\\
&\geq T_m(T_m(\mu_A(m_1), \mu_A(m_2)), 1 - S_m(\lambda_A(m_1), \lambda_A(m_2)))\\
&= T_m(T_m(\mu_A(m_2), (\mu_A(m_1)), T_m(1 - \lambda_A(m_1), 1 - \lambda_A(m_2)))\\
&= T_m(\mu_A(m_2), T_m(\mu_A(m_1), T_m(1 - \lambda_A(m_1), 1 - \lambda_A(m_2))))\\
&= T_m(\mu_A(m_2), T_m(T_m(\mu_A(m_1), 1 - \lambda_A(m_1)), 1 - \lambda_A(m_2)))\\
&= T_m(\mu_A(m_2), T_m(1 - \lambda_A(m_2), T_m(\mu_A(m_1), 1 - \lambda_A(m_1))))\\
&= T_m(T_m(\mu_A(m_2), 1 - \lambda_A(m_2)), T_m(\mu_A(m_1), 1 - \lambda_A(m_1)))\\
&\geq T_m(\alpha, \alpha) = \alpha
\end{aligned}$$

Thus, we have

$$T_m(\mu_A(m_1 - m_2), 1 -{}_A(m_1 - m_2)) \geq \alpha$$

Therefore, $m_1 - m_2 \in A_{T_m,\alpha}$. Also, let $x \in X$ and $m \in A_{T_m,\alpha}$. Then

$$T_m(\mu_A(m), 1 - \lambda_A(m)) \geq \alpha$$

Further,

$$T_m(\mu_A(xm), 1 - \lambda_A(xm)) \geq T_m(\mu_A(m), 1 - \lambda_A(m)) \geq \alpha$$

Therefore we have $T_m(\mu_A(xm), 1-\lambda_A(xm)) \geq \alpha$. Hence $xm \in A_{T_m,\alpha}$. Thus $A_{T_m,\alpha}$ is a is a BCK-submodule of M.

Definition 3.28. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X, let T and S be dual norms. Then $A_{T,S,\alpha}$ is a subset of X defined by

$$A_{T,S,1} = \{x \in XT(\mu_A(x), S(\mu_A(x), \lambda_A(x))) \geq \alpha\}$$

for every $\alpha \in [0,1]$.

Theorem 3.29. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T,S)-fuzzy BCK-submodule of M, then*

$$A_{T,S,1} = \{m \in M | T(\mu_A(m), S(\mu_A(m), \lambda_A(m))) = 1\}$$

is a BCK-submodule of M.

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T,S)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in A_{T,S,1}$, then

$$T(\mu_A(m_1), S(\mu_A(m_1), \lambda_A(m_1))) = 1$$

and

$$T(\mu_A(m_2), S(\mu_A(m_2), \lambda_A(m_2))) = 1.$$

Therefore $\mu_A(m_1) \geq 1$ and $\mu_A(m_2) \geq 1$ which mean that $\mu_A(m_1) = 1$ and $\mu_A(m_2) = 1$. From monotonicity of T, we have,

$$\begin{aligned}
&T(\mu_A(m_1 - m_2), S(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \\
&\geq T(T(\mu_A(m_1 - m_2)), T(\mu_A(m_1 - m_2))) \\
&\geq T(T(\mu_A(m), \mu_A(m)), T(\mu_A(m), \mu_A(m))) \\
&= T(T(1,1), T(1,1)) \\
&= T(1,1) = 1
\end{aligned}$$

Therefore, $T(\mu_A(m_1 - m_2), S(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) = 1$ implies $m_1, m_2 \in A_{T,S,1}$. Also, let $x \in X$ and $m \in A_{T,S,1}$. Then, $T(\mu_A(m), S(\mu_A(m), \lambda_A(m))) = 1$. which impliese $\mu_A(m) = 1$. Now,

$$\begin{aligned}
&T(\mu_A(xm), S(\mu_A(xm), \lambda_A(xm))) \\
&\geq T(\mu_A(xm), \mu_A(xm)) \\
&\geq T(\mu_A(m), \mu_A(m)) \\
&= T(1,1) = 1
\end{aligned}$$

Thus, we have, $T(\mu_A(xm), S(\mu_A(xm), \lambda_A(xm))) = 1$. Therefore, $xm \in A_{T,S,1}$. Hence, $A_{T,S,1}$ is a BCK-submodule of M.

Theorem 3.30. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of Msuch that T_m, S_m are dual. Then for every $\alpha \in [0,1]$,*

$$A_{T,S,\alpha} = \{m \in M | T(\mu_A(m), S(\mu_A(m), \lambda_A(m))) \geq \alpha\}$$

is a BCK-submodule of M.

Proof. Let $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in A_{T,S,\alpha}$, then

$$T_m(\mu_A(m_1), S_m(\mu_A(m_1), \lambda_A(m_1))) \geq \alpha$$

and

$$T_m(\mu_A(m_2), S_m(\mu_A(m_2), \lambda_A(m_2))) \geq \alpha.$$

Therefore $\mu_A(m_1) \geq \alpha$ and $\mu_A(m_2 \geq \alpha$. Due monotonicity of T_m, we have,

$$\begin{aligned}
&T_m(\mu_A(m_1 - m_2), S_m(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \\
&\geq T_m(\mu_A(m_1 - m_2)), (\mu_A(m_1 - m_2))) \\
&= \mu_A(m_1 - m_2) \\
&\geq T_m(\mu_A(m_1), \mu_A(m_2)) \\
&\geq T_m(\alpha, \alpha) \\
&= \alpha
\end{aligned}$$

Therefore, $T_m(\mu_A(m_1 - m_2), S_m(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \geq \alpha$ and hence $m_1 - m_2 \in A_{T_m,S_m,\alpha}$. Also, let $m \in A_{T_m,S_m,\alpha}$ and $x \in X$. Then,

$$T_m(\mu_A(m), S_m(\mu_A(m), \lambda_A(m))) \geq \alpha.$$

which impliese $\mu_A(m) \geq \alpha$. From monotonicity of T_m, we have,

$$\begin{aligned} & T_m(\mu_A(xm), S_m(\mu_A(xm), \lambda_A(xm)) \\ & \geq T_m(\mu_A(xm), \mu_A(xm)) \\ & = \mu_A(xm) \\ & \geq \mu_A(m) \\ & \geq \alpha \end{aligned}$$

Thus $T_m(\mu_A(xm), S_m(\mu_A(xm), \lambda_A(xm)) \geq \alpha$. Therefore, $xm \in A_{T_m,S_m,\alpha}$. Hence, $A_{T_m,S_m,\alpha}$ is a BCK-submodule of M.

4. Conclusion

One of the generalizations of fuzzy BCK-submodules, namely, intuitionistic (T,S)-fuzzy BCK- submodules was defined and some properties of intuitionistic (T,S)-fuzzy BCK-submodules are investigated. Also, some related results on level sets of an intuitionistic (T,S)-fuzzy BCK-submodule are investigated. These investigations of generalized fuzzy on BCK-modules could be enable us to discuss further study in this field.

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On strongly almost generalized difference lacunary ideal convergent sequences of fuzzy numbers

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Abstract

The purpose of this paper is to introduce some new sequence spaces of fuzzy numbers defined by lacunary ideal convergence using generalized difference matrix and Orlicz functions. We also study some algebraic and topological properties of these classes of sequences. Moreover, some illustrative examples are given in support of our results.

Keywords and phrases: Ideal convergence; fuzzy number; difference sequence; Orlicz function; lacunary sequence.
AMS subject classification (2010): 40A05; 40C05; 40G15; 06B99.

1 Introduction and preliminaries

The concept of ideal convergence is the dual (equivelant) to the notion of filter convergence introduced by Cartan [4]. The filter convergence is a generalization of the classical notion of convergence of sequences of real or complex numbers and it has been an important tool in the study of functional analysis. Nowadays many authors studied this notion from various aspects and applied this notion to various problems arising in the convergence theory. Kostyrko et al. [13] and Nuray and Ruckle [23] independently studied in detalis about the notion of ideal convergence which is based upon the structure of the admissible ideal I of subsets $\mathbb{N}$ of natural numbers. Later on it was further investigated by many authors, e.g. Tripathy and Hazarika [26], Mursaleen and Mohiuddine [22] and references therein.

Let S be a non-empty set. Then a non empty class $I \subseteq P(S)$ is said to be an *ideal* on S if and only if (i) $\phi \in I$; (ii) I is additive; (iii) hereditary. An ideal $I \subseteq P(S)$ is said to be *non trivial* if $I \neq \phi$ and $S \notin I$. A non-empty family of sets $F \subseteq P(S)$ is said to be a *filter* on S if and only if (i) $\phi \notin F$ (ii) for each $A, B \in F$ we have $A \cap B \in F$; (iii) for each $A \in F$ and each $B \supset A$, we have $B \in F$. For each ideal I, there is a filter $F(I)$ corresponding to I i.e. $F(I) = \{K \subseteq S : K^c \in I\}$, where $K^c = S - K$. We say that a non-trivial ideal $I \subseteq P(S)$ is an *admissible ideal* on S if and only if it contains all singletons, i.e. if it contains $\{\{s\} : s \in S\}$. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ (denoted by $I\text{-}\lim x_k = \ell$) if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$.

We used the standard notation $\theta = (k_r)$ to denote the *lacunary sequence*, where θ is a sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ $(r \neq 1)$ by q_r (see [8]).

The notion of lacunary ideal convergence for sequences of real numbers and fuzzy numbers, respectively, has been defined and studied in [27] and [9]. Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A real sequence

$x = (x_k)$ is said to be *lacunary I-convergent* to $L \in \mathbb{R}$, in symbol we shall write $I_\theta\text{-}\lim x = L$, if for every $\varepsilon > 0$, the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| \geq \varepsilon \right\} \in I.$$

Throughout the paper we use w to denotes the set of all real sequences $x = (x_k)$. The difference sequence spaces have been introduced by Kızmaz [12] by using the difference operator Δ as follows:

$$Z(\Delta) = \{(x_k) \in w : \Delta x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$ and $\Delta x_k = \Delta^1 x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$, where the standard notations ℓ_∞, c and c_0 are used to denote the set of bounded, convergent and null sequences, respectively. Later this idea was generalized by Et and Çolak [6] by considering Δ^n instead of Δ, where $(\Delta^n x_k) = \Delta^1(\Delta^{n-1} x_k)$ for $n \geq 2$ and all $k \in \mathbb{N}$. In case of $n = 0$ we obtain x_k. Tripathy et al. [28] presented another generalization of difference sequence spaces by introducing the operator Δ^n_m and is given by $\Delta^n_m x = (\Delta^n_m x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k+m})$ so that $\Delta^n_m x_k$ has the following binomial representation:

$$\Delta^n_m x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+m\nu},$$

for all $k \in \mathbb{N}$. If we take $n = 1$, then $Z(\Delta^n_m)$ is reduced to $Z(\Delta_m)$ which was introduced by Tripathy and Esi [25], in this case the operator $\Delta_m x$ is given by $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$ for all $k, m \in \mathbb{N}$. The choice of $m = 1$ in the definition of $Z(\Delta^n_m)$ gives us the difference sequence spaces introduced by Et and Colak [6]. Başar and Altay [1] introduced the generalized difference matrix $B(r, s) = (b_{nk}(r, s))$ by

$$b_{nk}(r, s) = \begin{cases} r, & \text{if } k = n; \\ s, & \text{if } k = n - 1; \\ 0, & \text{if } 0 \leq k < n - 1 \text{ or } k > n. \end{cases}$$

for all $k, n \in \mathbb{N}$ and all non-zero real numbers r, s. The generalized difference matrix B^n of order n has been recently defined by Başarir and Kayikçi [2] and its binomial representation is given by

$$B^n x_k = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{n-\nu} s^\nu x_{k-\nu},$$

for all $n \in \mathbb{N}$ and $r, s \in \mathbb{R} - \{0\}$. Another generalization of above difference matrix was given by Başarir et al. [3] as $B^n_{(m)}$, where $B^n_{(m)} x = (B^n_{(m)} x_k) = (r B^{n-1}_{(m)} x_k + s B^{n-1}_{(m)} x_{k-m})$ and $B^0_{(m)} x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$B^n_{(m)} x_k = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{n-\nu} s^\nu x_{k-m\nu}.$$

In [24], Orlicz introduced functions nowadays called Orlicz functions and constructed the sequence space (L^M). Krasnoselskii and Rutitsky further investigated the Orlicz space in [14]. Some recent related work we refer to Mohiuddine et al. [19, 20]. A function $M : [0, \infty) \to [0, \infty)$ is said to be an *Orlicz function* if it is non-decreasing, continuous, convex with $M(0) = 0$, $M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ (see [24]). It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda \in (0, 1)$. An Orlicz function M is said to be satisfy Δ_2-*condition* for all values of u, if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$ (see, Krasnoselskii and Rutitsky [14]).

Lindenstrauss and Tzafriri [16] introduced the sequence space ℓ_M by using the notion of Orlicz function by

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

and proved that this space is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Every space ℓ_M contains a subspace isomorphic to the classical sequence space ℓ_p for some $1 \leq p < \infty$. The space ℓ_p, $1 \leq p < \infty$ is itself an Orlicz sequence space with $M(t) = |t|^p$.

A sequence space E is said to be (i) *normal* (or *solid*) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, (ii) *symmetric* if $(x_{\pi(k)}) \in E$, whenever $(x_k) \in E$, where π is a permutation of $\mathbb{N}$.

Let E be a sequence space and $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$. A sequence space of the form $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$ is called a *K-step space* of E. A canonical preimage of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in w$ and is defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E. We say that E is *monotone* if E contains the canonical pre-image of all its step spaces. Note that every normal space is monotone (see [11], pp. 53).

A sequence $x = (x_k) \in \ell_\infty$ (the space of bounded sequences) is said to be *almost convergent*, denoted by $\widehat{c}$, if all of its Banach limits coincide. Lorentz [17] introduced this sequence space as follows:

$$\widehat{c} = \left\{ x \in \ell_\infty : \lim_k t_{jk}(x) \text{ exists uniformly in } j \right\},$$

where

$$t_{jk}(x) = \frac{x_j + x_{j+1} + ... + x_{j+k}}{k+1}.$$

It is clear that

$$t_{jk}(x) = \begin{cases} \frac{1}{k}\sum_{i=1}^{k} x_{j+i} & \text{for } k \geq 1; \\ x_j & \text{for } k = 0. \end{cases}$$

Zadeh [29] introduced the concept of fuzzy set theory and its applications can be found in many branches of mathematical and engineering sciences including management science, control engineering, computer science, artificial intelligence. Matloka [18] introduced the bounded and convergent sequences of fuzzy numbers and proved that every convergent sequence of fuzzy numbers is bounded. Later, various classes of sequences of fuzzy numbers have been defined and studied by Colak et al. [5], Et et al. [7], Mursaleen and Başarir [21], Hazarika [10] and references therein.

Now recalling some notions of fuzzy numbers which we will used to prove our main results. Throughout the paper we used w^F, ℓ_∞^F, c^F and c_0^F to denote the set of all, bounded, convergent and null sequence spaces of fuzzy numbers, respectively. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}$ i.e., a mapping $X : \mathbb{R} \to J(= [0, 1])$ associating each real number t with its grade of membership $X(t)$. A fuzzy number X is said to be (i) *upper-semi continuous* if for each $\varepsilon > 0, X^{-1}([0, a + \varepsilon))$ for all $a \in [0, 1]$ is

open in the usual topology of $\mathbb{R}$, (ii) *convex* if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$ for $s < t < r$ (iii) *normal* if there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$.

We used the notation X^α to denotes α-level set of a fuzzy number $X, 0 < \alpha \leq 1$ and is given by $X^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}$. The set of all normal, convex and upper semi-continuous fuzzy number with compact support will be denoted by $\mathbb{R}(J)$ and the fuzzy number we mean that the number belongs to $\mathbb{R}(J)$. We used the symbol D to denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on $\mathbb{R}$. For any two sets $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. A metric d on D is given by $d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. It is easy to see that (D, d) is a complete metric space. Also, the relation $\leq$ is a partial order on D.

The absolute value $|X|$ of $X \in \mathbb{R}(J)$ is given by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Suppose that $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}$ is a mapping such that $\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha)$. Then $(\mathbb{R}(J), \overline{d})$ is a complete metric space.

We define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$, for all $\alpha \in J$. By $\bar{0}$ and $\bar{1}$ we denotes the additive and multiplicative identities in $\mathbb{R}(J)$, respectively.

A sequence $u = (u_k)$ of fuzzy numbers is said to be (i) *bounded* if the set $\{u_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded, (ii) *convergent* to a fuzzy number u_0 if for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\bar{d}(u_k, u_0) < \varepsilon$, for all $k \geq n_0$, (iii) *I-convergent* (see [15]) if there exists a fuzzy number u_0 such that for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : \bar{d}(u_k, u_0) \geq \varepsilon\} \in I$. We write I-lim $u_k = u_0$, (iv) *I-bounded* if there exists $K > 0$ such that the set $\{k \in \mathbb{N} : \bar{d}(u_k, \bar{0}) \geq K\} \in I$.

2 Main results

Throughout the article we assume that I is an admissible ideal of $\mathbb{N}$. In this section, we introduce the following definitions. We introduce some new strongly almost ideal convergent sequence spaces using the generalized difference matrix $B^n_{(m)}$ and Orlicz function M. Let us consider a sequence $p = (p_k)$ of positive real numbers and let m, n be any nonnegative integers. For some $\rho > 0$, we define the following sequence spaces.

$$[\widehat{w}_0^{IF}(M, \theta, B^n_{(m)}, p)] = \Bigg\{(u_k) \in w^F : \Bigg\{r \in \mathbb{N} : \frac{1}{h_r} \times \sum_{k \in J_r} \left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\Bigg\} \in I, \text{uniformly in } j \in \mathbb{N}\Bigg\}$$

$$[\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)] = \Bigg\{(u_k) \in w^F : \Bigg\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\Bigg\} \in I, \text{uniformly in } j \in \mathbb{N} \text{ and for some } u_0 \in \mathbb{R}(J)\Bigg\}$$

$$[\widehat{w}^F_\infty(M, \theta, B^n_{(m)}, p)] = \Bigg\{(u_k) \in w^F : \sup_r \frac{1}{h_r}$$

$$\times \sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k} < \infty, \text{ uniformly in } j\in\mathbb{N}\Bigg\}$$

$$[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)] = \Bigg\{(u_k)\in w^F : \exists K>0 \; s.t. \Bigg\{r\in\mathbb{N} : \frac{1}{h_r}$$

$$\times \sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k} \geq K\Bigg\}\in I, \text{ uniformly in } j\in\mathbb{N}\Bigg\}.$$

Particular cases:

(i) If $p=(p_k)=1$ for all $k\in\mathbb{N}$, we denote $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_0(M,\theta,B^n_{(m)})]$, $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}(M,\theta,B^n_{(m)})]$, $[\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)})]$ and $[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)})]$.

(ii) If $M(x)=x$, we denote $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_0(\theta,B^n_{(m)},p)]$, $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}(\theta,B^n_{(m)},p)]$, $[\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{F}_\infty(\theta,B^n_{(m)},p)]$ and $[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_\infty(\theta,B^n_{(m)},p)]$.

(iii) If $\theta=(2^r)$, we denote $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_0(M,B^n_{(m)},p)]$, $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}(M,B^n_{(m)},p)]$, $[\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{F}_\infty(M,B^n_{(m)},p)]$ and $[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_\infty(M,B^n_{(m)},p)]$.

Throughout the manuscript, we will used the following well-known inequality. Suppose that $p=(p_k)$ is a sequence of positive real numbers with $0<p_k\leq \sup_k p_k = H$, $D=\max\{1,2^{H-1}\}$. Then

$$|a_k+b_k|^{p_k} \leq D(|a_k|^{p_k}+|b_k|^{p_k}) \text{ for all } k\in\mathbb{N} \text{ and } a_k,b_k\in\mathbb{C}.$$

Also $|a|^{p_k}\leq \max\{1,|a|^H\}$ for all $a\in\mathbb{C}$.

Now we are ready to give our main results as follows.

Theorem 2.1. *Let $p=(p_k)$ be a bounded sequence of positive real numbers. The spaces $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)]$, $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$, $[\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)},p)]$, and $[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)]$ are closed with respect to addition and scalar multiplication.*

Proof. We prove the result only for the space $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$. The others can be treated similarly. Let $u=(u_k)$ and $v=(v_k)$ be two elements of $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ and α_1,α_2 be scalars. Let $\varepsilon>0$ be given. Then there exist positive numbers ρ_1,ρ_2 such that

$$P=\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho_1}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}\in I \qquad \text{(uniformly in } j\in\mathbb{N})$$

and

$$Q=\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}v_k),v_0)}{\rho_2}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}\in I \qquad \text{(uniformly in } j\in\mathbb{N}).$$

Let $\rho_3=\max(2|\alpha_1|\rho_1, 2|\alpha_2|\rho_2)$. Since M is non-decreasing and convex function, we have

$$\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}(\alpha_1u_k+\alpha_2v_k)),\alpha_1u_0+\alpha_2v_0)}{\rho_3}\right)\right]^{p_k}$$

$$\leq \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\alpha_1\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho_3}\right)\right]^{p_k}+\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\alpha_2\overline{d}(t_{jk}(B^n_{(m)}v_k),v_0)}{\rho_3}\right)\right]^{p_k}$$
$$\leq \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho_1}\right)\right]^{p_k}+\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}v_k),v_0)}{\rho_2}\right)\right]^{p_k},$$

uniformly in j. Therefore, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}(\alpha_1u_k+\alpha_2v_k)),\alpha_1u_0+\alpha_2v_0)}{\rho_3}\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq P\cup Q\in I.$$

uniformly in j. This yields $(\alpha_1u+\alpha_2v)\in[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$. This completes the proof. □

Theorem 2.2. *Let M_1 and M_2 be two Orlicz functions. Then*

(i) $[Z(M_2,\theta,B^n_{(m)},p)]\subseteq[Z(M_1M_2,\theta,B^n_{(m)},p)]$.

(ii) $[Z(M_1,\theta,B^n_{(m)},p)]\cap Z(M_2,\theta,B^n_{(m)},p)]\subseteq[Z(M_1+M_2,\theta,B^n_{(m)},p)]$,

where $Z=\widehat{w}_0^{IF},\widehat{w}^{IF},\widehat{w}_\infty^{IF},\widehat{w}_\infty^{F}$.

Proof. (i) Let $u=(u_k)\in[\widehat{w}^{IF}(M_2,\theta,B^n_{(m)},p)]$ and let $\varepsilon>0$ be given. For some $\rho>0$, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\left[M_2\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in I, \tag{2.1}$$

uniformly in $j\in\mathbb{N}$. Choose λ with $0<\lambda<1$ such that $M_1(t)<\varepsilon$ for $0\leq t\leq\lambda$. We define

$$v_k=\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho}$$

and consider

$$\lim_{k\in\mathbb{N};0\leq v_k\leq\lambda}[M_1(v_k)]^{p_k}=\lim_{k\in\mathbb{N};v_k\leq\lambda}[M_1(v_k)]^{p_k}+\lim_{k\in\mathbb{N};v_k>\lambda}[M_1(v_k)]^{p_k}.$$

Therefore, one obtains

$$\lim_{k\in\mathbb{N};v_k\leq\lambda}[M_1(v_k)]^{p_k}\leq[M_1(2)]^H\lim_{k\in\mathbb{N};v_k\leq\lambda}[v_k]^{p_k},\qquad (H=\sup_k p_k). \tag{2.2}$$

For the second summation (i.e. $v_k>\lambda$), we go through the following procedure. We have

$$v_k<\frac{v_k}{\lambda}<1+\frac{v_k}{\lambda}.$$

It follows from the fact that M_1 is convex and non-decreasing,

$$M_1(v_k)<M_1\left(1+\frac{v_k}{\lambda}\right)\leq\frac{1}{2}M_1(2)+\frac{1}{2}M_1\left(\frac{2v_k}{\lambda}\right).$$

Since M_1 satisfies Δ_2-condition, we can write

$$M_1(v_k)<\frac{1}{2}K\frac{v_k}{\lambda}M_1(2)+\frac{1}{2}K\frac{v_k}{\lambda}M_1(2)=K\frac{v_k}{\lambda}M_1(2).$$

This yields the following estimates:

$$\lim_{k\in\mathbb{N};v_k>\lambda}[M_1(v_k)]^{p_k}\leq\max\left\{1,(K\lambda^{-1}M_1(2))^H\right\}\lim_{k\in\mathbb{N};v_k>\lambda}[v_k]^{p_k}. \tag{2.3}$$

It follows from (2.1), (2.2) and (2.3) that

$$(u_k) \in [\widehat{w}^{IF}(M_1.M_2, \theta, B^n_{(m)}, p)].$$

Hence, $[\widehat{w}^{IF}(M_2, \theta, B^n_{(m)}, p)] \subseteq [\widehat{w}^{IF}(M_1.M_2, \theta, B^n_{(m)}, p)]$.

(ii) Let $(u_k) \in [\widehat{w}^{IF}(M_1, \theta, B^n_{(m)}, p)] \cap [\widehat{w}^{IF}(M_2, \theta, B^n_{(m)}, p)]$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_1 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N})$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_2 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N}).$$

The rest of the proof follows from the following relation:

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[(M_1 + M_2) \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \\ &\qquad \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_1 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \\ &\qquad\quad \bigcup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_2 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}. \end{aligned}$$

□

Note that if we take $M_1(x) = M(x)$ and $M_2(x) = x$ for all $x \in [0, \infty)$ in the above theorem, then we obtain the following corollary:

Corollary 2.3. *One has* $[Z(\theta, B^n_{(m)}, p)] \subseteq [Z(M, \theta, B^n_{(m)}, p)]$, *where* $Z = \widehat{w}_0^{IF}$, $\widehat{w}^{IF}$, $\widehat{w}_\infty^{IF}$, $\widehat{w}_\infty^{F}$.

As in classical theory, the following is easy to prove.

Theorem 2.4. *(a) If* $M_1(x) \leq M_2(x)$ *for all* $x \in [0, \infty)$, *then* $[Z(M_1, \theta, B^n_{(m)}, p)] \subseteq [Z(M_2, \theta, B^n_{(m)}, p)]$ *for* $Z = \widehat{w}_0^{IF}, \widehat{w}^{IF}$ *and* $\widehat{w}_\infty^{F}$.

(b) If $n_1 < n_2$ *then* $[Z(\theta, B^{n_1}_{(m)}, p)] \subseteq [Z(\theta, B^{n_2}_{(m)}, p)]$ *for* $Z = \widehat{w}_0^{IF}, \widehat{w}^{IF}$ *and* $\widehat{w}_\infty^{F}$.

Theorem 2.5. *Let* M *be an Orlicz function. Then*

$$[\widehat{w}_0^{IF}(M, \theta, B^n_{(m)}, p)] \subset [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)] \subset [\widehat{w}_\infty^{F}(M, \theta, B^n_{(m)}, p)]$$

and the inclusions are proper.

Proof. Suppose that $(u_k) \in [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

Clearly,

$$M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right) \le \frac{1}{2}M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho}\right)+\frac{1}{2}M\left(\frac{\overline{d}(u_0,\overline{0})}{\rho}\right).$$

Taking supremum over k on both sides of above inequalities implies that $(u_k) \in [\widehat{w}^F_\infty(M,\theta,B^n_{(m)},p)]$. Thus, we have $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] \subset [\widehat{w}^F_\infty(M,\theta,B^n_{(m)},p)]$.

The inclusion $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] \subset [\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ is obvious.

We now show that the inclusion is strict in the above theorem by constructing the following illustrative example.

Example 2.1. Suppose that $\theta=(2^r)$ and $M(x)=x$ for all $x\in[0,\infty)$. Suppose also that $r=1$, $s=-1$, $n=1$, $m=2$. Let us define the sequence (u_k) of fuzzy numbers by

$$u_k(t)=\begin{cases}\frac{6}{k}t+1 & \text{if} \quad -\frac{k}{6}\le t\le 0;\\ -\frac{6}{k}t+1 & \text{if} \quad 0<t\le\frac{k}{6};\\ 0 & , \quad \text{otherwise,}\end{cases}$$

where $k=2^i$ $(i=1,2,3,...)$, otherwise $u_k(t)=\overline{0}$. For $\alpha\in(0,1]$, the α-level sets of u_k and $B^1_{(m)}u_k$ are

$$[u_k]^\alpha=\begin{cases}[\frac{k}{6}(\alpha-1),\frac{k}{6}(1-\alpha)] & \text{if} \quad k=2^i, i=1,2,3,...\\ [0,0] & , \quad \text{otherwise}\end{cases}$$

and

$$[B^1_{(2)}u_k]^\alpha=\begin{cases}[\frac{1}{3}(\alpha-1),\frac{1}{3}(1-\alpha)] & \text{for} \quad k=2^i\\ [0,0] & , \quad \text{otherwise .}\end{cases}$$

It is easy to prove that $-\frac{1}{3}<[T_j]^\alpha<\frac{1}{3}$ for $\alpha\in(0,1]$, where $[T_j]^\alpha=[t_{j,k}(B^1_{(2)}u_k)]^\alpha=[\frac{1}{j+1}\sum_{i=1}^{j}B^1_{(2)}u_k]^\alpha$. Because

$$[t_{j,k}(B^1_{(2)}u_k)]^\alpha=\begin{cases}\frac{1}{1+\frac{1}{j}}[\frac{1}{3}(\alpha-1),\frac{1}{3}(1-\alpha)] & \text{for} \quad k=2^i;\ j\ge1\\ [0,0] & , \quad \text{otherwise}\end{cases}$$

and

$$[t_{j,k}(B^1_{(2)}u_k)]^\alpha=\begin{cases}[\frac{1}{3}(\alpha-1),\frac{1}{3}(1-\alpha)] & \text{if} \quad j=0\\ [0,0] & , \quad \text{otherwise .}\end{cases}$$

Thus (T_j) is I-bounded but not I-convergent. □

Theorem 2.6. *The inclusions* $[Z(M,\theta,B^{n-1}_{(m)},p)]\subseteq[Z(M,\theta,B^n_{(m)},p)]$ *are strict for* $n\ge1$. *In general* $[Z(M,\theta,B^i_{(m)},p)]\subseteq[Z(M,\theta,B^n_{(m)},p)]$ $(i=1,2,...,n-1)$ *and the inclusion is strict, where* $Z=\widehat{w}^{IF}_0,\widehat{w}^{IF},\widehat{w}^{IF}_\infty,\widehat{w}^F_\infty$.

Proof. Suppose that $u=(u_k)\in[\widehat{w}^{IF}_0(M,\theta,B^{n-1}_{(m)},p)]$. Let $\varepsilon>0$ be given. Then there exists $\rho>0$ such that

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k}\ge\varepsilon\right\}\in I.$$

Since M is non-decreasing and convex it follows that

$$\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{2\rho}\right)\right]^{p_k}$$

$$\leq \left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{2\rho}\right)\right]^{p_k}$$
$$\leq D\left[\frac{1}{2}M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho}\right)\right]^{p_k} + D\left[\frac{1}{2}M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho}\right)\right]^{p_k}$$
$$\leq DK\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho}\right)\right]^{p_k} + DK\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho}\right)\right]^{p_k},$$

where $K = \max\{1, \left(\frac{1}{2}\right)^H\}$. Therefore we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n}_{(m)}u_k), \overline{0})}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : DK\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$$
$$\bigcup\left\{r \in \mathbb{N} : DK\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\},$$

i.e.,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n}_{(m)}u_k), \overline{0})}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Hence, $(u_k) \in [\widehat{w}_0^{IF}(M, \theta, B^n_{(m)}, p)]$.

We now show that the inclusion is strict in the above theorem (Theorem 2.6) by constructing the following illustrative example.

Example 2.2. Let $\theta = (2^r)$ and $M(x) = x$ for all $x \in [0, \infty)$ Suppose also that $r = 1$, $s = -1$, $n = 2$, $m = 2$ and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} -\frac{t}{k^2-1} + 1 & , \text{ if } k^2 - 1 \leq t \leq 0; \\ -\frac{t}{k^2+1} + 1 & , \text{ if } 0 < t \leq k^2 + 1; \\ 0 & , \text{ otherwise.} \end{cases}$$

For $\alpha \in (0, 1]$, the α-level sets of u_k, $B^1_{(2)}u_k$ and $B^2_{(2)}u_k$ are as follow:

$$[u_k]^\alpha = [(1-\alpha)(k^2-1), (1-\alpha)(k^2+1)],$$

and

$$[B^1_{(2)}u_k]^\alpha = [(1-\alpha)(4k-6), (1-\alpha)(4k-2)],$$
$$[B^2_{(2)}u_k]^\alpha = [4(1-\alpha), 12(1-\alpha)].$$

It is easy to verified that the sequence $[B^1_{(2)}u_k]^\alpha$ is not I-convergent but $[B^2_{(2)}u_k]^\alpha$ is I-convergent. $\square$

Theorem 2.7. *Let $0 < p_k \leq q_k < \infty$ for each k. Then $[Z(M, \theta, B^n_{(m)}, p)] \subseteq [Z(M, \theta, B^n_{(m)}, q)]$ for $Z = \widehat{w}_0^{IF}$ and $\widehat{w}^{IF}$.*

Proof. Let $(u_k) \in [\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},p)$. Then there exists a number $\rho > 0$ such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N}).$$

For sufficiently large k, since $p_k \leq q_k$ for each k, therefore we obtain

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{q_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I,$$

uniformly in $j \in \mathbb{N}$, i.e. $(u_k) \in [\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},q)]$.

Similarly, we can show that $]\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] \subseteq [\widehat{w}^{IF}(M,\theta,B^n_{(m)},q)]$. □

Corollary 2.8. *(a) Let $0 < \inf_k p_k \leq p_k \leq 1$. Then $[Z(M,\theta,B^n_{(m)},p)] \subseteq [Z(M,\theta,B^n_{(m)})]$ for $Z = \widehat{w}_0^{IF}$ and $\widehat{w}^{IF}$.*
(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then $[Z(M,\theta,B^n_{(m)})] \subseteq [Z(M,\theta,B^n_{(m)},p)]$ for $Z = \widehat{w}_0^{IF}$ and $\widehat{w}^{IF}$.

Theorem 2.9. *If I is an admissible ideal and $I \neq I_f$, then the sequence spaces $[\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},p)]$ and $\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ are neither normal nor monotone, where I_f denotes the class of all finite subsets of $\mathbb{N}$.*

Proof. To prove our result, we construct the following example.

Example 2.3. Suppose that $M(x) = x$ for all $x \in [0,\infty)$ and $r = 1$, $s = -1$, $n = 1$, $m = 1$. Consider that $I = I_\delta$, where $I_\delta = \{A \subset \mathbb{N} : \text{asymptotic density of } A \text{ (in symbol, } \delta(A)) = 0\}$ and note that I_δ is an ideal of $\mathbb{N}$, and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} 1+t-k & , \text{ if } t \in [k-1,k]; \\ 1-t+k & , \text{ if } t \in [k,k+1]; \\ 0 & , \text{ otherwise.} \end{cases}$$

Let us define

$$\alpha_k = \begin{cases} 1 & , \text{ if } k \text{ is odd;} \\ 0 & , \text{ if } k \text{ is even.} \end{cases}$$

Thus $(\alpha_k u_k) \notin [\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},p)]$ and $\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$. Therefore, we conclude that the spaces $[\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},p)]$ and $\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ are not normal and hence these spaces are not monotone. □

Theorem 2.10. *If I is an admissible ideal and $I \neq I_f$, then the sequence space $[Z(M,\theta,B^n_{(m)},p)]$ is not symmetric, where $Z = \widehat{w}_0^{IF}, \widehat{w}^{IF}$.*

Proof. We shall prove the result only for the space $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ with the help of the following example. For other space, the proof is similar so we omitted.

Example 2.4. Suppose that $M(x) = x$ for all $x \in [0, \infty)$ and $r = 1$, $s = -1$, $n = 1$, $m = 1$. Let $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} t - 4k + 1 & , \quad \text{if } t \in [4k-1, 4k]; \\ -t + 4k + 1 & , \quad \text{if } t \in [4k, 4k+1]; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Thus, we have $(u_k) \in [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$. But the rearrangement (v_k) of (u_k) defined as

$$v_k = \{u_1, u_4, u_2, u_9, u_3, u_{16}, u_5, u_{25}, u_6, ...\}.$$

This implies that $(v_k) \notin [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$. Hence $[\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$ is not symmetric. $\square$

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The Catalan Numbers: a Generalization, an Exponential Representation, and some Properties

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Abstract

In the paper, the authors establish an exponential representation for a function involving the gamma function and originating from investigation of the Catalan numbers in combinatorics, find necessary and sufficient conditions for the function to be logarithmically completely monotonic, introduce a generalization of the Catalan numbers, derive an exponential representation for the generalization, and present some properties of the generalization.

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1 Introduction

It is known [4, 21, 22] that, in combinatorics, the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. Explicit formulas of C_n for $n \ge 0$ include

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2^n(2n-1)!!}{(n+1)!} = \frac{1}{n}\binom{2n}{n-1} = {}_2F_1(1-n,-n;2;1) = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad (1)$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t$ for $\Re(z) > 0$ is the classical Euler gamma function and

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} \quad (2)$$

is the generalized hypergeometric series defined for $a_i \in \mathbb{C}$ and $b_i \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, for positive integers $p, q \in \mathbb{N}$, and in terms of the rising factorials $(x)_n = \prod_{k=0}^{n-1}(x+k)$. The asymptotic form for the Catalan function C_x is

$$C_x \sim \frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}} - \frac{9}{8}\frac{1}{x^{5/2}} + \frac{145}{128}\frac{1}{x^{7/2}} + \cdots\right),$$

see [3, 4, 21, 22, 24]. Recently, among other things, the formula

$$C_n = (-1)^n \frac{2^n}{n!} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{m=0}^{n-1} (\ell - 2m) = \frac{2^n}{n!} \sum_{k=0}^{n} \frac{k!}{2^k} \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!!$$

was found in [18, Theorem 3]. For more information on the Catalan numbers C_n, please refer to two monographs [2, 3] and references cited therein.

In the paper [20], motivated by the explicit expression (1), the authors established an integral representation of the Catalan function C_x for $x \ge 0$.

Theorem 1.1 ([20, Theorem 1])**.** *For $x \ge 0$, we have*

$$C_x = \frac{e^{3/2} 4^x (x+1/2)^x}{\sqrt{\pi}\,(x+2)^{x+3/2}} \exp\left[\int_0^\infty \frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\left(e^{-t/2} - e^{-2t}\right) e^{-xt} \,\mathrm{d}\, t\right]. \tag{3}$$

Recall from [8, Chapter XIII], [19, Chapter 1], and [25, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0 \le (-1)^k f^{(k)}(x) < \infty$ on I for all $k \ge 0$. Recall from [11] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if $0 \le (-1)^k [\ln f(x)]^{(k)} < \infty$ hold on I for all $k \in \mathbb{N}$. For more information on logarithmically completely monotonic functions, please refer to [14, 19].

The formula (3) can be rearranged as

$$\ln\left[\frac{\sqrt{\pi}\,(x+2)^{x+3/2}}{e^{3/2} 4^x (x+1/2)^x} C_x\right] = \int_0^\infty \frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\left(e^{-t/2} - e^{-2t}\right) e^{-xt} \,\mathrm{d}\, t. \tag{4}$$

Since the function $\frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)$ is positive on $(0, \infty)$, the right-hand side of (4) is a completely monotonic function on $(0, \infty)$. This means that the function

$$\frac{(x+2)^{x+3/2}}{4^x (x+1/2)^x} C_x \tag{5}$$

is logarithmically completely monotonic on $(0, \infty)$. Because any logarithmically completely monotonic function must be completely monotonic, see [14, Eq. (1.4)] and references therein, the function (5) is also completely monotonic on $(0, \infty)$.

By virtue of (1), the function (5) can be rewritten as

$$\frac{(x+2)^{x+3/2}\Gamma(x+1/2)}{(x+1/2)^x \Gamma(x+2)}, \quad x > 0. \tag{6}$$

Hence, the logarithmically complete monotonicity of (5) implies the logarithmically complete monotonicity of (6). The function (6) is the special case $F_{1/2,2}(x)$ of the general function

$$F_{a,b}(x) = \frac{\Gamma(x+a)}{(x+a)^x} \frac{(x+b)^{x+b-a}}{\Gamma(x+b)}, \quad a, b \in \mathbb{R}, \quad a \ne b \quad x > -\min\{a, b\}. \tag{7}$$

We notice that the function $F_{a,b}(x)$ does not appear in the expository and survey articles [9, 14] and plenty of references therein. Therefore, it is significant to naturally pose an open problem below.

Open Problem 1.1 ([20, Open Problem 1]). *What are the necessary and sufficient conditions on $a, b \in \mathbb{R}$ such that the function $F_{a,b}(x)$ defined by* (7) *is* (*logarithmically*) *completely monotonic in $x \in (-\min\{a, b\}, \infty)$?*

This problem was answered in [6, Theorem 2] as follows.

Theorem 1.2 ([6, Theorem 2]). *The sufficient conditions on a, b such that the function $[F_{a,b}(x)]^{\pm 1}$ defined by* (7) *is logarithmically completely monotonic in $x \in (-\min\{a, b\}, \infty)$ are $(a, b) \in D_\pm(a, b)$, where*

$$D_\pm(a,b) = \{(a,b) : a \gtrless b, a \ge 1\} \cup \left\{(a,b) : a \lessgtr b, a \le \frac{1}{2}\right\}.$$

The necessary conditions on a, b for the function $[F_{a,b}(x)]^{\pm 1}$ to be logarithmically completely monotonic in $x \in (-\min\{a, b\}, \infty)$ are $a(a-b) \gtrless \frac{a-b}{2}$.

The aims of this paper are to establish an exponential representation for the function $F_{a,b}(x)$, to find necessary and sufficient conditions on a, b for $[F_{a,b}(x)]^{\pm 1}$ to be logarithmically completely monotonic on $[0, \infty)$, to introduce a generalization of the Catalan numbers C_n, and to derive an exponential representation for the generalization of C_n.

The first main result in this paper can be stated as the following theorem.

Theorem 1.3. *For $a, b > 0$, the function $F_{a,b}(x)$ defined by* (7) *has the exponential representation*

$$F_{a,b}(x) = \exp\left[b - a + \int_0^\infty \frac{1}{t}\left(a + \frac{1}{t} - \frac{1}{1 - e^{-t}}\right)\left(e^{-bt} - e^{-at}\right)e^{-xt} \, \mathrm{d}\, t\right] \tag{8}$$

on $[0, \infty)$ and the function $[F_{a,b}(x)]^{\pm 1}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if $(a, b) \in D_\pm(a, b)$.

Comparing (3) with (8) hints and stimulates us to consider the three-variable function

$$C(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z \frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \ge 0. \tag{9}$$

Since $C\left(\frac{1}{2}, 2; n\right) = C_n$ for $n \ge 0$ is of the form (1), we can regard $C(a, b; x)$ as an analytical generalization of the Catalan numbers C_n. For uniqueness and convenience of referring to the quantity $C(a, b; x)$, we call $C(a, b; x)$ the Catalan–Qi function and, when taking $x = n \in \{0\} \cup \mathbb{N}$, call $C(a, b; n)$ the Catalan–Qi numbers.

By virtue of the integral representation (8) in Theorem 1.3, we immediately derive an integral representation for the Catalan–Qi function $C(a, b; x)$.

Theorem 1.4. *For $a, b > 0$ and $x \ge 0$, we have*

$$\begin{aligned} C(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x \frac{(x+a)^x}{(x+b)^{x+b-a}} \\ \times \exp\left[b - a + \int_0^\infty \frac{1}{t}\left(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\right)\left(e^{-at} - e^{-bt}\right)e^{-xt} \, \mathrm{d}\, t\right]. \end{aligned} \tag{10}$$

3

Remark 1.1. Can one give a combinatorial interpretation of the Catalan–Qi function $C(a,b;x)$ defined by (9) and its integral representation (10)?

In [22] and related references therein, the following simple properties of the Catalan numbers C_n are listed:

$$C_{n+1}=\frac{2(2n+1)}{n+2}C_n,\quad C_n=\frac{1}{(n+1)!}\prod_{k=1}^{n}(4k-2),\quad \sum_{n=1}^{\infty}\frac{C_n}{4^n}=1, \tag{11}$$

$$\sum_{n=0}^{\infty}C_n\frac{x^{2n}}{(2n)!}=\frac{I_1(2x)}{x},\quad e^{2x}\big[I_0(2x)-I_1(2x)\big]=\sum_{n=0}^{\infty}C_n\frac{x^n}{n!}, \tag{12}$$

where

$$I_\nu(z)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu}$$

for $\nu\in\mathbb{R}$ and $z\in\mathbb{C}$, see [1, p. 375, 9.6.10], is the modified Bessel function of the first kind. Corresponding to these properties, the following properties of the Catalan–Qi function $C(a,b;z)$ can be obtained.

Theorem 1.5. *For $n\ge 0$ and $\Re(z)\ge 0$, we have*

$$C(a,b;z+1)=\frac{b}{a}\frac{z+a}{z+b}C(a,b;z);\quad C(a,b;n)=\left(\frac{b}{a}\right)^n\prod_{k=0}^{n-1}\frac{a+k}{b+k};$$

$$\sum_{n=1}^{\infty}\left(\frac{a}{b}\right)^nC(a,b;n)=\frac{a}{b-a-1},\quad b>a+1>1;$$

$$\sum_{n=0}^{\infty}C(a,b;n)\frac{x^{2n}}{(2n)!}={}_1F_2\left(a;\frac{1}{2},b;\frac{b}{4a}x^2\right);\quad \sum_{n=0}^{\infty}C(a,b;n)\frac{x^n}{n!}={}_1F_1\left(a;b;\frac{b}{a}x\right).$$

Remark 1.2. When $a=\frac{1}{2}$ and $b=2$, the formulas in Theorem 1.5 become those listed in (11) and (12).

Remark 1.3. The last two formulas in Theorem 1.5 show that the functions ${}_1F_2\big(a;\frac{1}{2},b;\frac{b}{4a}x^2\big)$ and ${}_1F_1\big(a;b;\frac{b}{a}x\big)$ can be regarded as the generating functions of the Catalan–Qi numbers $C(a,b;n)$.

2 Proofs of Theorems 1.3 to 1.5

We are now start out to prove Theorem 1.3 by two approaches and to prove Theorems 1.4 and 1.5.

First proof of Theorem 1.3. Taking the logarithm of $F_{a,b}(x)$ gives

$$\ln F_{a,b}(x)=\ln\Gamma(x+a)-x\ln(x+a)-\ln\Gamma(x+b)+(x+b-a)\ln(x+b)\triangleq f_a(x)-f_a(x+b-a).$$

Differentiating twice with respect to the variable x of $f_a(x)$ yields

$$f_a'(x)=\psi(x+a)-\ln(x+a)+\frac{a}{x+a}-1\quad\text{and}\quad f_a''(x)=\psi'(x+a)-\frac{1}{x+a}-\frac{a}{(x+a)^2}.$$

By virtue of the formulas

$$\psi^{(n)}(z)=(-1)^{n+1}\int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}}\,\mathrm{d}\,t \quad\text{and}\quad \Gamma(z)=k^z\int_0^\infty t^{z-1}e^{-kt}\,\mathrm{d}\,t$$

for $\Re(z)>0$, $\Re(k)>0$, and $n\in\mathbb{N}$ in [1, p. 260, 6.4.1] and [1, p. 255, 6.1.1], we obtain

$$f_a''(x-a)=\psi'(x)-\frac{1}{x}-\frac{a}{x^2}=\int_0^\infty\biggl(\frac{1}{1-e^{-t}}-\frac{1}{t}-a\biggr)te^{-xt}\,\mathrm{d}\,t.$$

Accordingly, we have

$$\begin{aligned}[\ln F_{a,b}(x)]''=f_a''(x)-f_a''(x+b-a)&=\int_0^\infty\biggl(\frac{1}{1-e^{-t}}-\frac{1}{t}-a\biggr)t\bigl[e^{-(x+a)t}-e^{-(x+b)t}\bigr]\,\mathrm{d}\,t\\&=\int_0^\infty\biggl(\frac{1}{1-e^{-t}}-\frac{1}{t}-a\biggr)t\bigl(e^{-at}-e^{-bt}\bigr)e^{-xt}\,\mathrm{d}\,t.\end{aligned}\tag{13}$$

The famous Bernstein-Widder theorem, [25, p. 161, Theorem 12b], states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0,\infty)$ is that $f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\mu(t)$, where μ is a positive measure on $[0,\infty)$ such that the above integral converges on $(0,\infty)$. Hence, in order to find necessary and sufficient conditions on a,b such that the function $[\ln F_{a,b}(x)]''$ is completely monotonic on $(0,\infty)$, it is necessary and sufficient to discuss the positivity or negativity of the function

$$\biggl(\frac{1}{1-e^{-t}}-\frac{1}{t}-a\biggr)t\bigl(e^{-at}-e^{-bt}\bigr)\tag{14}$$

on $(0,\infty)$.

It is clear that the factor $e^{-at}-e^{-bt}$ is positive (or negative, respectively) if and only if $b>a$ (or $b<a$, respectively). Since the function $\frac{1}{1-e^{-t}}-\frac{1}{t}=\frac{1}{e^t-1}-\frac{1}{t}+1$ is strictly increasing on $(0,\infty)$ and has the limits $\lim_{t\to0^+}\bigl(\frac{1}{1-e^{-t}}-\frac{1}{t}\bigr)=\frac{1}{2}$ and $\lim_{t\to\infty}\bigl(\frac{1}{1-e^{-t}}-\frac{1}{t}\bigr)=1$, see [5, 15] and references therein, the factor $\frac{1}{1-e^{-t}}-\frac{1}{t}-a$ is positive (or negative, respectively) on $(0,\infty)$ if and only if $a\le\frac{1}{2}$ (or $a\ge1$, respectively). Consequently, the function (14) is

1. positive if and only if either $b>a$ and $a\le\frac{1}{2}$ or $b<a$ and $a\ge1$,
2. negative if and only if either $b<a$ and $a\le\frac{1}{2}$ or $b>a$ and $a\ge1$.

As a result, the function $\pm[\ln F_{a,b}(x)]''$ is completely monotonic on $(0,\infty)$ if and only if $(a,b)\in D_\pm(a,b)$.

By a straightforward computation, we see that

$$\lim_{x\to\infty}[\ln F_{a,b}(x)]'=\lim_{x\to\infty}\biggl[\psi(x+a)-\psi(x+b)+\ln\frac{x+b}{x+a}+\frac{a(b-a)}{(x+a)(x+b)}\biggr]=0\tag{15}$$

for all $a,b\in\mathbb{R}$. This implies that, if and only if $(a,b)\in D_\pm(a,b)$, the first logarithmic derivative satisfies $[\ln F_{a,b}(x)]'\lesseqgtr0$. By the definition of logarithmically completely monotonic functions, we conclude that, if and only if $(a,b)\in D_\pm(a,b)$, the function $[F_{a,b}(x)]^{\pm1}$ is logarithmically completely monotonic on $(0,\infty)$.

Integrating from u to ∞ with respect to x on the very ends of (13) and considering the limit (15) give

$$-[\ln F_{a,b}(u)]' = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\right)\bigl(e^{-at} - e^{-bt}\bigr)e^{-ut}\,\mathrm{d}\,t.$$

Further integrating with respect to u from x to ∞ on both sides of the above equality and employing the limit $\lim_{x\to\infty} F_{a,b}(x) = e^{b-a}$ reveal that

$$\ln F_{a,b}(x) = b - a + \int_0^\infty \frac{1}{t}\left(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\right)\bigl(e^{-at} - e^{-bt}\bigr)e^{-xt}\,\mathrm{d}\,t.$$

The first proof of Theorem 1.3 is thus complete. □

Second proof of Theorem 1.3. As did in the proof of [20, Theorem 1], employing the formula

$$\ln\Gamma(z) = \ln\bigl(\sqrt{2\pi}\, z^{z-1/2}e^{-z}\bigr) + \int_0^\infty \frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)e^{-zt}\,\mathrm{d}\,t$$

in [23, (3.22)] and utilizing $\ln\frac{b}{a} = \int_0^\infty \frac{e^{-au}-e^{-bu}}{u}\,\mathrm{d}\,u$ in [1, p. 230, 5.1.32] yield

$$\begin{aligned}
\ln F_{a,b}(x) &= b - a + \left(a - \frac{1}{2}\right)\ln\frac{x+a}{x+b} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-xt}}{t}\bigl(e^{-at} - e^{-bt}\bigr)\,\mathrm{d}\,t \\
&= b - a + \left(a - \frac{1}{2}\right)\int_0^\infty \frac{e^{-xt}}{t}\bigl(e^{-bt} - e^{-at}\bigr)\,\mathrm{d}\,t + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-xt}}{t}\bigl(e^{-at} - e^{-bt}\bigr)\,\mathrm{d}\,t \\
&= b - a + \int_0^\infty \frac{1}{t}\left(a - \frac{1}{2} - \frac{1}{2} + \frac{1}{t} - \frac{1}{e^t-1}\right)\bigl(e^{-bt} - e^{-at}\bigr)e^{-xt}\,\mathrm{d}\,t \\
&= b - a + \int_0^\infty \frac{1}{t}\left(a + \frac{1}{t} - \frac{1}{1-e^{-t}}\right)\bigl(e^{-bt} - e^{-at}\bigr)e^{-xt}\,\mathrm{d}\,t.
\end{aligned}$$

The rest of the second proof is the same as in the first proof after the equation (13). The second proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. This follows from straightforwardly combining (7) and (8) with (9). □

Proof of Theorem 1.5. It is easy to see that

$$C(a,b;z+1) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{z+1}\frac{\Gamma(z+a+1)}{\Gamma(z+b+1)} = \frac{b}{a}\frac{z+a}{z+b}\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{z}\frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{b}{a}\frac{z+a}{z+b}C(a,b;z).$$

Consequently, when taking $z = n-1$,

$$\begin{aligned}
C(a,b;n) &= \frac{b}{a}\frac{n+a-1}{n+b-1}C(a,b;n-1) = \left(\frac{b}{a}\right)^2\frac{n+a-1}{n+b-1}\frac{n+a-2}{n+b-2}C(a,b;n-2) \\
&= \cdots = \left(\frac{b}{a}\right)^n\frac{n+a-1}{n+b-1}\frac{n+a-2}{n+b-2}\cdots\frac{a+1}{b+1}\frac{a}{b}C(a,b;0) = \left(\frac{b}{a}\right)^n\prod_{k=0}^{n-1}\frac{a+k}{b+k}.
\end{aligned}$$

By (9), it follows that

$$\sum_{n=1}^{\infty}\left(\frac{a}{b}\right)^n C(a,b;n)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{n=1}^{\infty}\frac{\Gamma(n+a)}{\Gamma(n+b)}=\frac{\Gamma(b)}{\Gamma(a)}\frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)\Gamma(b-a)}=\frac{a}{b-a-1}.$$

The last two formulas in Theorem 1.5 can be straightforwardly derived from the definition (2) of the generalized hypergeometric series. The proof of Theorem 1.5 is complete. □

Remark 2.1. This paper is a companion of the articles [6, 7, 12, 13, 16, 18, 20] and the preprints [10, 18] and is a revised version of the preprint [17].

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Semiring structures based on meet and plus ideals in lower BCK-semilattices

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Abstract. The notion of the meet set based on two subsets of a lower BCK-semilattice X is introduced, and related properties are investigated. Conditions for the meet set to be a (positive implicative, commutative, implicative) ideal are discussed. The meet ideal based on subsets, and the plus ideal of two subsets in a lower BCK-semilattice X are also introduced, and related properties are investigated. Using meet operation and addition, the semiring structure is induced.

1. Introduction

Ideal theory has an important role in the development BCK/BCI-algebras (see [1, 3, 4]). It was shown in [5] that if X is a BCK-algebra then $(X, \leq)$ is a poset, and moreover if X is a commutative BCK-algebra, i.e., $x * (x * y) = y * (y * x)$ holds in X, then $(X, \leq)$ is a lower semilattice. Pałasiński [7] discussed properties of certain ideals in BCK-algebras which are lower semilattices.

In this paper, we introduce the notion of the meet set based on two subsets of a lower BCK-semilattice X and we discuss conditions for the meet set to be a (positive implicative, commutative, implicative) ideal. We also introduced the meet ideal based on subsets, and the plus ideal of two subsets in a lower BCK-semilattice X. We investigate several related properties, and we induce the semiring structure by using meet operation and addition.

2. Prliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-*algebra* if it satisfies the following conditions

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(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is called a *lower BCK-semilattice* (see [6]) if X is a lower semilattice with respect to the BCK-order.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [6]) if it satisfies

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X)\,(\forall y \in A)\,(x * y \in A \ \Rightarrow\ x \in A)\,. \tag{2.2}$$

Note that every ideal A of a BCK/BCI-algebra X satisfies the following implication (see [6]).

$$(\forall x, y \in X)\,(x \leq y,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.3}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A\rangle$. If A is finite, then we say that $\langle A\rangle$ is *finitely generated ideal* of X (see [6]).

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\,((x * y) * z \in A \ \Rightarrow\ x * (y * (y * x)) \in A)\,. \tag{2.4}$$

A subset A of a BCK-algebra X is called a *positive implicative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y, z \in X)\,((x * y) * z \in A,\, y * z \in A \ \Rightarrow\ x * z \in A)\,. \tag{2.5}$$

A subset A of a BCK-algebra X is called an *implicative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\,((x * (y * y)) * z \in A \ \Rightarrow\ x \in A)\,. \tag{2.6}$$

A proper ideal P of a lower BCK-semilattice X is said to be *prime* if it satisfies

$$(\forall a, b \in X)\,(a \wedge b \in P \ \Rightarrow\ a \in P \text{ or } b \in P)\,. \tag{2.7}$$

We refer the reader to the books [2, 6] for further information regarding BCK/BCI-algebras.

3. Meet and plus ideals

In what follows, let X be a lower BCK-semilattice unless otherwise specified. For any nonempty subsets A and B of X, we consider the set

$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$

where $a \wedge b$ is the greatest lower bound of a and b. We say that K is the *meet set* based on A and B. Note that $A \cap B \subseteq K$, but the reverse inclusion is not true as seen in the following example.

Example 3.1. (1) Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have

$$K := \{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 2\} \nsubseteq A \cap B.$$

(2) Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 2, 3\}$ and $B = \{1, 3, 4\}$ of X, we have

$$K := \{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 3\} \nsubseteq \{1, 3\} = A \cap B.$$

The following example shows that the set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ may not be an ideal of X for some subsets A and B of X.

Example 3.2. Let $X = \{0, 1, 2, 3, 4\}$ be a lower BCK-semilattice in Example 3.1(1). For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have

$$\{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 2\},$$

which is not an ideal of X.

We provide conditions for the meet set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ based on A and B to be an ideal.

Theorem 3.3. *If A and B are ideals of X, then so is the meet set*

$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$

based on A and B.

Proof. Obviously, $0 \in K$. Let $x \in K$ and $y*x \in K$ for $x, y \in X$. Then $x = a \wedge b$ and $y*x = a' \wedge b'$ where $a, a' \in A$ and $b, b' \in B$. Since $a \wedge b \leq a$ and A is an ideal, we have $x = a \wedge b \in A$. Similarly, we have

$$y * x = a' \wedge b' \leq a' \in A.$$

Since A is an ideal of X, it follows that $y \in A$. By the similar way, we get $y \in B$. Therefore,

$$y = y \wedge y \in \{a \wedge b \mid a \in A,\ b \in B\} = K$$

and K is an ideal of X. □

Lemma 3.4 ([6])**.** *For an ideal A of a BCK-algebra X, the following are equivalent.*

(i) *A is positive implicative.*
(ii) $(\forall x, y \in X)\,((x * y) * y \in A \ \Rightarrow\ x * y \in A)$.

Lemma 3.5 ([6])**.** *For an ideal A of a BCK-algebra X, the following are equivalent.*

(i) *A is commutative.*
(ii) $(\forall x, y \in X)\,(x * y \in A \ \Rightarrow\ x * (y * (y * x)) \in A)$.

Lemma 3.6 ([6])**.** *Let A be an ideal of a BCK-algebra X. Then A is implicative if and only if A is both positive implicative and commutative.*

Theorem 3.7. *If A and B are positive implicative (resp., commutative, implicative) ideals of X, then so is the meet set*

$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$

based on A and B.

Proof. Assume that A and B are positive implicative ideals of X. Then A and B are ideals of X, and so the set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ is an ideal of X by Theorem 3.3. Let $(x * y) * y \in K$ for every $x, y \in X$. Then $(x*y)*y = a \wedge b$ for some $a \in A$ and $b \in B$. Since $a \wedge b \leq a$ and A is an ideal, we have $(x * y) * y \in A$. Similarly, $(x * y) * y \in B$. Since A and B are positive implicative ideals, it follows from Lemma 3.4 that $x * y \in A$ and $x * y \in B$. Therefore

$$x * y = (x * y) \wedge (x * y) \in \{a \wedge b \mid a \in A,\ b \in B\} = K,$$

and so K is a positive implicative ideal of X by Lemma 3.4.

Now suppose that A and B are commutative ideals of X. Then A and B are ideals of X, and so the set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ is an ideal of X by Theorem 3.3. Let $x * y \in K$ for every $x, y \in X$. Then $x * y = a \wedge b$ for some $a \in A$ and $b \in B$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, it follows

that $x * y \in A \cap B$. Since A and B are commutative, we have $x * (y * (y * x)) \in A \cap B$ by Lemma 3.5. Hence

$$\begin{aligned} x * (y * (y * x)) &= (x * (y * (y * x))) \wedge (x * (y * (y * x))) \\ &\in \{a \wedge b \mid a \in A,\ b \in B\} = K, \end{aligned}$$

and therefore K is a commutative ideal if X.

Now, if A and B are implicative ideals of X, then they are both positive implicative and commutative by Lemma 3.6. Thus K is both a positive implicative ideal and a commutative ideal of X, and so it is an implicative ideal of X. □

Given two nonempty subsets A and B of X, we consider the ideal of X generated by the meet set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ based on A and B.

Definition 3.8. For any nonempty subsets A and B of X, we denote

$$A \wedge B := \langle \{a \wedge b \mid a \in A, b \in B\} \rangle$$

which is called the *meet ideal* of X generated by A and B. In this case, we say that the operation "$\wedge$" is a *meet operation*. If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.

Obviously, $A \wedge B = B \wedge A$ for any nonempty subsets A and B of X. If A and B are ideals of X, then

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\}.$$

Example 3.9. For two subsets $A = \{2, 3\}$ and $B = \{1, 4\}$ of X in Example 3.1, the meet ideal of X generated by A and B is $A \wedge B = \langle \{0, 1, 2\} \rangle = \{0, 1, 2, 3\}$.

For any nonempty subsets A, B and C of X, we have

$$A \subseteq B,\ A \subseteq C\ \Rightarrow\ A \subseteq B \wedge C. \tag{3.1}$$

The following example shows that there are subsets A, B and C of X such that $A \subseteq B$ and $A \subseteq C$, but $B \wedge C \nsubseteq A$.

Example 3.10. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{0, 1\}$, $B = \{0, 1, 2, 3\}$ and $C = \{0, 1, 2, 4\}$ of X, we have

$$B \wedge C = \langle \{b \wedge c \mid b \in B,\ c \in C\} \rangle = \{0, 1, 2\} \nsubseteq \{0, 1\} = A.$$

Proposition 3.11. *If A, B and C are ideals of X, then*

$$A \wedge \{0\} = \{0\}. \tag{3.2}$$

$$A \wedge B = A \cap B. \tag{3.3}$$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C) = \{a \wedge b \wedge c \mid a \in A, b \in B, c \in C\}. \tag{3.4}$$

Proof. It is clear that $A \wedge \{0\} = \{0\}$. Using (3.1), we have $A \cap B \subseteq A \wedge B$. Let $x \in A \wedge B$. Then there exist $a \in A$ and $b \in B$ such that $x = a \wedge b$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, we have $x \in A \cap B$ by (2.3). Hence $A \wedge B = A \cap B$. The result (3.4) is straightforward. □

Corollary 3.12. *If A, B and C are ideals of X, then the condition* (3.1) *is valid.*

By Proposition 3.11, we know that for ideals A_1, A_2, $\cdots$, A_n of X

$$\begin{aligned} \bigwedge_{i=1}^{n} A_i &:= A_1 \wedge A_2 \wedge \cdots \wedge A_n \\ &= \{a_1 \wedge a_2 \wedge \cdots \wedge a_n \mid a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\} \\ &= \bigcap_{i=1}^{n} A_i. \end{aligned} \tag{3.5}$$

For any nonempty subsets A and B of X, denote by $A + B$ the ideal generated by $A \cup B$, and is called the *plus ideal* of A and B. The operation "+" is called the *addition*. Obviously, $A, B \subseteq A + B$, $A + \{0\} = A$ and $A + B = B + A$.

Example 3.13. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 3\}$ and $B = \{2\}$ of X, we have

$$A + B = \langle A \cup B \rangle = \{0, 1, 2, 3\},$$

which is a plus ideal of X.

Proposition 3.14. *For any nonempty subsets A and B of X, we have $A \wedge B \subseteq A + B$.*

Proof. If $x \in A \wedge B$, then there exists $z_1, z_2, \cdots, z_n \in \{a \wedge b \mid a \in A,\ b \in B\}$ such that

$$(\cdots((x * z_1) * z_2) * \cdots) * z_n = 0. \tag{3.6}$$

For each $i \in \{1, 2, \cdots, n\}$, we have $z_i = a_i \wedge b_i$ where $a_i \in A$ and $b_i \in B$. Thus

$$a_i \wedge b_i \leq a_i \in A \subseteq A \cup B \subseteq A + B,$$

and so $z_i \in A + B$ for all $i \in \{1, 2, \cdots, n\}$. Since $0 \in A + B$, it follows from (3.6) and (2.2) that $x \in A + B$. Hence $A \wedge B \subseteq A + B$. □

Given two nonempty subsets A and B of X, we note that every ideal I of X is represented by the meet ideal based on some A and B, and every ideal J of X is represented by the plus ideal of A and B. But we know that they are different, that is, $I \neq J$ in general as seen in the following example.

Example 3.15. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For two subsets $A = \{1\}$ and $B = \{2, 3\}$ of X, the ideal $I = \{0, 1\}$ is represented by the meet ideal based on A and B as follows

$$I = \langle A \wedge B \rangle = \langle \{0, 1\} \rangle = \{0, 1\}.$$

Also the ideal $J = \{0, 1, 2, 3\}$ is represented by the plus ideal of A and B as follows:

$$J = A + B = \langle A \cup B \rangle = \langle \{1, 2, 3\} \rangle = \{0, 1, 2, 3\}.$$

We know that $I \neq J$.

The following example shows that the reverse inclusion in Proposition 3.14 is not true in general.

Example 3.16. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.13. For subsets $A = \{1, 2\}$ and $B = \{1, 3\}$ of X, we have

$$A \wedge B = \langle \{0, 1\} \rangle = \{0, 1\}$$

and

$$A + B = \langle \{1, 2, 3\} \rangle = \{0, 1, 2, 3\}.$$

Thus $A + B \nsubseteq A \wedge B$.

For any nonempty subsets A, B and C of X, consider the following condition.

$$A \subseteq C,\ B \subseteq C \ \Rightarrow A + B \subseteq C. \tag{3.7}$$

The following example shows that the condition (3.7) is not valid in general.

Example 3.17. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 3\}$, $B = \{2, 3\}$ and $C = \{1, 2, 3\}$ of X, we have

$$A + B = \langle A \cup B\rangle = \{0, 1, 2, 3\} \nsubseteq C.$$

We provide conditions for the implication (3.7) to be hold.

Proposition 3.18. *If A and B are nonempty subsets of X and C is an ideal of X, then the implication* (3.7) *is valid.*

Proof. Let A and B be subsets of X and C be an ideal of X such that $A \subseteq C$ and $B \subseteq C$. If $x \in A + B$, then

$$(\cdots((x * z_1) * z_2) * \cdots) * z_n = 0 \tag{3.8}$$

for some $z_1, z_2, \cdots, z_n \in A \cup B$. It follows that $z_i \in C$ for all $i = 1, 2, \cdots, n$ and $0 \in C$. Since C is an ideal of X, it follows from (3.8) and (2.2) that $x \in C$. Therefore $A + B \subseteq C$. □

Let A be an ideal of a BCI-algebra X and S be a subset of X with a nilpotent element. Then

$$x \in \langle A \cup S\rangle \text{ if and only if } (\cdots((x * s_1) * s_2) * \cdots) * s_n \in A$$

for some $s_1, s_2, \cdots, s_n \in S$ (see [2]). Since every element of a BCK-algebra is nilpotent, we can apply the result above to BCK-algebras as follows.

Lemma 3.19. *Let A an ideal of a BCK-algebra X. For any subset S of X, we have*

$$x \in \langle A \cup S\rangle \text{ if and only if } (\cdots((x * s_1) * s_2) * \cdots) * s_n \in A$$

for some $s_1, s_2, \cdots, s_n \in S$.

Lemma 3.20 ([2])**.** *Let X be a commutative BCK-algebra and $x, y, z \in X$. Then*

$$(x \wedge y) * (x \wedge z) = (x \wedge y) * z.$$

Theorem 3.21. *For any ideals A, B and C of a commutative BCK-algebra X, we have*

$$A \wedge (B + C) = (A \wedge B) + (A \wedge C) \text{ and } (B + C) \wedge A = (B \wedge A) + (C \wedge A).$$

Proof. Note that $A \wedge B \subseteq A$ and $A \wedge B \subseteq B \subseteq B + C$. It follows from (3.1) that

$$A \wedge B \subseteq A \wedge (B + C).$$

Similarly $A \wedge C \subseteq A \wedge (B + C)$, and thus

$$(A \wedge B) + (A \wedge C) \subseteq A \wedge (B + C)$$

by Proposition 3.18. Now let $x \in A \wedge (B + C)$. Then $x = a \wedge z$ for some $a \in A$ and $z \in B + C = \langle B \cup C \rangle$. It follows from Lemma 3.19 that there exist $c_1, c_2, \cdots, c_n \in C$ such that

$$(\cdots((z * c_1) * c_2) * \cdots) * c_n \in B. \tag{3.9}$$

Note that $a \wedge c_1, a \wedge c_2, \cdots, a \wedge c_n \in A \wedge C$. Using Lemma 3.20 and (a3) induces

$$\begin{aligned} ((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2) &= ((a \wedge z) * c_1) * (a \wedge c_2) \\ &= ((a \wedge z) * (a \wedge c_2)) * c_1 \\ &= ((a \wedge z) * c_2) * c_1 \\ &= ((a \wedge z) * c_1) * c_2 \end{aligned}$$

which implies from Lemma 3.20 and (a3) again that

$$\begin{aligned} &(((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2)) * (a \wedge c_3) \\ &= (((a \wedge z) * c_1) * c_2) * (a \wedge c_3) \\ &= (((a \wedge z) * (a \wedge c_3)) * c_1) * c_2 \\ &= (((a \wedge z) * c_3) * c_1) * c_2 \\ &= (((a \wedge z) * c_1) * c_2) * c_3. \end{aligned}$$

By the mathematical induction, we conclude that

$$\begin{aligned} &(\cdots(((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2)) * \cdots) * (a \wedge c_n) \\ &= (\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n. \end{aligned} \tag{3.10}$$

The inequality $a \wedge z \leq z$ implies from (a2) that

$$(\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n \leq (\cdots((z * c_1) * c_2) * \cdots) * c_n. \tag{3.11}$$

Since $(\cdots((z * c_1)) * c_2) * \cdots) * c_n \in B$ and B is an ideal, it follows from (2.3) that

$$(\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n \in B. \tag{3.12}$$

Note that $(\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n \leq a \wedge z \leq a$ and $a \in A$, and so

$$(\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n \in A. \tag{3.13}$$

Combining (3.10), (3.12) and (3.13), we have

$$(\cdots(((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2)) * \cdots) * (a \wedge c_n) \in A \wedge B. \tag{3.14}$$

Since $a \wedge c_1, a \wedge c_2, \cdots, a \wedge c_n \in A \wedge C$, it follows from Lemma 3.20 that

$$x = a \wedge z \in \langle (A \wedge B) \cup (A \wedge C) \rangle = (A \wedge B) + (A \wedge C). \tag{3.15}$$

Consequently $A \wedge (B + C) = (A \wedge B) + (A \wedge C)$. Similarly we have $(B + C) \wedge A = (B \wedge A) + (C \wedge A)$. $\square$

Through our discussion above, we make a semiring as follows.

Theorem 3.22. *Let $\mathcal{I}(X)$ be the set of all ideals of a commutative BCK-algebra X. Then $(\mathcal{I}(X), +, \wedge)$ is a semiring, that is, two operations $+$ and $\wedge$ are associative on $\mathcal{I}(X)$ such that*

(i) *addition $+$ is a commutative operation,*
(ii) *there exist $\{0\} \in \mathcal{I}(X)$ such that $A + \{0\} = A$ and $A \wedge \{0\} = \{0\} \wedge A = \{0\}$ for each $A \in \mathcal{I}(X)$, and*
(iii) *the meet operation $\wedge$ distributes over addition $(+)$ both from the left and from the right.*

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The solutions of some types of q-shift difference differential equations *

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Abstract

In this paper, we investigate some properties of solutions of some types of q-shift difference differential equations. In addition, we also generalize the Rellich-Wittich-type theorem about differential equations to the case of q-shift difference differential equations. Moreover, we give some example to show the existence and growth of some q-shift difference differential equations.

Key words: q-shift; difference differential equation; zero order.
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1 Introduction and Some Results

The main purpose of this paper is to investigate some properties of solutions of some q-shift difference differential equations by using Nevanlinna theory in the fields of complex analysis. Thus, we firstly assume that readers are familiar with the basic results and the notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r,f)$, $N(r,f)$, $T(r,f)$, $\cdots$, (see Hayman [15], Yang [33] and Yi and Yang [34]). For a meromorphic function f, we use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set of finite logarithmic measure, $\mathbb{S}(f)$ denotes the family of all meromorphic function $a(z)$ such that $T(r,a)=S(r,f)=o(T(r,f))$, where $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure. Besides, we use $S_1(r,f)$ to denote any quantity satisfying $S_1(r,f)=o(T(r,f))$ for all r on a set F of logarithmic density 1, the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty}\frac{1}{\log r}\int_{[1,r]\cap F}\frac{1}{t}dt.$$

For convenience, we claim that the set F of logarithmic density can be not necessarily the same at each occurrence.

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About forty years ago, F. Rellich, H. Wittich and I. Laine investigated the existence or growth of solutions of some differential equations (see [17, 18, 20, 22]) and obtained the following results.

Theorem 1.1 *(see [17, Rellich]). Let the differential equation be the following form*

$$w'(z) = f(w), \tag{1}$$

If $f(w)$ is transcendental meromorphic function of w, then equation (1) has no non-constant entire solution.

Theorem 1.2 *(see [26, Wittich]). Let*

$$\Phi(z, w) = \sum a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n}$$

be differential polynomial, with coefficients $a_{(i)}(z)$ are polynomials of z. If the right-hand side of the differential equation

$$\Phi(z, w) = f(w), \tag{2}$$

$f(w)$ is the transcendental meromorphic function of w, then equation (2) has no non-constant entire solution.

Remark 1.1 *H. Wittich [26] studied the more general differential equation than equation (1).*

Later, Yanagihara and Shimomura extended the above type theorem to the case of difference equations (see [25, 31, 32]), and obtained the following two results

Theorem 1.3 *(see [25, Shimomura]). For any non-constant polynomial $P(w)$, the difference equation*

$$w(z + 1) = P(w(z))$$

has a non-trivial entire solution.

Theorem 1.4 *(see [31, Yanagihara]). For any non-constant rational function $R(w)$, the difference equation*

$$w(z + 1) = R(w(z))$$

has a non-trivial meromorphic solution in the complex plane.

After theirs work, by using Nevanlinna theory in complex difference equations (see [1, 3, 7, 8, 11, 12, 14]), many mathematicians have done a lot of researches in difference equations, difference product and q-difference in the complex plane $\mathbb{C}$, there were a number of articles (including [5, 13, 16, 19, 24, 36]) focused on the existence and growth of solutions of difference equations. In addition, K. Liu, H.Y. Xu and X. G. Qi investigated some properties of complex q-shift difference equations [23, 24, 28]. Inspired by these papers, the purpose of this paper is to study the above Rellich-Wittich-type theorem of q-shift difference differential equation.

Definition 1.1 *We call the equation as q-shift difference differential equation if a equation contains the q-shift term $f(z+c)$, q-difference term $f(qz)$ and differential term $f'(z)$ of one function $f(z)$ at the same time.*

We consider the q-shift difference differential equation of the form

$$\Omega(z,w) := \sum_J a_J(z) \prod_{j=1}^{n} \left(w^{(j)}(q_j z + c_j)^{i_j} \right) = P_s[f(w)], \tag{3}$$

where $a_J(z)$ are polynomials of z and $q_j, c_j \in \mathbb{C} \setminus \{0\}$, $P_m[f]$ is a polynomial of f of degree m,

$$P_m[f] = d_m(z)f^m + d_{m-1}(z)f^{m-1} + \cdots + d_0(z),$$

and $d_m(z), \ldots, d_0(z)$ are polynomials of z, and obtain the following results.

Theorem 1.5 *For equation (3), if $s \geq 1$ and f is a transcendental meromorphic function, then equation (3) has no non-constant transcendental entire solution with zero order.*

Theorem 1.6 *Under the assumptions of Theorem 1.5, the q-shift difference differential equation*

$$\sum_J a_J(z) \prod_{j=1}^{n} (w^{(j)}(q_j z + c_j))^{i_j} = \frac{P_s[f(w)]}{Q_t[f(w)]},$$

has no non-constant transcendental entire solution with zero order, where $s \geq 1$, and $P_s[f]$ and $Q_t[f]$ are irreducible polynomials in f.

In 2012, Beardon [4] studied entire solutions of the generalized functional equation

$$f(qz) = qf(z)f'(z), \qquad f(0) = 0, \tag{4}$$

where q is a non-zero complex number. Beardon [4] obtained the main theorem as follows.

Theorem 1.7 *[4]. Any transcendental solution f of equation (4) is of the form*

$$f(z) = z + z(bz^p + \cdots),$$

where p is a positive integer, $b \neq 0$ and $q \in \mathcal{K}_p$. In particular, if $q \notin \mathcal{K}$, then the only formal solutions of (4) are $\mathcal{O}$ and $\mathcal{I}$, where $\mathcal{K}, \mathcal{K}_p, \mathcal{O}$ and $\mathcal{I}$ were stated as in [4].

In 2013, Zhang [35] further the growth of solutions of equation (4) and obtained the following theorem

Theorem 1.8 *[35, Theorem 1.1]. Suppose that f is a transcendental solution of (4) for $q \in \mathcal{K}$, then we have*

$$\rho(f) \leq \frac{\log 2}{\log |q|},$$

where

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log T(r,f)}{\log r},$$

where $\mathcal{K}$ is stated as in Theorem 1.7.

Inspired by the ideas of Xu [27, 30] and Beardon [4], we investigate the growth of solutions of some q-shift difference differential equations and obtain the following results.

Theorem 1.9 *Suppose that f is a solution of*

$$f(qz+c) = \eta f(z)f'(z), \tag{5}$$

where $q, c, \eta \in \mathbb{C} \setminus \{0\}$ and $|q| > 1$. If f is a transcendental entire function, then we have

$$\rho(f) \leq \frac{\log 2}{\log |q|}.$$

Furthermore, if f is a polynomial, then f is a polynomial of degree 1, that is, $f(z) = a_1 z + a_0$, where

$$a_1 = \frac{q}{\eta}, \qquad a_0 = \frac{qc}{\eta(1+q)}.$$

The following example shows that equation (5) had a transcendental entire solution.

Example 1.1 *Let $q = 2, c = 2\pi$ and $\eta = 2$. Then $f(z) = \sin z$ satisfies equation*

$$f(2z+2\pi) = 2f(z)f'(z),$$

and

$$\rho(f) = 1 = \frac{\log 2}{\log 2}.$$

We also investigate the existence and growth of solutions of equation (5) when the constant η in equation (5) is replaced by a function, and obtain the following result.

Theorem 1.10 *Let f be a transcendental solution of equation*

$$f(qz+c)^n = R(z)f(z)[f^{(j)}(z)]^s, \tag{6}$$

where $q, c, \in \mathbb{C}$ and $|q| > 1$, n, j, s are positive integers and $R(z)$ is rational function in z. If f is an entire function, then $n \leq s+1$ and

$$\rho(f) \leq \frac{\log(s+1) - \log n}{\log |q|}.$$

Furthermore, if $n = 1$ and f is a meromorphic function with infinitely many poles, then we have

$$\frac{\log(s+1)}{\log |q|} \leq \mu(f) \leq \rho(f) \leq \frac{\log(sj+s+1)}{\log |q|}.$$

The following example shows that equation (6) has transcendental entire and meromorphic solutions.

Example 1.2 *Let $q = 2, c = 2\pi i$, $n = 1$ and $s = 1$, then $f(z) = ze^z$ satisfies system*

$$f(2z+2\pi i) = \frac{2z+2\pi i}{z(z+1)} f(z)f'(z).$$

and

$$\rho(f) = 1 \leq \frac{\log 2}{\log 2}.$$

Example 1.3 *Let $q = 2, c = \pi i$, $n = 1$ and $s = 1$, then $f(z) = \frac{e^{2z}}{z^2}$ satisfies equation*

$$f(2z + 2\pi i) = \frac{z^5}{(2z-2)(2z+2\pi i)^2} f(z)f'(z),$$

and

$$\frac{\log 2}{\log 2} = 1 \leq \mu(f) = \rho(f) = 1 \leq \frac{\log 3}{\log 2}.$$

Theorem 1.11 *Let f be a transcendental solution of the equation*

$$f(qz + c)^n = \varphi(z)f(z)[f^{(j)}(z)]^s, \tag{7}$$

where $q, c, \in \mathbb{C}$ and $|q| > 1$, n, j, s are positive integers and $\varphi(z)$ is a small function with respect of f. If f is a meromorphic function with $\overline{N}(r, f) = S(r, f)$, then $n < s + 1$ and f satisfies

$$\rho(f) \leq \frac{\log(s+1) - \log n}{\log |q|}.$$

Furthermore, if $n = 1$ and f has infinitely many poles, and the number of distinct common poles of f and $\frac{1}{\varphi}$ is finite, then we have

$$\rho(f) = \frac{\log(s+1)}{\log |q|}.$$

The following example shows that equation (7) has transcendental meromorphic solution f with the order $\rho(f) = \frac{\log(s+1)}{\log |q|}$.

Example 1.4 *Let $n = j = s = 1$ and $q = \sqrt{2}, c = \frac{1}{2\sqrt{2}}$, then $f(z) = e^{z^2}$ satisfies equation*

$$f(2z + \frac{1}{2\sqrt{2}}) = \frac{1}{2z} e^{\frac{1}{8}} e^z f(z)f'(z).$$

Thus, $\varphi(z) = \frac{1}{2z} e^{\frac{1}{8}} e^z$ with $T(r, \varphi) = S(r, f)$ and the order of $f(z)$ satisfies

$$\rho(f) = 2 = \frac{\log 2 - \log 1}{\frac{1}{2} \log 2}.$$

2 Some Lemmas

Lemma 2.1 *(Valiron-Mohon'ko). [18] Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z)f(z)^i}{\sum_{j=0}^{n} b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Lemma 2.2 *(see [23]). Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz+\eta)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.3 *(see [28]). Let $f(z)$ be a transcendental meromorphic function of zero order and q, η be two nonzero complex constants. Then*

$$T(r, f(qz+\eta)) = T(r, f(z)) + S_1(r, f), \quad N(r, f(qz+\eta)) \leq N(r, f) + S_1(r, f).$$

Lemma 2.4 *(see [34, p.37] or [33]). Let $f(z)$ be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then*

$$N(r, f^{(l)}) = N(r, f) + l\overline{N}(r, f), \quad T(r, f^{(l)}) \leq T(r, f) + l\overline{N}(r, f) + S(r, f).$$

Lemma 2.5 *Let $q, c \in \mathbb{C} \setminus \{0\}$ and $f(z)$ be a nonconstant meromorphic function with zero order. Then for any positive finite integer k, we have*

$$m\left(r, \frac{f^{(k)}(qz+c)}{f(z)}\right) = S_1(r, f),$$

and

$$m\left(r, f^{(k)}(qz+c)\right) \leq m(r, f) + S_1(r, f).$$

Proof: It follows from Lemma 2.2 that

$$m\left(r, \frac{f^{(k)}(qz+c)}{f(z)}\right) \leq m\left(r, \frac{f^{(k)}(qz+c)}{f(qz+c)}\right) + m\left(r, \frac{f(qz+c)}{f(z)}\right) = S_1(r, f).$$

Moreover, we have

$$m\left(r, f^{(k)}(qz+c)\right) = m\left(r, \frac{f^{(k)}(qz+c)}{f(z)} f(z)\right) \leq m(r, f) + S_1(r, f).$$

This completes the proof of Lemma 2.5. □

Lemma 2.6 *(see [11]). Let $\Phi : (1, \infty) \to (0, \infty)$ be a monotone increasing function, and let f be a nonconstant meromorphic function. If for some real constant $\alpha \in (0, 1)$, there exist real constants $K_1 > 0$ and $K_2 \geq 1$ such that*

$$T(r, f) \leq K_1 \Phi(\alpha r) + K_2 T(\alpha r, f) + S(\alpha r, f),$$

then the order of growth of f satisfies

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \limsup_{r \to +\infty} \frac{\log \Phi(r)}{\log r}.$$

Lemma 2.7 *(see [9]). Let $f(z)$ be a transcendental meromorphic function and $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_1 z + p_0$ be a complex polynomial of degree $k > 0$. For given $0 < \delta < |p_k|$, let $\lambda = |p_k| + \delta, \mu = |p_k| - \delta$, then for given $\varepsilon > 0$ and for r large enough,*

$$(1 - \varepsilon) T(\mu r^k, f) \leq T(r, f \circ p) \leq (1 + \varepsilon) T(\lambda r^k, f).$$

Lemma 2.8 *(see [2, 10] or [6]). Let $g:(0,+\infty)\to R, h:(0,+\infty)\to R$ be monotone increasing functions such that $g(r)\le h(r)$ outside of an exceptional set E with finite linear measure, or $g(r)\le h(r)$, $r\notin H\cup(0,1]$, where $H\subset(1,\infty)$ is a set of finite logarithmic measure. Then, for any $\alpha>1$, there exists r_0 such that $g(r)\le h(\alpha r)$ for all $r\ge r_0$.*

3 Proofs of Theorems 1.5 and 1.6

3.1 The proof of Theorem 1.5

Suppose that w be non-constant entire solution of equation (3) with zero order. Let $E_1=\{z:|w(z)|>1\}$ and $E_2=\{z:|w(z)|\le 1\}$, then we have

$$\begin{aligned}|\Omega(z,w)| &= \left|\sum_J a_J(z)(w(z))^{\lambda_i}\left(\frac{w'(q_1z+c_1)}{w(z)}\right)^{i_1}\cdots\left(\frac{w'(q_{n_1}z+c_{n_1})}{w(z)}\right)^{i_{n_1}}\right|\\ &\le \begin{cases} |w(z)|^{\lambda}\sum_J|a_J(z)|\left|\frac{w'(q_1z+c_1)}{w(z)}\right|^{i_1}\cdots\left|\frac{w'(q_{n_1}z+c_{n_1})}{w(z)}\right|^{i_{n_1}}, & if\ \ z\in E_1,\\ \sum_J|a_J(z)|\left|\frac{w'(q_1z+c_1)}{w(z)}\right|^{i_1}\cdots\left|\frac{w'(q_{n_1}z+c_{n_1})}{w(z)}\right|^{i_{n_1}}, & if\ \ z\in E_2,\end{cases}\end{aligned}$$

where $\lambda=\max\{\lambda_i\},\lambda_i=i_1+\cdots+i_{n_1}$. It follows from Lemma 2.2 and Lemma 2.5 that

$$m(r,\Omega(z,w))=\frac{1}{2\pi}\left(\int_{E_1}+\int_{E_2}\right)\log^+|\Omega(z,w)|d\theta\le\lambda m(r,w)+S_1(r,w).$$

And since $w(z)$ is a non-constant entire function, we have $N(r,w)=0$. Thus, we have $N(r,\Omega(z,w))=0$ and

$$T(r,\Omega)=m(r,\Omega)\le\lambda m(r,w)+S_1(r,w)=\lambda T(r,w)+S_1(r,w). \tag{8}$$

Since $P_s[f(w)]$ is a polynomial of $f(w)$, we can take a complex constant α such that

$$P_s[f(w)]-\alpha=[f(w)-\alpha_1]\cdots[f(w)-\alpha_s],$$

where $\alpha_1,\ldots,\alpha_s$ are complex constants, and there at least exists a constant $\beta\in\{\alpha_1,\ldots,\alpha_s\}$, which is not a Picard exceptional value of $f(w)$. Let $\xi_j, j=1,2,\ldots,p$ be the zeros of $f(w)-\beta$, where p is an any positive integer with $p\ge1$. Then it follows

$$\sum_{j=1}^{p}N(r,\frac{1}{w-\xi_j})\le N(r,\frac{1}{f(w)-\beta})\le N(r,\frac{1}{P_s[f(w)]-\alpha}). \tag{9}$$

Thus, by using the second main theorem and (8), (9), we can get that

$$\begin{aligned}(p-2)T(r,w) &\leq \sum_{j=1}^{p} N(r,\frac{1}{w-\xi_j}) + S(r,w)\\ &\leq N(r,\frac{1}{P_s[f(w)]-\alpha}) + S(r,w)\\ &\leq T(r,P_s[f(w)]) + S(r,w)\\ &\leq T(r,\Omega(z,w)) + S(r,w)\\ &\leq \lambda T(r,w) + S_1(r,w). \qquad (10)\end{aligned}$$

It follows from (8) and (10) that

$$(p-2-\lambda)T(r,w) \leq S_1(r,w). \qquad (11)$$

Since w is transcendental and p is arbitrary, we can get a contradiction with (11). Hence, we complete the proof of Theorem 1.5.

3.2 The proof of Theorem 1.6

By using the same argument as in Theorem 1.5, and applying Lemma 2.1, we can prove the conclusion of Theorem 1.6 easily.

4 The proof of Theorem 1.9

Suppose that f is a solution of (5). If f is a polynomial of degree $m \geq 1$, let

$$f(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0,$$

where $a_m, \ldots, a_0$ are complex constants. From (5), we have

$$\begin{aligned}&a_m(qz+c)^m + a_{m-1}(qz+c)^{m-1} + \cdots + a_0\\ =&\eta(a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0)[ma_m z^{m-1} + (m-1)a_{m-1}z^{m-1} + \cdots + a_1]. \qquad (12)\end{aligned}$$

By computing the degree of two sides in z in (12), we can get that $m = 2m-1$, that is, $m = 1$. Thus, $f(z)$ can be rewritten as $f(z) = a_1 z + a_0$. It follows

$$a_1(qz+c) + a_0 = \eta(a_1 z + a_0)a_1,$$

that is,

$$a_1 q = \eta a_1^2, \quad a_1 c + a_0 = \eta a_1 a_0.$$

Thus, we have $a_1 = \frac{q}{\eta}, \quad a_0 = \frac{qc}{\eta(1+q)}$.

If f is a transcendental entire function, from Lemma 2.4, we have

$$T(r, f(qz+c)) \leq 2T(r,f) + S(r,f) \leq 2(1+\varepsilon)T(\beta r, f), \qquad (13)$$

for sufficiently large r and any given $\beta > 1, \varepsilon > 0$. By Lemma 2.7 and (13), for $\theta = |q| - \delta(0 < \delta < |q|, 0 < \theta < 1), i = 1, 2$ and sufficiently larger r, we get

$$(1-\varepsilon)T(\theta r, f) \leq 2(1+\varepsilon)T(\beta r, f),$$

outside of a possible exceptional set E of finite linear measure. From Lemma 2.8, for any given $\gamma > 1$ and sufficiently large r, we obtain

$$(1-\varepsilon)T(\theta r, f) \leq 2(1+\varepsilon)T(\gamma\beta r, f). \tag{14}$$

that is,

$$\frac{(1-\varepsilon)}{2(1+\varepsilon)}T(r, f) \leq T\left(\frac{\beta\gamma}{\theta}r, f\right). \tag{15}$$

Since $|q| > 1$, we can choose $\delta > 0$ such that $\theta > 1$, and let $\varepsilon \to 0$, $\delta \to 0, \beta \to 1$, $\gamma \to 1$, and for sufficiently large r, by Lemma 2.6, we have

$$\rho(f) \leq \frac{\log 2}{\log |q|}.$$

Thus, this completes the proof of Theorem 1.9.

5 Proofs of Theorems 1.10 and 1.11

5.1 The Proof of Theorem 1.10

Since $R(z)$ is a rational function, then we have $T(r, R(z)) = O(\log r)$. If f is a transcendental entire function, similar to the argument as in Theorem 1.9, we can get $\rho(f) \leq \frac{\log(s+1)-\log n}{\log |q|}$ easily.

If f is a meromorphic function, by Lemma 2.1 and Lemma 2.4, it follows from (6) that

$$T(r, f(qz+c)) \leq \frac{sj+s+1}{n}T(r, f(z)) + S(r, f).$$

Since $|q| > 1$, by Lemma 2.7 and using the same argument as in Theorem 1.9, we have $\rho(f) \leq \frac{\log(sj+s+1)-\log n}{\log |q|}$.

Suppose that $n = 1$. Since $R(z)$ is a rational function, we can choose a sufficiently large constant $R(> 0)$ such that $R(z)$ has no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$. Since f has infinitely many poles, we can choose a pole z_0 of f of multiplicity $\tau \geq 1$ satisfying $|z_0| > R$. Thus, it follows that the right side of the equation (6) has a pole of multiplicity $\tau_1 = (s+1)\tau + sj$ at z_0, and f has a pole of multiplicity τ_1 at $qz_0 + c$. Replacing z by $qz_0 + c$ in equation (6), we have that f has a pole of multiplicity $\tau_2 = (s+1)\tau_1 + sj$ at $q^2z_0 + qc + c$. We proceed to follow the step above. Since $R(z)$ has no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$ and f has infinitely many poles again, we may construct poles $\zeta_k = q^k z_0 + q^{k-1}c + \cdots + c, k \in \mathbb{N}_+$ of f of multiplicity τ_k satisfying

$$\tau_k = (s+1)\tau_{k-1} + sj = (s+1)^k\tau + sj[(s+1)^{k-1} + \cdots + 1],$$

as $k \to \infty, k \in \mathbb{N}$. Since $|q| > 1$, then $|\zeta_k| \to \infty$ as $k \to \infty$, for sufficiently large k, we have

$$\begin{aligned}\tau(s+1)^k &\leq (\tau+j)(s+1)^k - j = \tau_k \leq \tau + \tau_1 + \cdots + \tau_k \leq n(|\zeta_k|, f) \\ &\leq n(|q|^k|z_0| + |C|(|q|^{k-1} + \cdots + |q| + 1), f).\end{aligned} \tag{16}$$

Thus, for each sufficiently large r, there exists a $k \in \mathbb{N}_+$ such that

$$r \in [|q|^k|z_0| + |C|\sum_{i=0}^{k-1}|q|^i, |q|^{(k+1)}|z_0| + |C|\sum_{i=0}^{k}|q|^i),$$

that is,

$$k > \frac{\log r - \log(|z_0| + \frac{|c|}{|q|-1}) - \log\frac{|c|}{|q|-1} - \log|q|}{\log|q|}. \tag{17}$$

Thus, it follows from (17) that

$$n(r, f) \geq \tau(s+1)^k \geq K_1(s+1)^{\frac{\log r}{\log|q|}}, \tag{18}$$

where

$$K_1 = \tau(s+1)^{\frac{-\log(|z_0| + \frac{|c|}{|q|-1}) - \log\frac{|c|}{|q|-1} - \log|q|}{\log|q|}}.$$

Since for all $r \geq r_0$,

$$K_1(s+1)^{\frac{\log r}{\log|q|}} \leq n(r, f) \leq \frac{1}{\log 2}N(2r, f) \leq \frac{1}{\log 2}T(2r, f),$$

it follows from (18) that

$$\rho(f) \geq \mu(f) \geq \frac{\log(s+1)}{\log|q|}.$$

Thus, this completes the proof of Theorem 1.10.

5.2 The proof of Theorem 1.11

By using the same argument as in Theorem 1.10, we can prove the conclusion of Theorem 1.11 easily.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

HW and HYX completed the main part of this article. All authors read and approved the final manuscript.

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Numerical method for solving inequality constrained matrix operator minimization problem☆

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Abstract

In this paper, we considered a matrix inequality constrained linear matrix operator minimization problems with a particular structure, some of whose reduced versions can be applicable to image restoration. We present an efficient iteration method to solve this problem. The approach belongs to the category of Powell-Hestense-Rockafellar augmented Lagrangian method, and combines a nonmonotone projected gradient type method to minimize the augmented Lagrangian function at each iteration. Several propositions and one theorem on the convergence of the proposed algorithm were established. Numerical experiments are performed to illustrate the feasibility and efficiency of the proposed algorithm, including when the algorithm is tested with randomly generated data and on image restoration problems with some special symmetry pattern images.

Key words:
matrix equation, matrix minimization problem, matrix inequality, augmented lagrangian method, image restoration.
2000 MSC: 65F30, 65H15, 15A24

1. Introduction

Let m, n, l_1, s_1, l_2, s_2 be positive integers. Let $\mathcal{A}(X; A_1, \cdots, A_p)$ be a linear mapping from $\mathbb{R}^{m\times n}$ onto $\mathbb{R}^{l_1\times s_1}$ and $\mathcal{G}(X; E_1, \cdots, E_q)$ be a linear mapping from $\mathbb{R}^{m\times n}$ onto $\mathbb{R}^{l_2\times s_2}$, where A_i $(i = 1, \ldots, p)$ and E_j $(j = 1, \ldots, q)$ with suitable sizes are the parameter matrices. In this paper we are interested in solving the following constrained matrix minimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\Big\|\mathcal{A}\Big(X; A_1, \cdots, A_p\Big) - C\Big\|^2 \\ \text{subject to} & X \in \mathcal{S} \\ & L \le \mathcal{G}\Big(X; E_1, \cdots, E_q\Big) \le U. \end{array} \tag{1.1}$$

where $\|\cdot\|$ denotes the Frobenius norm, the symbol $\geq$ means nonnegative, the set $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$ shows the constraint, $C \in \mathbb{R}^{l_1\times s_1}$ and $L, U \in \mathbb{R}^{l_2\times s_2}$ are given matrices. In general, $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$ is a linear space

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possessing special structures, such as symmetry/skew-symmetry, centrosymmetry/centro skew-symmetry, mirror-symmetry/mirror-skew-symmetry, P-commuting symmetry/ skew-symmetry with respect to a given symmetric matrix P, Toeplitz matrix and so on. It is obvious that the linear operator equation in (1.1) is quite general and includes several linear matrix equations such as the Lyapunov and Sylvester matrix equations which are shown in Table 1. For an instant, the Lyapunov matrix equation

$$A_1^T X A_2 + A_2^T X A_1 = -C$$

is equivalent to the linear operator equation in (1.1), if we define the operator $\mathcal{A}$ as:

$$\mathcal{A} : X \to A_1^T X A_2 + A_2^T X A_1.$$

Table 1: *One-sided and two-sided Lyapunov and Sylvester matrix equations.*

Name	Matrix equation
Continuous-time (CT) Lyapunov	$A_1 X + X A_1^T + BB^T = 0$
Generalized continuous-time (CT) Lyapunov	$A_1^T X A_2 + A_2^T X A_1 = -C$
Generalized discrete-time (CT) Lyapunov	$A_1^T X A_1 + A_2^T X A_2 = -C$
Continuous-time (CT) Sylvester	$A_1 X + X A_2 = C$
Discrete-time (DT) Sylvester	$A_1 X A_2^T + X = C$
Generalized Sylvester	$A_1 X A_2^T + A_3 X A_4^T = C$

Throughout we always assume that the matrix operator inequality in model (1.1) is consistent with these given matrices E_j, L, U and unknown $X \in \mathcal{S}$, then we known that the solution set of Problem (1.1) is nonempty.

The interest that we have in this problem stems from the following reasons. Firstly, by using the vec operator $vec(.)$ and the Kronecker produc $\otimes$, the model (1.1) can be equivalently rewritten as the convex linearly constrained quadratic programming(LCQP) in the vector-form

$$\begin{aligned} &\text{minimize } f(x) = \frac{1}{2} x^T Q x + g^T x + c \\ &\text{subject to } \ l \le Gx \le u, \end{aligned} \tag{1.2}$$

where

$$Q = P^T M^T M P, \ \ g = -P^T M^T vec(C), \ \ c = \frac{1}{2} vec(C)^T vec(C) \tag{1.3}$$

and

$$Px = vec(X), \ \ l = vec(L), \ \ u = vec(U). \tag{1.4}$$

The matrices M and G are the Kronecker product of the parameter matrices $\{A_i\}_{i=1}^p$ and $\{E_j\}_{j=1}^q$ which satisfies $vec(\mathcal{A}(X; A_1, \ldots, A_p)) = Mvec(X)$ and $vec(\mathcal{G}(X; E_1, \ldots, E_q)) = Gvec(X)$, respectively. Specifically, in (1.3)-(1.4), P is the matrix that characterizes the elements $X \in \mathcal{S}$ by $vec(X) = Px$ in terms of its independent parameter vector x of X[18]. In theory, the model (1.2) can be solved by some classical optimization methods, such as interior point method, active set method, trust region method, Newton method, and other available methods. In particular, Delbos F. in [2] considered the vector LCQP(1.2) by using an augmented Lagrangian method and given a global linear convergence of the proposed algorithm. However, using this transformation will on the one hand destroy the original structure of the unknown matrix $X \in \mathcal{S}$ if the linear

subspace $\mathcal{S}$ has some special symmetrical structure. On the other hand, using this transformation will result in a coefficient matrix in large scale, and then increase computational complexity and storage requirement. Indeed, taking $l = m = n = s = p = q = 200$ in (1.1), then the matrices Q and G in the transformed model (1.2) have sizes of about 40000×40000. For these reasons, it cannot be a practicable method for solving Problem (1.1) by the vec operator and the Kronecker produc if the system scale is large. In this paper we will consider directly from the perspective of matrices.

Secondly, various simplified versions of Problem (1.1) have been studied extensively. If we drop the matrix inequality constraint, then Problem (1.1) is reduced to the minimization problem with special structures. Methods proposed for solving such problems can be broadly classified into two classes, including factorization techniques for small size problems, based on the special structure of the linear subspace $\mathcal{S}$ that produce a low-dimensional problems that are then solved using direct methods[3, 4, 5, 6, 7, 8, 9, 10, 11], and iterative schemes, for large-scale problems, based on Krylov subspace-type methods, such as the well-known Jacobi and Gauss-Seidel iterations[12, 13], the conjugate gradient-type methods[14, 15] and the least squares QR(LSQR) methods[16, 17, 18] and so on. On the other hand, if we simplify the general matrix inequality constraint in (1.1) into the nonnegative constraint $X \geq \mathbf{0}$ or the bound constraint $L \leq X \leq U$, then the similar problem has been studied with Dykstra's alternating projection algorithm[19, 20] and spectral projection gradient method[21]. In particular, Problem (1.1) can be regarded as a natural generalization of the problems in [21, 22, 23]. The authors in [21] considered the following constrained minimization problem

$$\text{Minimize } \left\| \sum_{i=1}^{q} A_i X B_i - C \right\|^2 \quad \text{subject to } X \in \Omega = \{X \in \mathbb{R}^{m\times n} : L \leq X \leq U\}. \tag{1.5}$$

They propose a globalized variants projected gradient method and apply the left and right preconditioning strategies to solve (1.5). While the authors in [22, 23] devoted to solve the matrix equation $AX = B$ or minimize $\|AX - B\|$ with special structures under the constraint $CXD \geq E$, respectively. The problems considered in [22] and [23] can be transformed into least nonnegative correction problems based on the fact that close-form optimal solutions of $AX = B$ or minimizing $\|AX - B\|$ with special structures can be readily derived, and then some fixed point-like algorithms can be applied to solve these transformed problems. However, all these previous ideas show difficulties when dealing with the Problem (1.1), due to the generalization of the objection function and the matrix operator inequality, so that either the projection onto the set $\{X \in \mathbb{R}^{m\times n} | L \leq \mathcal{G}(X) \leq U\}$ is not available, or a close-form optimal solution of minimizing the objection function in (1.1) with $X \in \mathcal{S}$ is not tractable.

Thirdly, we consider the application of the model (1.1) in image restoration. In fact, the authors in [21, 24] consider the problem of image restoration, combined with a Tikhonov regularization term, as a convex constrained minimization problem by use a Kronecker decomposition of the blurring matrix and the Tikhonov regularization matrix. And then they propose and show the effectiveness of their approaches, a globalized variants projected gradient method [21] and a conditional gradient-type method[21], to restore some blurred and highly noisy images. However, in this paper, we are only concerned with the restoration problems with some special symmetric pattern images, which have not yet studied in [21, 24]. Moreover, to the best of our knowledge, this class of image restoration problems have received little attention in the other literature. The main difficult is due to the fact that the restore image should preserve the same special symmetric structure with the original images. In this paper we undertake some significant attempts in this field.

In this paper, we will propose and study an algorithm in the framework of the classic Powell-Hestenes-Rockafellar augmented Lagrangian method, first suggested by Hestenes [25] and Powell [26], and developed by E.G. Birgin [27, 28] for solving Problem (1.1). The classic PHR-AL method is a fundamental and

effective approach in inequality-constrained optimization. The algorithm effectively combines a nonmonotone projected gradient type method to minimize the augmented Lagrangian function at each iteration. We will give several propositions and one theorem on the convergence of the proposed algorithm, and apply it to solving Problem (1.1) with randomly generated data and comparing it with existing methods. We also apply our approach, combined with a Tikhonov regularization term, to restore some blurred and highly noisy symmetric pattern images.

Throughout this paper, we use the following notations. Let e_i be the ith column of the identity matrix I_k and $S_k = (e_k, e_{k-1}, \dots, e_1)$, i.e., the kth backward identity matrix. Let $\mathbf{0}$ be the zero matrix of suitable size and $P_{\mathcal{S}}$ be the Euclidean projection onto set $\mathcal{S}$.We write $\varepsilon_k \downarrow 0$ to indicate that ε_k is a (not necessarily decreasing) sequence of non-negative numbers that tends to zero. We denote $\mathbb{N} = \{0, 1, 2, \dots\}$. For $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, A_+(or A_-) be the matrix with the (i, j)-entry equals to $\max\{0, a_{ij}\}$(or $\min\{0, a_{ij}\}$), respectively. For $A, B \in \mathbb{R}^{m\times n}$, $\{A, B\}_-$ denotes a matrix with the ijth entry being equal to $\min\{a_{ij}, b_{ij}\}$, $\langle A, B\rangle = trace(B^T A)$ denotes the inner product of matrices A and B. Then $\mathbb{R}^{m\times n}$ is a Hilbert inner product space and the norm generated is the Frobenius norm $\|\cdot\|$. For any linear operator $\mathcal{L}$ form $\mathbb{R}^{m\times n}$ onto $\mathbb{R}^{l_1\times s_1}$, there is another operator called the adjoint of $\mathcal{L}$, written $\mathcal{L}^T$: $\mathbb{R}^{l_1\times s_1} \to \mathbb{R}^{m\times n}$. What defines the adjoint is that for any two matrices $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{l_1\times s_1}$,

$$\langle \mathcal{L}(X), Y\rangle = \langle X, \mathcal{L}^T(Y)\rangle.$$

The rest of this paper is organized as follows. In section 2, we will briefly characterize the application of model (1.1) in image restoration. Based on the classic augmented Lagrangian method, in section 3 we propose, analyze and test an algorithm for solving the inequality-constrained matrix minimization problem (1.1). Some numerical results are reported in section 4 to verify the efficiency of the proposed algorithm. Numerical tests on the proposed algorithm applied to some special image restoration problems are also reported in this section.

2. The application of model (1.1) in image restoration

For completeness, in this section we briefly characterize how to apply the model (1.1) into image restoration and we refer to [21, 24] for detailed description. Consider solving the following model in image restoration with Tikhonov regularization:

$$\min_{l\le x\le u} \frac{1}{2}\|Hx - g\|_2^2 + \frac{\lambda^2}{2}\|Tx\|_2^2, \tag{2.6}$$

where $\|\cdot\|_2$ is the 2-norm. In image restoration, H will be the blurring operator, g the observed image, T the regularization operator, λ the regularization parameter, and x the restored image to be sought. The constraints represent the dynamic range of the image.

The minimizer of (2.6) can be computed by the following linear system

$$H_\lambda x = H^T g, \quad \text{where } H_\lambda = H^T H + \lambda^2 T^T T. \tag{2.7}$$

In some practical problems in image restoration, often the system (2.7) may not be consistent due to measurement errors in the data matrices, and hence it is useful to consider the following minimization problem with constraints

$$\min_{l\le x\le u} \frac{1}{2}\left\|H_\lambda x - H^T g\right\|_2^2. \tag{2.8}$$

4

Here we assume that the matrices H and T can be separated as Kronecker product of matrices with a smaller size, i.e., $H = H_1 \otimes H_2$ and $T = T_1 \otimes T_2$. In the case of nonseparable, one can still obtain an approximation solution of H_1 and H_2 by solving the Kronecker product approximation problem (KPA) of the form $(H_1, H_2) = \text{argmin}_{\hat{H}_1,\hat{H}_2} \|H - \hat{H}_1 \otimes \hat{H}_2\|$[29]. Then, (2.8) can be written as

$$\min_{L\le X\le U} \frac{1}{2} \left\| \left\{ (H_1{}^T H_1) \otimes (H_2{}^T H_2) + \lambda^2 (T_1{}^T T_1) \otimes (T_2{}^T T_2) \right\} vec(X) - (H_1 \otimes H_2)^T vec(G) \right\|^2, \tag{2.9}$$

where X, G, L and U are the matrices such that $vec(X) = x$, $vec(G) = g$, $vec(L) = l$ and $vec(U) = u$. If some special symmetry pattern images are considered, by using some properties of the Kronecker product, (2.9) is then written as

$$\begin{aligned} \min \quad & \frac{1}{2} \left\| A_1 X B_1 + \lambda^2 A_2 X B_2 - C \right\|^2 \\ \text{subject to} \quad & L \le X \le U, \quad X \in \mathcal{S}, \end{aligned} \tag{2.10}$$

with $A_1 = H_2{}^T H_2,\ B_1 = H_1{}^T H_1,\ A_2 = T_2{}^T T_2,\ B_2 = T_1{}^T T_1$ and $C = H_2{}^T G H_1$ and $\mathcal{S}$ is the matrix set whose elements have the same symmetry structure with the original images. The parameter λ in (2.10) is a scalar need to be determined, and its optimal value can be obtained by the classical Generalized cross-validation (GCV) method[21, 24], which is chosen to minimize the GCV function defined by

$$GCV(\lambda) = \frac{\|H\hat{x}_\lambda - g\|_2^2}{\{trace(I - HH_\lambda^{-1}H^T)\}^2} = \frac{\|(I - HH_\lambda^{-1}H^T)g\|_2^2}{\{trace(I - HH_\lambda^{-1}H^T)\}^2},$$

where $H_\lambda = H^T H + \lambda^2 T^T T$. Then, the method proposed for solving Problem (1.1) could be applied directly to the model (2.10) by considering the linear matrix operators $\mathcal{A}(X) = A_1 X B_1 + \lambda^2 A_2 X B_2$ and $\mathcal{G}(X) = X$.

3. Augmented Lagrangian method for solving Problem (1.1)

In this section we propose a matrix-form iteration method, in the framework of the classic Powell-Hestense-Rockafellar augmented Lagrangian(PHR-AL) method, to compute the solution of Problem (1.1). We then prove some convergence results for the proposed algorithm at the end of this section. For convenience, the two linear matrix operators will be simply denote by $\mathcal{A}(X)$ and $\mathcal{G}(X)$ in the following discussion.

Lemma 1. *Assume x^* is a local minimizer of the quadratic program*

$$\min_{x\in R^s} f(x) = \frac{1}{2} x^T M x + g^T x + c \ \text{ subject to } \ Gx \ge b,$$

then there exists a vector y^* such that

$$Mx^* + g - G^T y^* = 0,\ \ Gx^* \ge b,\ \ \langle y^*, Gx^* - b\rangle = 0,\ \ y^* \ge \mathbf{0}.$$

Theorem 1. *Matrix $X^* \in \mathbb{R}^{m\times n}$ is a solution of Problem (1.1) if and only if there exists nonnegative matrices $Y_1^*, Y_2^* \in \mathbb{R}^{l_2\times s_2}$ such that the following conditions are satisfied:*

$$\begin{cases} P_{\mathcal{S}}\left\{ \mathcal{A}^T(\mathcal{A}(X^*) - C) - \mathcal{G}^T(Y_1^* - Y_2^*) \right\} = \mathbf{0} \\ \mathcal{G}(X^*) - L \ge \mathbf{0} \\ U - \mathcal{G}(X^*) \ge \mathbf{0} \\ \langle Y_1^*, \mathcal{G}(X^*) - L\rangle = 0 \\ \langle Y_2^*, U - \mathcal{G}(X^*)\rangle = 0. \end{cases} \tag{3.11}$$

Proof. Assume that there are nonnegative matrices $Y_1^*, Y_2^* \in \mathbb{R}^{l_2 \times s_2}$ such that the conditions (3.11) are satisfied. Let

$$f(X) = \frac{1}{2}\left\|\mathcal{A}\big(X\big) - C\right\|^2$$

and

$$\widetilde{f}(X) = f(X) + \langle Y_1^*, L - \mathcal{G}(X)\rangle + \langle Y_2^*, \mathcal{G}(X) - U\rangle.$$

Then for any $\widetilde{W} \in \mathcal{S}$, we have

$$\begin{aligned}
&\widetilde{f}(X^* + \widetilde{W})\\
&= \tfrac{1}{2}\|\mathcal{A}(X^* + \widetilde{W}) - C\|^2 + \Big\langle Y_1^*, L - \mathcal{G}(X^* + \widetilde{W})\Big\rangle + \Big\langle Y_2^*, \mathcal{G}(X^* + \widetilde{W}) - U\Big\rangle\\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \Big\langle \mathcal{A}(\widetilde{W}), \mathcal{A}(X^*) - C\Big\rangle - \Big\langle Y_1^* - Y_2^*, \mathcal{G}(\widetilde{W})\Big\rangle\\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \Big\langle \widetilde{W}, \mathcal{A}^T(\mathcal{A}(X^*) - C) - \mathcal{G}^T(Y_1^* - Y_2^*)\Big\rangle\\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \tfrac{1}{2}\Big\langle \widetilde{W}, P_{\mathcal{S}}\Big(\mathcal{A}^T(\mathcal{A}(X^*) - C) - \mathcal{G}^T(Y_1^* - Y_2^*)\Big)\Big\rangle\\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2\\
&\geq \widetilde{f}(X^*).
\end{aligned}$$

This implies that X^* is a global minimizer of the function $\widetilde{f}(X)$. Since $\langle Y_1^*, \mathcal{G}(X^*) - L\rangle = 0$, $\langle Y_2^*, U - \mathcal{G}(X^*)\rangle = 0$ and $\widetilde{f}(X) \geq \widetilde{f}(X^*)$ for all $X \in \mathcal{S}$, we have

$$\begin{aligned}
f(X) \;&\geq f(X^*) + \Big\langle Y_1^*, L - \mathcal{G}(X^*)\Big\rangle + \Big\langle Y_2^*, \mathcal{G}(X^*) - U\Big\rangle - \Big\langle Y_1^*, L - \mathcal{G}(X)\Big\rangle - \Big\langle Y_2^*, \mathcal{G}(X) - U\Big\rangle\\
&= f(X^*) - \Big\langle Y_1^*, L - \mathcal{G}(X)\Big\rangle - \Big\langle Y_2^*, \mathcal{G}(X) - U\Big\rangle.
\end{aligned}$$

Hence, we have from $Y_1^* \geq \mathbf{0}$ and $Y_2^* \geq \mathbf{0}$ that $f(X) \geq f(X^*)$ for all $X \in \mathcal{S}$ with $\mathcal{G}(X) - L \geq \mathbf{0}$ and $U - \mathcal{G}(X) \geq \mathbf{0}$. Hence X^* is a solution to Problem (1.1).

Conversely, assuming that X^* is a solution to Problem (1.1), then X^* certainly satisfies the Karush-Kuhn-Tucker conditions of Problem (1.1). That is, there exists a nonnegative matrix Y^* such that satisfies conditions (3.11).

We now define the following Powell-Hestenes-Rockafellar(PHR) Augmented Lagrangian function

$$L_\rho(X, Z_1, Z_2) = \frac{1}{2}\|\mathcal{A}(X) - C\|^2 + \frac{\rho}{2}\left\|\Big(L - \mathcal{G}(X) + \frac{Z_1}{\rho}\Big)_+\right\|^2 + \frac{\rho}{2}\left\|\Big(\mathcal{G}(X) - U + \frac{Z_2}{\rho}\Big)_+\right\|^2, \tag{3.12}$$

where $Z_1 \geq \mathbf{0}$ and $Z_2 \geq \mathbf{0}$ are the Lagrangian multiplier matrices and $\rho > 0$ is the penalty parameter. Clearly, the partial derivative of function $L_\rho(X, Z_1, Z_2)$ with respect to X is given by

$$\nabla_X L_\rho(X, Z_1, Z_2) = \mathcal{A}^T\Big(\mathcal{A}(X) - C\Big) - \rho\mathcal{G}^T\left(\Big(L - \mathcal{G}(X) + \frac{Z_1}{\rho}\Big)_+ - \Big(\mathcal{G}(X) - U + \frac{Z_2}{\rho}\Big)_+\right).$$

The augmented Lagrangian method proposed by E.G. Birgin et al in in [27, 28] (with necessary modifications) to solve Problem (1.1) can be described as follows:

Algorithm PHR-AL. *(The PHR-AL method for solving Problem (1.1).)*

1. Input coefficient matrices $A_i, B_i (i = 1, \ldots p)$ in the linear operator $\mathcal{A}$ and matrices $E_i, E_j (i = 1, \ldots q)$ in the linear operator $\mathcal{G}$. Input matrices C, L, U and a large parameter matrix $Z_{max} > \mathbf{0}$. Input $\gamma > 1$, $r \in (0, 1)$, $\rho_1 > 0$, a small tolerance $\varepsilon > 0$ and tolerance $\varepsilon_k \downarrow 0$. Choose initial matrices $\overline{Z}_1^1$ and $\overline{Z}_2^1$ with $\mathbf{0} \leq \overline{Z}_1^1, \overline{Z}_2^1 \leq Z_{max}$. Set $k \leftarrow 1$.

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2. Compute X^k as an approximate stationary point of

$$\text{minimize} \quad L_{\rho_k}(X, \overline{Z}_1^k, \overline{Z}_2^k) \quad \text{subject to } \mathrm{X} \in \mathcal{S}. \tag{3.13}$$

That is, compute X^k such that $\left\| P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\| < \varepsilon_k$.

3. Define

$$Z_1^k = \left(\overline{Z}_1^k + \rho_k(L - \mathcal{G}(X^k))\right)_+, \quad Z_2^k = \left(\overline{Z}_2^k + \rho_k(\mathcal{G}(X^k) - U)\right)_+.$$

4. If $k = 1$ or

$$\begin{aligned} &\left(\left\|\{\mathcal{G}(X^k) - L, Z_1^k\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^k),\ Z_2^k\}_-\right\|^2\right)^{1/2} \\ &\qquad \le r\left(\left\|\{\mathcal{G}(X^{k-1}) - L, Z_1^{k-1}\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^{k-1}),\ Z_2^{k-1}\}_-\right\|^2\right)^{1/2}, \end{aligned} \tag{3.14}$$

define $\rho_{k+1} = \rho_k$. Else, define $\rho_{k+1} = \gamma\rho_k$.

5. If

$$\left(\left\|P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\|^2 + \left\|\{\mathcal{G}(X^k) - L, Z_1^k\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^k),\ Z_2^k\}_-\right\|^2\right)^{1/2} < \varepsilon,$$

then stop.

6. Update $\overline{Z}_1^{k+1}$ and $\overline{Z}_2^{k+1}$ with $\mathbf{0} \le \overline{Z}_1^{k+1}, \overline{Z}_2^{k+1} \le Z_{max}$ in such a way that $(\overline{Z}_1^{k+1})_{ij} = (Z_1^k)^{ij}$ and $(\overline{Z}_2^{k+1})_{ij} = (Z_2^k)^{ij}$ if $0 \le (Z_1^k)_{ij}, (Z_2^k)_{ij} \le (Z_{max})_{ij}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$.
7. Set $k \leftarrow k + 1$ and go to step 2.

Problem (3.13) in Algorithm PHR-AL is a linear constrained matrix minimization problem. It is certainly solvable for all the known matrices and the scalar ρ_k. Here we will use the spectral projected gradient (SPG) method to compute the approximation stationary point X^k of problem (3.13). The SPG method is a nonmonotone projected gradient type method for minimizing general smooth functions on convex sets[27]. The SPG method is simple, easy to code, and does not require matrix factorizations. Moreover, it overcomes the traditional slowness of the gradient method by incorporating a spectral step length and a nonmonotone globalization strategy. The main steps of SPG algorithm (with necessary modifications) to compute an approximate stationary point of problem (3.13) can be described as follows:

Algorithm SPG. *(Compute an approximate stationary point of problem* (3.13)*)*

1. Input matrices $\overline{Z}_1^k$ and $\overline{Z}_2^k$; an integer $M > 1$, parameters $\alpha_{\min} > 0$, $\alpha_{\max} > \alpha_{\min}$, $\widetilde{\gamma} \in (0, 1)$, $0 < \sigma_1 < \sigma_2 < 1$ and $\alpha_1 \in [\alpha_{\min}, \alpha_{\max}]$. Choose an initial matrix $X_1 \in \mathcal{S}$ and let $i \leftarrow 1$.
2. If $\left\|P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\| < \varepsilon_k$, stop. (In this case, X_i is an approximate stationary point of problem (3.13).)
3. Compute $dX_i = -\alpha_i P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\right\}$. Let $\lambda = 1$.
4. Compute $\check{X} = X_i + \lambda dX_i$.
5. If

$$L_{\rho_k}(\check{X}, \overline{Z}_1^k, \overline{Z}_2^k) \le \max_{1 \le j \le \min\{i, M\}} L_{\rho_k}(X_{i-j}, \overline{Z}_1^k, \overline{Z}_2^k) + \widetilde{\gamma}\lambda\left\langle dX_i, \nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\right\rangle, \tag{3.15}$$

define $\lambda_i = \lambda$, $X_{i+1} = \check{X}$, $s_i = X_{i+1} - X_i$, $y_i = \nabla L_{\rho_k}(X_{i+1}, \overline{Z}_1^k, \overline{Z}_2^k) - \nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)$. Then goto step 6. If (3.15) does not hold, define $\lambda_{new} \in [\sigma_1\lambda, \sigma_2\lambda]$, Let $\lambda = \lambda_{new}$ and goto step 4.

6. Compute $b_i = \langle s_i, y_i\rangle$. If $b_i \le 0$, let $\alpha_i = \alpha_{\max}$, otherwise, compute

$$a_i = \langle s_i, s_i\rangle, \quad \alpha_i = \min\{\alpha_{\max}, \max\{\alpha_{\min}, a_i/b_i\}\}.$$

7. Let $i \leftarrow i + 1$ and goto step2.

In the practical implementation of Algorithm PHR-AL, similarly to [27], we take the parameters $\gamma = 5$, $r = 0.5$, $\rho_1 = 1$, and the large matrix $Z_{\max}$ with all elements equal to 10^{10}. The initial matrices $\overline{Z}_1^1$ and $\overline{Z}_2^1$ are chosen as $\overline{Z}_1^1 = \overline{Z}_2^1 = \mathbf{0}$. For the implementation of Algorithm SPG, similarly to [30], we take the parameters $M = 10$, $\gamma = 10^{-4}$, $\alpha_{min} = 10^{-30}$, $\alpha_{max} = 10^{30}$, $\sigma_1 = 0.1$, $\sigma_2 = 0.9$, $\lambda_{new} = (\sigma_1\lambda + \sigma_2\lambda)/2$ and $\alpha_0 = 1$. The initial matrix X_1 is chosen as the $(k-1)$th approximate solution of Algorithm PHR-AL.

Lemma 2. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL and the sequence ρ_k is bounded, then we have*

$$L \le \mathcal{G}(X^*) \le U.$$

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that $\lim_{k\in\mathbb{K}} X^k = X^*$. Since $\lim_{k\to\infty} \rho_k = \infty$ when (3.14) does not hold, the boundedness of ρ_k implies that there exists $k_0 \in \mathbb{N}$ such that (3.14) takes place for all $k \ge k_0$. Therefore,

$$\lim_{k\in\mathbb{K}} \left\|\{\mathcal{G}(X^k) - L, Z_1^k\}_-\right\| = 0 \quad \text{and} \quad \lim_{k\in\mathbb{K}} \left\|\{U - \mathcal{G}(X^k), Z_2^k\}_-\right\| = 0.$$

Note that $Z_1^k \ge \mathbf{0}$ and $Z_2^k \ge \mathbf{0}$ for all $k \in \mathbb{N}$, we have

$$\lim_{k\in\mathbb{K}} (L - \mathcal{G}(X^k))_+ = \mathbf{0} \quad \text{and} \quad \lim_{k\in\mathbb{K}} (\mathcal{G}(X^k) - U)_+ = \mathbf{0},$$

that is, $\mathcal{G}(X^*) - L \ge \mathbf{0}$ and $U - \mathcal{G}(X^*) \ge \mathbf{0}$.

Lemma 3. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL, then X^* is a first-order stationary point of the problem*

$$\text{minimize} \quad \frac{1}{2}\left\{\left\|(L - \mathcal{G}(X^*))_+\right\|^2 + \left\|(\mathcal{G}(X^*) - U)_+\right\|^2\right\} \quad \text{subject to } \mathrm{X} \in \mathcal{S}. \tag{3.16}$$

In other words, $X^* \in \mathcal{S}$ satisfies

$$P_{\mathcal{S}}\left\{\mathcal{G}^T\left((L - \mathcal{G}(X^*))_+ - (\mathcal{G}(X^*) - U)_+\right)\right\} = \mathbf{0}.$$

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that $\lim_{k\in\mathbb{K}} X^k = X^*$. Consider first the case in which the sequence ρ_k is bounded. By the proof of Lemma 2, we have that

$$\lim_{k\in\mathbb{K}} \left\|(L - \mathcal{G}(X^k))_+\right\| = 0 \quad \text{and} \quad \lim_{k\in\mathbb{K}} \left\|(\mathcal{G}(X^k) - U)_+\right\| = 0.$$

Note that

$$\left\|\mathcal{G}^T\left(\left(L - \mathcal{G}(X^*)\right)_+\right)\right\| \le \left\|\left(L - \mathcal{G}(X^*)\right)_+\right\| \quad \text{and} \quad \left\|\mathcal{G}^T\left(\left(\mathcal{G}(X^*) - U\right)_+\right)\right\| \le \left\|\left(\mathcal{G}(X^*) - U\right)_+\right\|,$$

we have that

$$\lim_{k\in\mathbb{K}} \left\|\mathcal{G}^T\left((L - \mathcal{G}(X^k))_+ - (\mathcal{G}(X^k) - U)_+\right)\right\| = 0.$$

Since $X^k \in \mathcal{S}$ for all k, this implies the desired result in the case that $\{\rho_k\}$ is bounded.

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Assume now that $\{\rho_k\}$ is not bounded. Therefore there exists an infinite sequence of indices $\mathbb{K}' \subset \mathbb{K}$ such that $\lim_{k\in\mathbb{K}'} \rho_k = \infty$. Note that $\varepsilon_k \downarrow 0$ and $\left\|P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\| < \varepsilon_k$, we have

$$\lim_{k\in\mathbb{K}} \left\| P_{\mathcal{S}} \left\{ \mathcal{A}^T\left(\mathcal{A}(X^k) - C\right) - \rho_k \mathcal{G}^T \left(\left(L - \mathcal{G}(X^k) + \frac{\overline{Z}_1^k}{\rho_k}\right)_+ - \left(\mathcal{G}(X^k) - U + \frac{\overline{Z}_2^k}{\rho_k}\right)_+ \right) \right\} \right\| = 0.$$

Therefore we have

$$\lim_{k\in\mathbb{K}} \left\| P_{\mathcal{S}} \left\{ \mathcal{A}^T\left(\mathcal{A}(X^k) - C\right)\big/\rho_k - \mathcal{G}^T \left(\left(L - \mathcal{G}(X^k) + \frac{\overline{Z}_1^k}{\rho_k}\right)_+ - \left(\mathcal{G}(X^k) - U + \frac{\overline{Z}_2^k}{\rho_k}\right)_+ \right) \right\} \right\| = 0.$$

Since $\{X^k\}$, $\{\overline{Z}_1^k\}$ and $\{\overline{Z}_2^k\}$ are bounded, we obtain

$$\left\| P_{\mathcal{S}}\left\{\mathcal{G}^T \left((L - \mathcal{G}(X^*))_+ - (\mathcal{G}(X^*) - U)_+ \right)\right\} \right\| = 0.$$

This implies that X^* is a stationary point of (3.16).

Theorem 2. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL and the sequence $\{\rho_k\}$ is bounded, then X^* is a solution to Problem* (1.1).

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that

$$\lim_{k\in\mathbb{K}} X^k = X^*, \quad \lim_{k\in\mathbb{K}} \rho_k = \rho^*, \quad \lim_{k\in\mathbb{K}} \overline{Z}_1^k = \overline{Z}_1^* \quad \text{and} \quad \lim_{k\in\mathbb{K}} \overline{Z}_2^k = \overline{Z}_2^*.$$

By Lemma 2, we have $L \le \mathcal{G}(X^*) \le U$. Since

$$\left\|P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\| < \varepsilon_k$$

holds for all $\varepsilon_k \downarrow 0$, we have

$$\left\|P_{\mathcal{S}}\left\{\nabla L_{\rho^*}(X^*, \overline{Z}_1^*, \overline{Z}_2^*)\right\}\right\| = 0. \tag{3.17}$$

Let

$$Y_1^* = \rho^* \left(L - \mathcal{G}(X^*) + \overline{Z}_1^*/\rho^*\right)_+ \quad \text{and} \quad Y_2^* = \rho^* \left(\mathcal{G}(X^*) - U + \overline{Z}_2^*/\rho^*\right)_+,$$

then $Y_1^* \ge \mathbf{0}$ and $Y_2^* \ge \mathbf{0}$, and, from (3.17), we have

$$P_{\mathcal{S}}\left\{\mathcal{A}^T\left(\mathcal{A}(X^*) - C\right) - \mathcal{G}^T(Y^* - Z^*)\right\} = \mathbf{0}.$$

Since $\{\rho_k\}$ is bounded, then there exists $k_0 \in \mathbb{N}$ such that (3.14) takes place for all $k \ge k_0$. Hence, we have

$$\lim_{k\to\infty} \{\mathcal{G}(X^k) - L,\ Z_1^k\}_- = \{\mathcal{G}(X^*) - L,\ Z_1^*\}_- = \mathbf{0}$$

and

$$\lim_{k\to\infty} \{U - \mathcal{G}(X^k),\ Z_2^k\}_- = \{U - \mathcal{G}(X^*),\ Z_2^*\}_- = \mathbf{0},$$

which imply that $\langle \mathcal{G}(X^*) - L,\ Z_1^*\rangle = 0$ and $\langle U - \mathcal{G}(X^*),\ Z_2^*\rangle = 0$. By the definition of Z_1^k, Z_2^k and Y_1^*, Y_2^* we know that $(Z_1^*)_{ij} > 0$ if and only if $(Y_1^*)_{ij} > 0$ and $(Z_2^*)_{ij} > 0$ if and only if $(Y_2^*)_{ij} > 0$ ($i = 1, 2, \ldots, l_2$, $j = 1, 2, \ldots, s_2$). So we have $\langle \mathcal{G}(X^*) - L, Y_1^*\rangle = 0$ and $\langle U - \mathcal{G}(X^*), Y_2^*\rangle = 0$. Hence X^* satisfies conditions (3.11). By Theorem 1, we know that X^* is a solution to Problem (1.1).

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4. Numerical examples

In this section, we first report some numerical results when Algorithm PHR-AL is implemented to solve Problem (1.1) with random data, and then we illustrate the applicability when the algorithm is applied to solve the model (2.10) in image restoration. All the tested algorithms were coded by MATLAB 7.8 (R2009a) and all our computational experiments were run on a personal computer with an Intel(R) Core i3 processor at 2.13 GHz with 2.00 GB of memory.

4.1. Tested with random data

In this example, we test the two linear operators as $\mathcal{A}(X) = A_1XB_1 + A_2XB_2$ and $\mathcal{G}(X) = E_1XF_1$, and $\mathcal{S}$ as the set of all real $m \times n$ rectangular centrosymmetric matrices[31].

Example 1. Given the matrices $A_1, B_1, A_2, B_2, E_1, F_1, C, L$ and U in Matlab style as follows:

$$\begin{aligned}
&A_1 = randn(l_1, m), \;\; B_1 = randn(n, s_1), \;\; A_2 = randn(l_1, m), \;\; B_2 = randn(n, s_1), \\
&E_1 = rand(l_2, m), \;\; F_1 = rand(n, s_2), \;\; C = A_1\overline{X}B_1 + A_2\overline{X}B_2, \\
&L = E_1\overline{X}F_1 - 10 * ones(l_2, s_2), \;\; U = E_1\overline{X}F_1 + 10 * ones(l_2, s_2),
\end{aligned}$$

where $\overline{X} = Z + S_mZS_n$ with $Z = rand(m, n)$. Matrices L, U and C are chosen in this way to guarantee that Problem (1.1) is solvable.

Note that the Algorithm PHR-AL involve an outer iteration and an inner iteration, the convergence stopping criterion of the outer iterations are all set to be $\varepsilon = 10^{-8}$, and the small tolerance ε_k in the inner iterations is set to

$$\varepsilon_0 = 10^0 \quad \text{and} \quad \varepsilon_k = \begin{cases} 0.1\varepsilon_{k-1} & \text{if } \varepsilon_{k-1} > \varepsilon, \\ \varepsilon_{k-1} & \text{if } \varepsilon_{k-1} < \varepsilon. \end{cases} \tag{4.18}$$

The largest number of the inner iteration is set to be 200. We consider the following two cases to be tested: (a) $l_1 \geq m$ and $s_1 \geq n$ and (b) $l_1 < m$ and $s_1 < n$.

Table 2: Numerical results for the case (a) $l_1 \geq m$ and $s_1 \geq n$ in Example 1.

l_1, m, n, s_1, l_2, s_2	CPU	$\frac{\lVert X^*-\overline{X}\rVert}{\lVert\overline{X}\rVert}$
10,10,10,10,10,10	0.1248	5.1294×10^{-11}
30,18,20,30,25,30	0.3588	3.4006×10^{-13}
50,50,50,50,50,50	3.4476	1.2540×10^{-12}
80,60,70,100,80,80	4.0404	6.7827×10^{-14}
100,100,100,100,100,100	13.3537	6.7580×10^{-14}
150,100,100,150,120,120	10.1401	4.8226×10^{-15}
150,150,150,150,150,150	44.2263	4.7307×10^{-14}
200,180,180,200,150,150	53.3367	1.2976×10^{-14}
250,250,250,250,200,200	161.7106	1.1052×10^{-13}

For case $l_1 \geq m$ and $s_1 \geq n$, Problem (1.1) has unique solution and the true solution is $\overline{X}$. Therefore in Table 2, we report the mean computing time in seconds and the mean relative error based on their average values of 10 repeated tests with randomly generated matrices A_1, B_1, A_2, B_2, E_1 and F_1 for each problem size. Here the relative error is defined as $Re = \frac{\lVert X^*-\overline{X}\rVert}{\lVert\overline{X}\rVert}$, where X^* is the estimated solution.

For case $l < n$ and $s < n$, as Problem(1.1) has multiple solutions, the algorithm is not guaranteed to converge to the solution $\overline{X}$, it is not meaningful to record the relative errors. In this case, we report the mean

Table 3: Numerical results for the case (b)$l_1 < m$ and $s_1 < n$ in Example 1.

l_1, m, n, s_1, l_2, s_2	CPU	$\|A_1XB_1 + A_2XB_2 - C\|$
6,10,10,6,10,10	0.1560	9.3373×10^{-9}
15,30,25,15,20,20	0.7644	1.4437×10^{-9}
30,60,75,35,50,50	4.2432	3.2598×10^{-10}
50,120,125,65,80,80	17.6749	2.6637×10^{-10}
50,200,200,50,100,100	43.9299	7.1933×10^{-11}
70,150,150,70,120,120	44.9595	1.7993×10^{-10}
100,200,200,100,150,150	132.8817	1.7718×10^{-10}
100,300,300,100,180,180	348.3970	5.1966×10^{-11}

computing time in seconds and the mean residual $\|A_1XB_1 + A_2XB_2 - C\|$ (see Table 3) based on 10 repeated tests with randomly generated matrices A, B, E and F for each problem size in each test.

4.2. Application to image restoration with some special symmetry pattern images

In this subsection, we test the efficiency when Algorithm PHR-AL is applied to solve the model (2.10) in image restoration. We only focus on some special symmetry pattern images. The original image is denoted by $\widehat{X}$ in each example and it consists of $m \times n$ grayscale pixel values in the range $[0, d]$ with $d = 255$ is the maximum possible pixel value of the image. Let $\hat{x} = vec(\widehat{X})$ denotes the vector obtained by stacking the columns of $\widehat{X}$ and H represents the blurring matrix. The vector $\hat{g} = H\hat{x}$ represents the associated blurred and noise-free image. In our tests, similarly to [24], we generated a blurred and noisy image g by

$$g = \hat{g} + \mathbf{n}_0 \times \sigma_{\hat{x}} \times 10^{-\frac{SNR}{20}},$$

where $\mathbf{n}_0$ is a random vector noise with a zero mean and a variance equal to one, and SNR is the signal to noise ratio defined by

$$SNR = 10\log_{10}\Big(\frac{\sigma_{\hat{x}}^2}{\sigma_{\mathbf{n}}^2}\Big),$$

where $\sigma_{\hat{x}}^2$ and $\sigma_{\mathbf{n}}^2$ are the variance of the noise and the original image, respectively. The performance of the Algorithm PHR-AL and its comparison are evaluated by the peak signal-to-noise ratio (PSNR) in decible (dB):

$$PSNR(X) = 10\log_{10}\Big(\frac{d^2mn}{\|\hat{x} - x\|_2^2}\Big) = 10\log_{10}\Big(\frac{d^2mn}{\|\widehat{X} - X\|^2}\Big).$$

In all the tests, the largest number of the involved inner iteration(Algorithm SPG) in the Algorithm PHR-AL is set to be 20. The algorithm started with the degraded images and terminated when the relative difference between the successive iterates of the restored image satisfy

$$R_{error} = \frac{\|X^{k+1} - X^k\|}{\|X^k\|} \le 0.5 \times 10^{-2}.$$

Example 2. In the first example, we consider the *"butterfly"* original image of size 192×254 and is shown on the left side of Figure 1. The original image has perfectly mirror-symmetry[32], that is, the pixel value

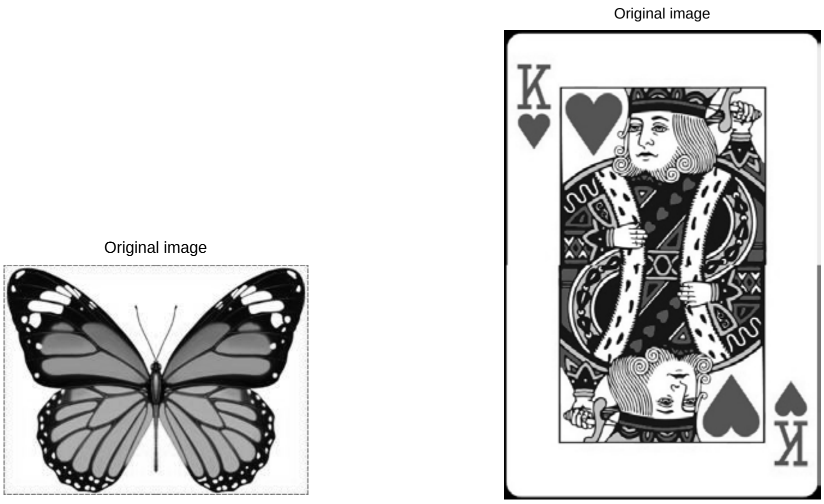

Figure 1: Original images. Left: "*Butterfly*"(mirror-symmetric). Right: "*PlayCard-K-Heart*" (centro-symmetric).

matrix $\widehat{X}$ can be expressed as $\widehat{X} = (X_L, X_L S_n)$, where X_L is the left half of the matrix $\widehat{X}$. Actually, we have $\|\widehat{X} - \mathcal{P}_{\mathcal{S}}(\widehat{X})\| = 0$, where $\mathcal{S}$ is the set of all real 192×254 column mirror-symmetry matrices and

$$\mathcal{P}_{\mathcal{S}}(X) = \left(\frac{X_L + X_R S_n}{2}, \ \frac{X_L S_n + X_R}{2}\right), \quad \forall X \in \mathbb{R}^{192\times 254}$$

where X_R is the left half and the right half of X. The blurring matrix H is chosen to be $H = H_1 \otimes H_2 \in \mathbb{R}^{192^2 \times 254^2}$, where $H_1 = [h_{ij}^{(1)}] \in \mathbb{R}^{192\times 192}$ and $H_2 = [h_{ij}^{(2)}] \in \mathbb{R}^{254\times 254}$ are the Toeplitz matrices whose entries are given by

$$h_{ij}^{(1)} = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(i-j)^2}{2\sigma^2}\right), & |i-j| \le r, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h_{ij}^{(2)} = \begin{cases} \frac{1}{2r-1}, & |i-j| \le r, \\ 0, & \text{otherwise.} \end{cases}$$

In this example we choose the band $r = 3$ and the variance $\sigma = 0.4$. A random Gaussian noise, with $SNR = 15dB$, was added to produce a blurred and noisy image G with $PSNR(G) = 8.1411$. The blurred and noisy image is shown on the left side of Figure 4. The restoration of the image from the degraded image is obtained by solving the minimization problem (2.10) using the PHR-AL algorithm. The regularization matrix T is chosen to be $T = T_1 \otimes T_2 \in \mathbb{R}^{192^2 \times 254^2}$, where $T_1 = I_{192}$ and T_2 is the tridiagonal matrix, of size 254×254, generated by vector $(1, 2, 1)$. The optimal value of the parameter $\lambda = 0.015$ was obtained by using the GCV method. The corresponding GCV curve is plotted on the right side of Figure 2.

The restored image obtained by using Algorithm PHR-AL is given on the left of Figure 4, the relative error was $Re(X) = 1.2521 \times 10^{-1}$ with $PSNR(X) = 21.0231$, and the iterations are terminated after 3 iterations with a cpu time of 13.9309 s. Table 1 reports on more results for three levels of noise corresponding to different $SNR = 5, 10, 15$ and to different values of $\sigma = 0.35, 0.55, 0.85$ given in the definition of the blurring matrices H_1 and H_2 in Example 2.

Example 3. In the second example, the original image is the "*PlayCard-K-Heart*" image of size 628×423 and is shown on the right side of Figure 1. The original image is centrosymmetric, that is, the pixel value

Table 4: *Results for Example 3.*

σ	$SNR(dB)$	λ_{opt}	$PSNR(G)(dB)$	$PSNR(X)$(dB)	$Re(X)$	CPU-times(s)
0.35	5	0.036	5.3075	19.6357	1.4690×10^{-1}	23.4002
	10	0.025	6.0042	20.9344	1.2650×10^{-1}	17.8621
	15	0.017	6.4097	21.3394	1.2073×10^{-1}	18.0337
	20	0.011	6.6397	21.6077	1.1706×10^{-1}	18.3145
	25	0.007	6.7709	21.9395	1.1267×10^{-1}	19.5781
0.55	5	0.036	8.3142	18.7410	1.6284×10^{-1}	29.6090
	10	0.025	9.3290	21.1153	1.2389×10^{-1}	40.3419
	15	0.018	9.9286	21.8547	1.1378×10^{-1}	38.4386
	20	0.012	10.2655	21.9397	1.1267×10^{-1}	28.2830
	25	0.008	10.4569	21.1417	1.2351×10^{-1}	18.8137
0.85	5	0.035	8.4387	18.5387	1.6667×10^{-1}	38.4542
	10	0.026	9.4712	20.7428	1.2932×10^{-1}	39.1875
	15	0.019	10.0763	20.9952	1.2561×10^{-1}	27.9086
	20	0.014	10.4170	20.5296	1.3253×10^{-1}	12.9949
	25	0.010	10.6154	20.7946	1.2855×10^{-1}	18.8137

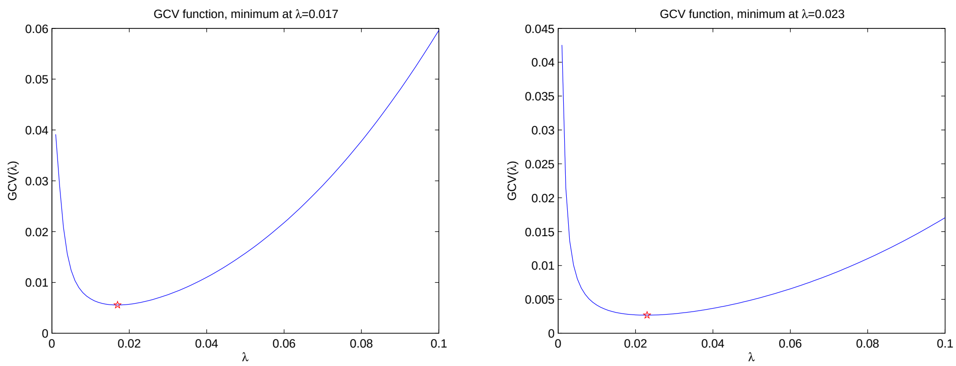

Figure 2: The GCV curve for the Example 2 with the optimal value of $\lambda = 0.017$ (left) and the GCV curve for the Example 3 with the optimal value of $\lambda = 0.023$.

Figure 3: The blurred and noisy image (left) with $PSNR(G) = 8.1411$, $r = 3$ and $\sigma = 0.45$ and the restored image (right) with $PSNR(X) = 21.0231$ and $Re(X) = 1.2521 \times 10^{-1}$.

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matrix $\widehat{X}$ satisfies $\widehat{X} = S_{628}\widehat{X}S_{423}$. Actually, we have $\|\widehat{X}-P_{\mathcal{S}}(\widehat{X})\| = 0$, where $\mathcal{S}$ is the set of all real 628×423 rectangle centrosymmetry matrices and $P_{\mathcal{S}} = \frac{1}{2}(X + S_{628}XS_{423})$ for any $X \in \mathbb{R}^{628\times423}$. The blurring matrix H is chosen to be $H = H_1 \otimes H_2 \in \mathbb{R}^{256^2\times256^2}$, where $H_1 = I_{628}$ is the identity matrix and $H_2 = [h_{ij}^{(2)}]$ is the Toeplitz matrices of dimension 423×423 given by

$$h_{ij}^{(2)} = \begin{cases} \frac{1}{2r-1}, & |i-j| \le r, \\ 0, & \text{otherwise}. \end{cases}$$

The blurring matrix H models a uniform blur. The regularization matrix T is chosen to be $T = T_1 \otimes T_2 \in$

Figure 4: The blurred and noisy image (left) with $PSNR(G) = 8.0481$, $r = 3$ and $\sigma = 0.45$ and the restored image (right) with $PSNR(X) = 20.1459$ and $Re(X) = 1.5784 \times 10^{-1}$.

$\mathbb{R}^{256^2\times256^2}$, where T_1 and T_2 are similar to the ones given in Example 2. In this example we set $r = 3$ and a random Gaussian noise, with $SNR = 15dB$, was added to produce a blurred and noisy image G with $PSNR(G) = 8.0481$. The obtained image is shown on the middle of Figure 2. The optimal value of the parameter $\lambda = 0.023$ was obtained by using the GCV method. The corresponding GCV curve is plotted on the right side of Figure 2.

The restored image obtained by using our proposed Algorithm PHR-AL is also denoted by X and it is given on the right side of Figure 4. The relative error was $Re(X) = 1.5784 \times 10^{-1}$ with the $PSNR(X) = 20.1459$. The iterations are terminated after 5 iterations with a cpu time of 86.9699s.

5. Conclusion

In this paper, we consider solving a class of inequality constrained matrix-form minimization problems, whose various simplified versions have been studied extensively. These matrix-form minimization problems problem can be transformed into the convex linearly constrained quadratic programming in the vector-form by using the vec operator $vec(.)$ and the Kronecker produc $\otimes$. However, using this transformation will destroy the preindicated linear structure of the unknown matrix and will increase computational complexity and storage requirement. In this paper we will consider the problem from a general point of view and

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directly from the perspective of matrices. We propose, analyze and test a matrix-form iteration algorithm framework with the augmented Lagrangian method for solving this problem and its reduced versions which are applicable in image restoration. The numerical results, including when the algorithm is tested with some randomly generated data and on some image restoration problems with special symmetry pattern images, illustrate the effectiveness of the proposed algorithm.

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Some properties on non-admissible and admissible functions sharing some sets in the unit disc *

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Abstract

In this paper, we deal with the uniqueness problem of two non-admissible functions sharing some values and sets in the unit disc, and also investigate the problem on an admissible function and a non-admissible function sharing some values and sets. Some theorems of this paper improve the results given by Fang. In addition, the results in this paper analogous version of the uniqueness theorems of meromorphic functions sharing some sets on the whole complex plane which given by Yi and Cao.

Key words: uniqueness; meromorphic function; admissible; non-admissible.

Mathematical Subject Classification (2010): Primary 30D 35.

1 Introduction and main results

We should assume that reader is familiar with the basic results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see Hayman [6] , Yang [14] and Yi and Yang [18]). For a meromorphic function f, we use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set of finite logarithmic measure, and use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}} := \mathbb{C}\bigcup\{\infty\}$ to denote the extended complex plane, and $\mathbb{D} = \{z : |z| < 1\}$ to denote the unit disc.

R. Nevanlinna [10] proved the following well-known theorems.

Theorem 1.1 (see [10]) *If f and g are two non-constant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 IM in $\mathbb{C}$, then $f(z) \equiv g(z)$.*

After this work, the uniqueness of meromorphic functions with shared sets and values attracted many investigations (see [18]). Moreover, the uniqueness theory of meromorphic functions is an important subject in the value distribution theory. In this paper, we mainly investigate the uniqueness of meromorphic functions with slow growth sharing some sets in the unit disc.

We firstly introduce the following basic notations and definitions of meromorphic functions in $\mathbb{D}$(see [2, 4, 7, 12, 8, 13, 22]).

Definition 1.1 *(see [12]). Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to1^-} T(r, f) = \infty$. Then*

$$D(f) := \limsup_{r\to1^-} \frac{T(r, f)}{-\log(1 - r)}$$

is called the (upper) index of inadmissibility of f. If $D(f) = \infty$, f is called admissible.

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Definition 1.2 *(see [12]). Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f)=\infty$. Then*

$$\rho(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}$$

is called the order (of growth) of f.

The Second Main Theorem for admissible functions (see [12, Theorem 3]) is very important in studying the uniqueness of two admissible functions in the unit disc $\mathbb{D}$, which was proved by in 2005.

Theorem 1.2 (see [12, Theorem 3]). *Let f be an admissible meromorphic function in $\mathbb{D}$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

where $E \subset (0,1)$ is a possibly occurring exceptional set with $\int_E \frac{dr}{1-r} < \infty$. If the order of f is finite, the remainder $S(r,f)$ is a $O\left(\log\frac{1}{1-r}\right)$ without any exceptional set.

In 2005, Titzhoff [12] also obtained the five values theorem for admissible functions in the unit disc $\mathbb{D}$ as follows.

Theorem 1.3 (see [5, 12]). *If two admissible functions f, g share five distinct values, then $f \equiv g$.*

From Theorem 1.2(see [12, Theorem 3]), we can easily obtain a lot of theorems similar to meromorphic functions in the complex plane. In 1999, Fang [5] investigated the uniqueness of admissible functions sharing two sets and three sets and obtained a series of theorems. In 2015, Xu, Yang and Cao [15] investigated the problem on shared values of admissible function and non-admissible function, and obtained some interesting results. Inspired by Xu, Yang and Cao [15] and Fang[5], we further study the problem on shared-sets of admissible function and non-admissible function in the unit disc.

The following theorem also plays a very important role in studies non-admissible functions sharing some sets in this paper.

Theorem 1.4 (see [12, Theorem 2]). *Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f) = \infty$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + \log\frac{1}{1-r} + S(r,f).$$

Remark 1.1 *In contrast to admissible functions, the term $\log\frac{1}{1-r}$ in Theorem 1.4 does not necessarily enter the remainder $S(r,f)$ because the non-admissible function f may have $T(r,f) = O\left(\log\frac{1}{1-r}\right)$.*

Remark 1.2 *We can see that $S(r,f) = o\left(\log\frac{1}{1-r}\right)$ holds in Theorem 1.4 without a possible exception set when $0 < D(f) < \infty$.*

The following lemma for non-admissible functions in the unit disc is used in this paper.

Lemma 1.1 *(see [15]). Let $f(z)$ be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f)=\infty$, $a_j(j=1,2,\ldots,q)$ be q distinct complex numbers, and $k_j(j=1,2,\ldots,q)$ be positive integers or ∞. If f is a non-admissible function, then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} \frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f-a_j}\right) + \sum_{j=1}^{q}\frac{1}{k_j+1}N\left(r,\frac{1}{f-a_j}\right) + \log\frac{1}{1-r} + S(r,f),$$

and

$$\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T(r,f) \le \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f-a_j}\right) + \log\frac{1}{1-r} + S(r,f),$$

where $\overline{n}_{k)}(r,\frac{1}{f-a})$ is used to denote the zeros of $f-a$ in $|z|\le r$, whose multiplicities are no greater than k and are counted only once, $\overline{N}_{k)}(r,\frac{1}{f-a})$ is the corresponding counting functions, and $\frac{k_j}{k_j+1}=1, \overline{N}_{k_j)}(r,\frac{1}{f-a_j})=\overline{N}(r,\frac{1}{f-a_j})$ and $\frac{1}{k_j+1}=0$ if $k_j=\infty$, $S(r,f)$ is stated as in Theorem 1.2.

The main purpose of this paper is to deal with the problem of two non-admissible functions sharing some sets, and an admissible function sharing some sets with an non-admissible function. *Section 2, the uniqueness of two non-admissible functions sharing some sets in $\mathbb{D}$ are investigated and some results showed that the number and weight of sharing sets is related with the index of inadmissibility of functions in $\mathbb{D}$. In section 3, the problem of an admissible function and a non-admissible function sharing some sets is studied, and one of those results shows that admissible function and non-admissible function can share at most five distinct values with reduced weighted 1.*

2 The uniqueness and sharing sets of non-admissible functions in the unit disc

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\mathbb{X}\subseteq\mathbb{C}$. Define

$$E(S,\mathbb{D},f) = \bigcup_{a\in S}\{z\in\mathbb{D}|f_a(z)=0,\quad \textit{counting}\quad \textit{multiplicities}\},$$

$$\overline{E}(S,\mathbb{D},f) = \bigcup_{a\in S}\{z\in\mathbb{D}|f_a(z)=0,\quad \textit{ignoring}\quad \textit{multiplicities}\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

For two non-constant meromorphic functions f,g, we say f and g share the set S CM(counting multiplicities) in $\mathbb{D}$ if $E(S,\mathbb{D},f)=E(S,\mathbb{D},g)$; we say f and g share the set S IM(ignoring multiplicities) in $\mathbb{D}$ if $\overline{E}(S,\mathbb{D},f)=\overline{E}(S,\mathbb{D},g)$. In particular, as $S=\{a\}$ and $a\in\widehat{\mathbb{C}}$, we say f and g share the value a CM in $\mathbb{D}$ if $E(a,\mathbb{D},f)=E(a,\mathbb{D},g)$, and we say f and g share the value a IM in $\mathbb{D}$ if $\overline{E}(a,\mathbb{D},f)=\overline{E}(a,\mathbb{D},g)$. We use $\overline{E}_{k)}(a,\mathbb{D},f)$ to denote the set of zeros of $f-a$ in $\mathbb{D}$, with multiplicities no greater than k, in which each zero counted only once. We say that $f(z)$ and $g(z)$ share the value a with reduced weight k in $\mathbb{D}$, if $\overline{E}_{k)}(a,\mathbb{D},f)=\overline{E}_{k)}(a,\mathbb{D},g)$. If $\mathbb{D}=\mathbb{C}$, we have the simple notation as before, $E(S,f),\overline{E}(S,f),\overline{E}_{k)}(a,f)$ and so on(see [18]).

The deficiency of $a\in\widehat{\mathbb{C}}$ with respect to a meromorphic function f on the unit disc $\mathbb{D}$ is defined by

$$\delta(a,f)=\delta(0,f-a)=\liminf_{r\to 1^-}\frac{m(r,\frac{1}{f-a})}{T(r,f)} = 1-\limsup_{r\to 1^-}\frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

and the reduced deficiency by

$$\Theta(a,f)=\Theta(0,f-a)=1-\limsup_{r\to 1^-}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

We now show our main theorems. The first theorem can be called five values theorem of non-admissible functions.

Theorem 2.1 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $1<D(f_1),D(f_2)<\infty$, and f_1,f_2 share $a_j (j=1,2,3,4,5)$ IM. Then $f_1(z)\equiv f_2(z)$.*

Remark 2.1 *From Theorem 2.1, we can get that $f_1(z)\equiv f_2(z)$ if f_1,f_2 share five distinct values and $D(f_1),D(f_2)>1$. However, the conclusion holds in Theorem 1.3 under the condition which f_1,f_2 are admissible functions, that is, $D(f_1)=\infty$, and $D(f_2)=\infty$. Thus, we can see that Theorem 2.1 is a greatly improvement of Theorem 1.3.*

In order to prove Theorem 2.1, we will prove the following general results of two non-admissible functions sharing some sets.

Theorem 2.2 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and $q>2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}$, where $[x]$ denotes the largest integer less than or equal to x. Let k_j $(j=1,2,\ldots,q)$ be positive integers or ∞ satisfying

$$k_1\geq k_2\geq\cdots\geq k_q \tag{1}$$

and

$$\overline{E}_{k_j)}(S_j,\mathbb{D},f_1)=\overline{E}_{k_j)}(S_j,\mathbb{D},f_2),\quad (j=1,2,\ldots,q). \tag{2}$$

Furthermore, let

$$\Theta(f_i)=\sum_a\Theta(0,f_i-a)-\sum_{j=1}^{q}\sum_{s=0}^{l-1}\Theta(0,f_i-(a_j+sb)),(i=1,2),$$

and

$$\begin{aligned}A_1 &= \frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(a_j+sb))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(a_j+sb))}{k_j+1}\\ &\quad+\frac{(lm-3l+1)k_m}{k_m+1}-\frac{(2l-1)k_n}{k_n+1}+\Theta(f_1)-2,\end{aligned}$$

$$\begin{aligned}A_2 &= \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(a_j+sb))}{k_j+1}\\ &\quad+\frac{(ln-3l+1)k_n}{k_n+1}-\frac{(2l-1)k_m}{k_m+1}+\Theta(f_2)-2,\end{aligned}$$

where m and n are positive integers in $\{1,2,\ldots,q\}$ and a is an arbitrary complex number or ∞. If

$$\min\{A_1,A_2\}\geq\frac{2}{D(f_1)+D(f_2)},\quad and\quad \max\{A_1,A_2\}>\frac{2}{D(f_1)+D(f_2)}. \tag{3}$$

Then $f_1(z)\equiv f_2(z)$.

By letting $l = 1$, $q = 5$ and $k_1 = k_2 = \cdots = k_5 = \infty$ in Theorem 2.2, we can get Theorem 2.1 easily. Now, we start to prove Theorem 2.2 as follows.

Proof of Theorem 2.2: Suppose that $f_1(z) \not\equiv f_2(z)$. From the second fundamental theorem in the unit disc (Theorem 1.4) we have

$$\begin{aligned}(ql+p-2)T(r,f_1) &< \sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\sum_{k=1}^{p}\overline{N}\left(r,\frac{1}{f_1-d_k}\right)\\ &\quad+\log\frac{1}{1-r}+S(r,f_1).\end{aligned}$$

By definition we have

$$\overline{N}\left(r,\frac{1}{f_1-d_k}\right)<(1-\Theta(0,f_1-d_k))\,T(r,f_1)+S(r,f_1).$$

From Lemma 1.1 and the definition of deficiency, it follows that for $s \in \{0,1,\ldots,l-1\}$

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ \le\ & \frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\frac{1}{k_j+1}N\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ <\ & \frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\frac{1}{k_j+1}\left(1-\delta(0,f_1-(a_j+sb))\right)T(r,f_1)\\ &+S(r,f_1).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&(ql+p-2)T(r,f_1)\\ <\ & \left\{\sum_{k=1}^{p}(1-\Theta(0,f_1-d_k))\right\}T(r,f_1)+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})\\ &+\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1}{k_j+1}\left(1-\delta(0,f_1-(a_j+sb))\right)\right\}T(r,f_1)+\log\frac{1}{1-r}+S(r,f_1).\end{aligned}$$

Since $\Theta(0, f-a) \ge 0$ for any meromorphic function f and any complex number $a \in \widehat{\mathbb{C}}$. Without loss of generality, we assume that there exist infinitely many d such that $\Theta(0, f_1 - d) > 0$ and $d \notin \{a_j + sb : j = 1, 2, \ldots, q \text{ and } s = 0, 1, \ldots, l-1\}$. We denote them by d_k $(k = 1, 2, \ldots, \infty)$. Obviously, $\Theta(f_1) = \Sigma_{k=1}^{\infty}\Theta(0, f_1 - d_k)$. Thus there exits a p such that $\Sigma_{k=1}^{p}\Theta(0, f_1 - d_k) > \Theta(f_1) - \varepsilon$ holds for any given ε (> 0). Noting that

$$1\ge\frac{k_1}{k_1+1}\ge\frac{k_2}{k_2+1}\ge\cdots\ge\frac{k_q}{k_q+1}\ge\frac{1}{2},$$

we can deduce that

$$\begin{aligned}&(ql+p-2)T(r,f_1)\\ <\ & (p-\Theta(f_1)+\varepsilon)\,T(r,f_1)+\frac{k_m}{k_m+1}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ &+\left\{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\left(\frac{k_j}{k_j+1}-\frac{k_m}{k_m+1}\right)\left(1-\delta(0,f_1-(a_j+sb))\right)\right\}T(r,f_1)\\ &+\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1-\delta(0,f_1-(a_j+sb))}{k_j+1}\right\}T(r,f_1)+\log\frac{1}{1-r},\end{aligned}$$

namely,

$$\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})$$
$$+\log\frac{1}{1-r},$$

where

$$B_1=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(a_j+sb))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(a_j+sb))}{k_j+1}+\Theta(f_1)-2.$$

By a similar discussion as above, we also have

$$\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_2-(a_j+sb)}\right)+\log\frac{1}{1-r},$$

where

$$B_2=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(a_j+sb))}{k_j+1}+\Theta(f_2)-2.$$

Hence

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)\\ <\ &\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(a_j+sb)})\\ &+2\log\frac{1}{1-r}.\end{aligned}$$

We now assert that $f_1(z)-f_2(z)\not\equiv sb$, $s=1,2,\ldots,l-1$. Otherwise, we get that a_j $(j=1,2,\ldots,q)$ are the Picard exceptional values of f_1, and that $a_j+(l-1)b$ $(j=1,2,\ldots,q)$ are the Picard exceptional values of f_2. By $q>2+\frac{1}{D(f_1)}$ and Theorem 1.4, we get a contradiction. Similarly, we have $f_2(z)-f_1(z)\not\equiv sb$, $s=1,2,\ldots,l-1$.

By condition (2) and the first fundamental theorem, we have

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ &\le\overline{N}\left(r,\frac{1}{f_1-f_2}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_1-f_2-sb}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_2-f_1-sb}\right)\\ &\le(2l-1)(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_2-(a_j+sb)}\right)\\ &\le\overline{N}\left(r,\frac{1}{f_1-f_2}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_1-f_2-sb}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_2-f_1-sb}\right)\\ &\le(2l-1)(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)\\ <\quad &(2l-1)\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T(r,f_1)+T(r,f_2))+2\log\frac{1}{1-r},\end{aligned}$$

namely,

$$(A_1-\varepsilon)\,T(r,f_1)+(A_2-\varepsilon)\,T(r,f_2)\le 2\log\frac{1}{1-r}. \tag{4}$$

Since $0<D(f_1),D(f_2)<\infty$, we have $S(r,f_1)=o\left(\log\frac{1}{1-r}\right), S(r,f_2)=o\left(\log\frac{1}{1-r}\right)$. And from the definition of index, for any ε satisfying

$$0<2\varepsilon<\min\left\{D(f_1),D(f_2),\max\{A_1,A_2\}-\frac{2}{D(f_1)+D(f_2)}\right\}, \tag{5}$$

there exists a sequence $\{r_t\}\to 1^-$ such that

$$T(r_t,f_1)>(D(f_1)-\varepsilon)\log\frac{1}{1-r_t},\quad T(r_t,f_2)>(D(f_2)-\varepsilon)\log\frac{1}{1-r_t}, \tag{6}$$

for all $t\to\infty$. From (4)-(6), we have

$$[(D(f_1)-\varepsilon)(A_1-\varepsilon)+(D(f_2)-\varepsilon)(A_2-\varepsilon)-2]\log\frac{1}{1-r_t}<o\left(\log\frac{1}{1-r_t}\right). \tag{7}$$

From (7) and ε being arbitrary, the above inequality contradicts to (3). Therefore, the proof of Theorem 2.2 is completed.

We can get the following corollaries from Theorem 2.2.

Corollary 2.1 *Let k_j $(j=1,2,\ldots,q)$ be positive integers or ∞ satisfying (1), and let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$ and (2). Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and $q>2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}$, where $[x]$ denotes the largest integer less than or equal to x. If

$$\sum_{j=3}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}+\frac{(2-2l)k_3}{k_3+1}>2+\frac{2}{D(f_1)+D(f_2)}.$$

Then $f_1(z)\equiv f_2(z)$.

Proof: Let $m=n=3$. Noting that $\Theta(f_i)\ge 0$ and $\delta(0,f_i-(a_j+sb))\ge 0$ for $j=1,2,\ldots,q$ and $i=1,2$, one can deduce from Theorem 2.2 that Corollary 2.1 follows. □

The following corollary is an analog of a result due to H.-X. Yi (Theorem 10.7 in [18], see also [21]) on $\mathbb{C}$.

Corollary 2.2 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and

$$q>\max\left\{4+\frac{2}{(D(f_1)+D(f_2))l},2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}\right\}.$$

If $\overline{E}(S_j,\mathbb{D},f_1)=\overline{E}(S_j,\mathbb{D},f_2),(j=1,2,\ldots,q)$. Then $f_1(z)\equiv f_2(z)$.

Proof: Let $k_1 = k_2 = \ldots = k_q = \infty$. One can deduce from Corollary 2.1 that Corollary 2.2 follows immediately. $\square$

Let $l = 1$. Then it is easily derived the following corollary from Corollary 2.1, which is an analog of the Corollary of Theorem 3.15 in [18].

Corollary 2.3 *Let a_j $(j = 1, 2, \ldots, q)$ be q distinct complex numbers in $\widehat{\mathbb{C}}$, and k_j $(j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1), and let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$ and $\overline{E}_{k_j)}(a_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(a_j, \mathbb{D}, f_2)$. Set $D := \min\{D(f_1),\ D(f_2)\}$. Then*

(i) if $D > 1$, $q = 7$ and $k_7 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(ii) if $D > 1$, $q = 6$ and $k_6 \geq 4$, then $f_1(z) \equiv f_2(z)$;
(iii) if $D > 2$ and $q = 7$, then $f_1(z) \equiv f_2(z)$;
(iv) if $D > 3$, $q = 6$ and $k_3 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(v) if $D > 6$, $q = 5$, $k_3 \geq 3$ and $k_5 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(vi) if $D > 10$, $q = 5$ and $k_4 \geq 4$, then $f_1(z) \equiv f_2(z)$;
(vii) if $D > 12$, $q = 5$, $k_3 \geq 5$ and $k_4 \geq 3$, then $f_1(z) \equiv f_2(z)$;
(viii) if $D > 42$, $q = 5$, $k_3 \geq 6$ and $k_4 \geq 2$, then $f_1(z) \equiv f_2(z)$.

We now state another main theorem.

Theorem 2.3 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\}, \quad j = 1, 2, \ldots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \ldots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$ and $q > 2 + \max\left\{\left[\frac{1}{D(f_1)}\right], \left[\frac{1}{D(f_2)}\right]\right\}$. Let k_j $(j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1), and

$$\overline{E}_{k_j)}(S_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(S_j, \mathbb{D}, f_2), \quad (j = 1, 2, \ldots, q). \tag{8}$$

Furthermore, let

$$\Theta(f_i) = \sum_a \Theta(0, f_i - a) - \sum_{j=1}^{q} \sum_{s=0}^{l-1} \Theta(0, f_i - (c + a_j w^s)), (i = 1, 2),$$

and

$$\begin{aligned} A_3 &= \frac{\sum_{j=1}^{m-1} \sum_{s=0}^{l-1} \delta(0, f_1 - (c + a_j w^s))}{k_m + 1} + \sum_{j=m}^{q} \sum_{s=0}^{l-1} \frac{k_j + \delta(0, f_1 - (c + a_j w^s))}{k_j + 1} \\ &\quad + \frac{l(m-2)k_m}{k_m + 1} - \frac{lk_n}{k_n + 1} + \Theta(f_1) - 2, \end{aligned}$$

$$\begin{aligned} A_4 &= \frac{\sum_{j=1}^{n-1} \sum_{s=0}^{l-1} \delta(0, f_2 - (c + a_j w^s))}{k_n + 1} + \sum_{j=n}^{q} \sum_{s=0}^{l-1} \frac{k_j + \delta(0, f_2 - (c + a_j w^s))}{k_j + 1} \\ &\quad + \frac{l(n-2)k_n}{k_n + 1} - \frac{lk_m}{k_m + 1} + \Theta(f_2) - 2, \end{aligned}$$

where m and n are positive integers in $\{1, 2, \ldots, q\}$ and a is an arbitrary complex number or ∞. If

$$\min\{A_3, A_4\} \geq \frac{2}{D(f_1) + D(f_2)}, \quad \text{and} \quad \max\{A_3, A_4\} > \frac{2}{D(f_1) + D(f_2)}. \tag{9}$$

Then $(f_1(z) - c)^l \equiv (f_1(z) - c)^l$.

Proof: We assume that $(f_1(z)-c)^l \not\equiv (f_2(z)-c)^l$. Without loss of generality, we assume that there exist infinitely many d such that $\Theta(0, f_1-d)>0$ and $d \notin \{c+a_jw^s : j=1,2,\ldots,q \text{ and } s=0,1,\ldots,l-1\}$. We denote them by d_k $(k=1,2,\ldots,\infty)$. Obviously, $\Theta(f_1)=\sum_{k=1}^{\infty}\Theta(0,f_1-d_k)$. Thus there exits a p such that $\sum_{k=1}^{p}\Theta(0,f_1-d_k)>\Theta(f_1)-\varepsilon$ holds for any given ε (>0).

Using a similar discussion as in the proof of Theorem 2.2, we obtain

$$\begin{aligned}
&\left(\frac{l(m-1)k_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_4-\varepsilon\right)T(r,f_2)\\
<\ &\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(c+a_jw^s)})\\
&+2\log\frac{1}{1-r},
\end{aligned}$$

where

$$B_3=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(c+a_jw^s))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(c+a_jw^s))}{k_j+1}+\Theta(f_1)-2.$$

$$B_4=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(c+a_jw^s))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(c+a_jw^s))}{k_j+1}+\Theta(f_2)-2.$$

Furthermore, from condition (8) and the first fundamental theorem, we have

$$\begin{aligned}
\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)}) &< \overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l})\\
&\le l(T(r,f_1)+T(r,f_2))+O(1).
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(c+a_jw^s)}) &< \overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l})\\
&\le l(T(r,f_1)+T(r,f_2))+O(1).
\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}
&\left(\frac{l(m-1)k_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_4-\varepsilon\right)T(r,f_2)\\
<\ &l\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T(r,f_1)+T(r,f_2))+2\log\frac{1}{1-r},
\end{aligned}$$

namely,

$$(A_3-\varepsilon)\,T(r,f_1)+(A_4-\varepsilon)\,T(r,f_2)<2\log\frac{1}{1-r}. \tag{10}$$

Since $0<D(f_1),D(f_2)<\infty$, we have $S(r,f_1)=o\left(\log\frac{1}{1-r}\right), S(r,f_2)=o\left(\log\frac{1}{1-r}\right)$. And from the definition of index, for any ε satisfying

$$0<2\varepsilon<\min\left\{D(f_1),D(f_2),\max\{A_3,A_4\}-\frac{2}{D(f_1)+D(f_2)}\right\}, \tag{11}$$

there exists a sequence $\{r_t\} \to 1^-$ such that

$$T(r_t, f_1) > (D(f_1) - \varepsilon) \log \frac{1}{1 - r_t}, \quad T(r_t, f_2) > (D(f_2) - \varepsilon) \log \frac{1}{1 - r_t}, \tag{12}$$

for all $t \to \infty$. From (10)-(12), we have

$$[(D(f_1) - \varepsilon)(A_3 - \varepsilon) + (D(f_2) - \varepsilon)(A_4 - \varepsilon) - 2] \log \frac{1}{1 - r_t} < o\left(\log \frac{1}{1 - r_t}\right). \tag{13}$$

From (13) and ε being arbitrary, the above inequality contradicts to (9).

Therefore, the proof of Theorem 2.3 is completed. □

We have an analog of a result due to H.-X. Yi (Theorem 10.8 in [18], see also [21]).

Corollary 2.4 *let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\}, \quad j = 1, 2, \ldots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \ldots, q)$, $q > 2 + \frac{2}{l} + \frac{2}{D(f_1)+D(f_2)}$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. If $\overline{E}(S_j, \mathbb{D}, f_1) = \overline{E}(S_j, \mathbb{D}, f_2)$ for $j = 1, 2, \ldots, q$, then $(f_1(z) - c)^l \equiv (f_2(z) - c)^l$.

Proof: Let $m = n = 1$ and $k_1 = k_2 = \ldots = \infty$. Noting that $\Theta(f_i) \geq 0$ and $\delta(0, f_i - (a_j + sb)) \geq 0$ for $j = 1, 2, \ldots, q$ and $i = 1, 2$, Then Corollary 2.4 follows immediately from Theorem 2.2. □

3 The problem of sharing sets of admissible function and non-admissible function in the unit disc

We now show that an admissible function can share sufficiently many sets concerning multiple values with another non-admissible function as follows.

Theorem 3.1 *If f_1 is admissible and f_2 is a non-admissible satisfying $\lim_{r\to 1^-} T(r, f_2) = \infty$, $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers, and let $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1). Then*

$$\overline{E}_{k_j)}(a_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(a_j, \mathbb{D}, f_2), \quad (j = 1, 2, \ldots, q).$$

and

$$\sum_{j=m+1}^{q} \frac{k_j}{k_j + 1} + \frac{(m-1)k_m}{k_m + 1} - 2 > 0$$

do not hold at same time.

Theorem 3.2 *If f_1 is admissible and f_2 is a non-admissible satisfying $\lim_{r\to 1^-} T(r, f_2) = \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\}, \quad j = 1, 2, \ldots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \ldots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. Then $\overline{E}(S_j, \mathbb{D}, f_1) = \overline{E}(S_j, \mathbb{D}, f_2)$ for $j = 1, 2, \ldots, q$, and $q > 1 + \frac{2}{l}$ can not hold at the same time.

To prove the above theorems, we require the following lemmas.

Lemma 3.1 (see [12, Lemma 1]). *Let $f(z), g(z)$ satisfy $\lim_{r\to 1^-} T(r, f) = \infty$ and $\lim_{r\to 1^-} T(r, g) = \infty$. If there is a $K \in (0, \infty)$ with*

$$T(r, f) \leq KT(r, g) + S(r, f) + S(r, g),$$

then each $S(r, f)$ is also an $S(r, g)$.

Lemma 3.2 *If f_1 is admissible and f_2 is a non-admissible satisfying* $\lim_{r\to 1^-} T(r,f_2)=\infty$, $a_j (j=1,2,\dots,q)$ *be q distinct complex numbers, and let $k_j (j=1,2,\dots,q)$ be positive integers or ∞ satisfying (1). Set $A_5 = B_1 + \frac{[(m-3)l+1]k_m}{k_m+1}$. Then (2) and $A_5>0$ do not hold at same time, where $B_1, S_j (j=1,2,\dots,q)$ are stated as in Theorem 2.1.*

Proof: Suppose that (2) and $A_5>0$ can hold at the same time. Since $f_1(z)$ is an admissible function, using the same argument as in Theorem 2.2 and from Theorem 1.2 and Lemma 1.1, for any $\varepsilon(0<2\varepsilon<A_5)$, we have

$$\left(\frac{(m-1)lk_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})+S(r,f_1),$$

where B_1 is stated as in Section 2.

Since f_1 is admissible and f_2 is non-admissible, we can get that $f_1(z)\not\equiv f_2(z)$. Thus, by condition (2) and the first fundamental theorem, we have

$$\begin{aligned}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\le&\overline{N}\left(r,\frac{1}{f_1-f_2}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_1-f_2-sb}\right)\\&+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_2-f_1-sb}\right)\\\le&(2l-1)(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

From the two above inequality, we get

$$\left(\frac{[(m-3)l+1]k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)\le\frac{(2l-1)k_m}{k_m+1}T(r,f_2). \tag{14}$$

Since $0<\varepsilon<A_5$, we have $\frac{[(m-3)l+1]k_m}{k_m+1}+B_1-\varepsilon>0$. From (14), we have

$$T(r,f_1)\le\frac{1}{A_5-\varepsilon}\frac{(2l-1)k_m}{k_m+1}T(r,f_2). \tag{15}$$

From Lemma 3.1, (15) and $\frac{1}{A_5-\varepsilon}\frac{(2l-1)k_m}{k_m+1}>0$, we can get that each $S(r,f_1)$ is also an $S(r,f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r,f_2)=S(r,f_1)$. Thus, we have

$$T(r,f_2)=S(r,f_1)=S(r,f_2)=o(T(r,f_2)).$$

This is a contradiction. Hence, we can get that (2) and $A_5>0$ do not hold at the same time. □

Lemma 3.3 *If f_1 is admissible and f_2 is a non-admissible satisfying* $\lim_{r\to 1^-} T(r,f_2)=\infty$, $a_j (j=1,2,\dots,q)$ *be q distinct complex numbers, and let $k_j (j=1,2,\dots,q)$ be positive integers or ∞ satisfying (1). Set $A_6 = B_3 + \frac{(m-2)lk_m}{k_m+1}$. Then (8) and $A_6>0$ do not hold at same time, where $B_3, S_j (j=1,2,\dots,q)$ are stated as in Theorem 2.3.*

Proof: Suppose that (8) and $A_6>0$ can hold at the same time. Since $f_1(z)$ is an admissible function, using the same argument as in Theorem 2.3 and from Theorem 1.1 and Lemma 1.1, for any $\varepsilon(0<\varepsilon<A_6)$, we have

$$\left(\frac{(m-1)lk_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)})+S(r,f_1),$$

where B_3 is stated as in Section 2.

From the assumptions of Lemma 3.3, we can get that $(f_1(z)-c)^l \not\equiv (f_2(z)-c)^l$. Thus, by condition (8) and the first fundamental theorem, we have

$$\begin{aligned}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)}) &< \overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l}) \\ &\leq l(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

From the two above inequality, we get

$$\left(\frac{(m-2)lk_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)\leq \frac{lk_m}{k_m+1}T(r,f_2). \tag{16}$$

Since $0<\varepsilon<A_6$, we have $\frac{(m-2)lk_m}{k_m+1}+B_3-\varepsilon>0$. From (16), we have

$$T(r,f_1)\leq \frac{1}{A_5-\varepsilon}\frac{(2l-1)k_m}{k_m+1}T(r,f_2). \tag{17}$$

From Lemma 3.1, (17) and $\frac{1}{A_6-\varepsilon}\frac{lk_m}{k_m+1}>0$, we can get that each $S(r,f_1)$ is also an $S(r,f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r,f_2)=S(r,f_1)$. Thus, we have

$$T(r,f_2)=S(r,f_1)=S(r,f_2)=o(T(r,f_2)).$$

This is a contradiction. Hence, we can get that (8) and $A_6>0$ do not hold at the same time.

Thus, the proof of Lemma 3.3 is completed. □

Proof of Theorem 3.1: Let $l=1$, and since $\Theta(f_i)\geq 0$ $(i=1,2)$ and $\delta(0,f_1-a_j)\geq 0$ $(j=1,2,\dots,q)$, the assertion follows from Lemma 3.2.

Proof of Theorem 3.2: Let $k_1=k_2=\cdots=k_q=\infty$, and since $\Theta(f_i)\geq 0$ $(i=1,2)$ and $\delta(0,f_1-a_j)\geq 0$ $(j=1,2,\dots,q)$, the assertion follows from Lemma 3.3.

It is very interesting to consider distinct small functions instead of distinct complex numbers (see [9, 11, 17],etc). *Thus it may be interesting to consider the following questions:*

Question 3.1 *What condition on two non-admissible functions in the unit disc $\mathbb{D}$ sharing small functions will guarantee that the two non-admissible functions are identical?*

Question 3.2 *How many small functions can an admissible function and non-admissible function in the unit disc $\mathbb{D}$ share at most?*

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THE FIXED POINT ALTERNATIVE TO THE STABILITY OF AN ADDITIVE (α, β)-FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we solve the additive (α, β)-functional equation

$$f(x) + f(y) + 2f(z) = \alpha f(\beta(x + y + 2z)), \tag{0.1}$$

where α, β are fixed real or complex numbers with $\alpha \neq 4$ and $\alpha\beta = 1$.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [18] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. See [5, 7, 14, 15, 20, 21, 19, 22, 23, 19, 25] for more information on functional equations.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [2, 6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several

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functional equations have been extensively investigated by a number of authors (see [3, 4, 12, 13, 16, 17]).

In Section 2, we solve the additive (α,β)-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the additive (α,β)-functional equation (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive (α,β)-functional equation (0.1) in Banach spaces by using the direct method.

Throughout this paper, assume that X is a normed space and that Y is a Banach space. Let α,β be fixed real or complex numbers with $\alpha\neq 4$ and $\alpha\beta=1$.

2. Additive (α,β)-functional equation (0.1) in Banach spaces I

We solve the additive (α,β)-functional equation (0.1) in vector spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X\to Y$ satisfies*

$$f(x)+f(y)+2f(z)=\alpha f(\beta(x+y+2z)) \tag{2.1}$$

for all $x,y,z\in X$, then $f : X\to Y$ is additive.

Proof. Assume that $f : X\to Y$ satisfies (2.1).

Letting $x=y=z=0$ in (2.1), we get $4f(0)=\alpha f(0)$. So $f(0)=0$.

Letting $y=-x$ and $z=0$ in (2.1), we get $f(x)+f(-x)=0$ and so $f(-x)=-f(x)$ for all $x\in X$.

Letting $x=-2z$ and $y=0$ in (2.1), we get $f(-2z)+2f(z)=0$ and so $f(2z)=2f(z)$ for all $z\in X$. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f(x)$$

for all $x\in X$.

Letting $z=-\frac{x+y}{2}$ in (2.1), we get

$$f(x)+f(y)-f(x+y)=f(x)+f(y)+2f\left(-\frac{x+y}{2}\right)=0$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x,y\in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (α,β)-functional equation (2.1) in Banach spaces.

Theorem 2.2. *Let $\varphi : X^3\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)\le\frac{L}{2}\varphi(x,y,z) \tag{2.2}$$

for all $x,y,z\in X$. Let $f : X\to Y$ be a mapping satisfying $f(0)=0$ and

$$\|f(x)+f(y)+2f(z)-\alpha f\left(\beta(x+y+2z)\right)\|\le\varphi(x,y,z) \tag{2.3}$$

for all $x,y,z\in X$. Then there exists a unique additive mapping $A : X\to Y$ such that

$$\|f(x)-A(x)\|\le\frac{L}{2(1-L)}\left(\varphi(x,x,-x)+\varphi(2x,0,-x)\right) \tag{2.4}$$

for all $x\in X$.

Proof. Letting $y = x$ and $z = -x$ in (2.3), we get

$$\|2f(x) + 2f(-x)\| \le \varphi(x, x, -x) \tag{2.5}$$

for all $x \in X$.

Replacing x by $2x$ and letting $y = 0$ and $z = -x$ in (2.3), we get

$$\|f(2x) + 2f(-x)\| \le \varphi(2x, 0, -x) \tag{2.6}$$

for all $x \in X$.

It follows from (2.5) and (2.6) that

$$\|f(2x) - 2f(x)\| \le \varphi(x, x, -x) + \varphi(2x, 0, -x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf \left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu(\varphi\,(x, x, -x) + \varphi\,(2x, 0, -x)),\ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [11]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon(\varphi\,(x, x, -x) + \varphi\,(2x, 0, -x))$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\left(\varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(x, 0, -\frac{x}{2}\right)\right) \\ &\le 2\varepsilon\frac{L}{2}(\varphi\,(x, x, -x) + \varphi\,(2x, 0, -x)) = L\varepsilon(\varphi\,(x, x, -x) + \varphi\,(2x, 0, -x))\end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\begin{aligned}\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| &\le \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(x, 0, -\frac{x}{2}\right) \\ &\le \frac{L}{2}(\varphi(x, x, -x) + \varphi(2x, 0, -x))\end{aligned}$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\,(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \leq \mu(\varphi(x, x, -x) + \varphi(2x, 0, -x))$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)}(\varphi(x, x, -x) + \varphi(2x, 0, -x))$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|A(x) + A(y) + 2A(z) - \alpha A(\beta(x + y + 2z))\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + 2f\left(\frac{z}{2^n}\right) - \alpha f\left(\beta\left(\frac{x+y+2z}{2^n}\right)\right)\right\| \\
&\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0
\end{aligned}$$

for all $x, y, z \in X$. So

$$A(x) + A(y) + 2A(z) - \alpha A(\beta(x + y + 2z)) = 0$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x) + f(y) + 2f(z) - \alpha f(\beta(x + y + 2z))\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{2.9}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r + 4}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(1-L)}(\varphi(x, x, -x) + \varphi(2x, 0, -x))$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2} (\varphi(x, x, -x) + \varphi(2x, 0, -x))$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{4 + 2^r}{2 - 2^r} \theta \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

3. Additive (α, β)-functional equation (0.1) in Banach spaces II

In this section, using the direct method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (2.1) in Banach spaces.

Theorem 3.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty,$$

$$\|f(x) + f(y) + 2f(z) - \alpha f(\beta(x + y + 2z))\| \leq \varphi(x, y, z) \tag{3.1}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2} (\Psi(x, x, -x) + \Psi(2x, 0, -x)) \tag{3.2}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(x, 0, -\frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \left(2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right) + 2^j \varphi\left(\frac{x}{2^j}, 0, -\frac{x}{2^{j+1}}\right) \right) \end{aligned} \tag{3.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, -\frac{x}{2^q}\right) + 2^q \Psi\left(\frac{2x}{2^q}, 0, -\frac{x}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r + 4}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. □

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (3.1) *and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}(\Psi(x, x, -x) + \Psi(2x, 0, -x)) \tag{3.4}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}(\varphi(x, x, -x) + \varphi(2x, 0, -x))$$

for all $x \in X$. Hence

$$\begin{aligned}\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \left(\frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, -2^j x) + \frac{1}{2^{j+1}} \varphi(2^{j+1} x, 0, -2^j x) \right) \qquad (3.5)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get (3.4).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{4 + 2^r}{2 - 2^r} \theta \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. □

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The approximation problem of Dirichlet series with regular growth *

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Abstract

By introducing the concept of β_U-order functions, we study the error in approximating Dirichlet series of infinite order in the half plane by Dirichlet polynomials. Some necessary and sufficient conditions on the error and regular growth of finite β_U-order of these functions have been obtained.

Key words: β-order, β_U-order, Regular growth, Dirichlet series.
2010 Mathematics Subject Classification: 30B50, 30D15.

1 Introduction and basic notes

Consider Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}a_ne^{\lambda_ns},\qquad s=\sigma+it, \tag{1}$$

where

$$0\leq\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\lambda_n\to\infty\ \ as\ \ n\to\infty; \tag{2}$$

$s=\sigma+it$ (σ,t are real variables); a_n are nonzero complex numbers and

$$\limsup_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h<+\infty, \tag{3}$$

$$\limsup_{n\to+\infty}\frac{\log^+|a_n|}{\lambda_n}=0, \tag{4}$$

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then from (2) and (3), by using the similar method in [19] or [15], we can get

$$\limsup_{n\to\infty}\frac{n}{\lambda_n}=E<+\infty,\qquad \limsup_{n\to\infty}\frac{\log n}{\lambda_n}=0. \tag{5}$$

Then the abscissas of convergence and absolutely convergence is 0, that is, $f(s)$ is an analytic function in the left half plane $H=\{s=\sigma+it:\sigma<0,t\in\mathbb{R}\}$.

We denote D to be the class of all functions $f(s)$ satisfying (2)-(4) and analytic in $Res<0$, denote $\overline{D}_\alpha$ to be the class of all functions $f(s)$ satisfying (2)-(3) and analytic in $Re\leq\alpha$ where $-\infty<\alpha<+\infty$. Thus, if $-\infty<\alpha<0$ and $f(s)\in D$, then $f(s)\in\overline{D}_\alpha$; if $0<\alpha<+\infty$ and $f(s)\in\overline{D}_\alpha$, then $f(s)\in D$. We denote Π_k to be the class of all exponential polynomial of degree almost k, that is,

$$\Pi_k=\left\{\sum_{j=1}^{k}b_je^{\lambda_js}:(b_1,b_2,\ldots,b_k)\in\mathbb{C}^k\right\}.$$

For $f(s)\in D$,

$$M(\sigma,f)=\max_{-\infty<t<\infty}|f(\sigma+it)|,\qquad m(\sigma,f)=\max_{n\geq1}\{|a_n|e^{\sigma\lambda_n}\}$$

are called, respectively, the maximum modulus, the maximum term of $f(s)$ for $Res=\sigma<0$.

Definition 1.1 *Let $f(s)\in D$, the order of $f(s)$ can be defined by*

$$\rho=\limsup_{\sigma\to0^-}\frac{\log\log^+M(\sigma,f)}{-\log(-\sigma)},$$

where $\log^+x=\begin{cases}\log x & x\geq1\\ 0 & x<1\end{cases}$

For $\rho=0,0<\rho<\infty,\rho=\infty$, $f(s)$ can be called, respectively, zero order, finite order, infinite order Dirichlet series. Considerable attention has been paid to the growth and the value distribution of analytic functions defined by Dirichlet series; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18] for some results.

For $f(s)\in\overline{D}_\alpha,-\infty<\alpha<+\infty$, we denote $E_n(f,\alpha)$ by the error in approximating the function $f(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(f,\alpha)=\inf_{p\in\Pi_n}\parallel f-p\parallel_\alpha,\quad n=1,2,\ldots,$$

where

$$\parallel f-p\parallel_\alpha=\max_{-\infty<t<+\infty}|f(\alpha+it)-p(\alpha+it)|.$$

In 2010, the authors [17] investigated the relations between the error $E_n(f,\alpha)$ and the growth order of $f(s)$, and obtained some equivalence relation between $E_n(f,\alpha)$ and the regular growth of $f(s)$ with finite order as follows:

Theorem 1.1 (see [17]). *Let $f(s)\in D$ be of finite order ρ, then for any real number $-\infty<\alpha<0$, we have*

$$\lim_{\sigma\to0^-}\frac{\log^+M(\sigma,f)}{U_1(-\frac{1}{\sigma})}=1\Longleftrightarrow\limsup_{n\to+\infty}\frac{\log^+\left[E_n(f,\alpha)e^{-\alpha\lambda_{n+1}}\right]}{BU_1\left(\frac{\lambda_{n+1}}{\log^+\left[E_n(f,\alpha)e^{-\alpha\lambda_{n+1}}\right]}\right)}=1;$$

and there exists a increasing, positive integer sequence $\{n_\nu\}$ satisfying

$$\lim_{\nu\to+\infty}\frac{\log^+\left[E_{n_\nu}(f,\alpha)e^{-\alpha\lambda_{n_\nu+1}}\right]}{BU_1\left(\frac{\lambda_{n_\nu+1}}{\log^+\left[E_{n_\nu}(f,\alpha)e^{-\alpha\lambda_{n_\nu+1}}\right]}\right)}=1,\qquad\lim_{\nu\to+\infty}\frac{\lambda_{n_{\nu+1}}}{\lambda_{n_\nu}}=1,$$

where $B = \frac{(1+\rho)^{1+\rho}}{\rho^\rho}$ *and* $U_1(r) = r^{\rho(r)}$, $\rho(r)$ *satisfies the following conditions:*

(i) there exists a real number $r_0 > 0$, $\rho(r)$ *is nonnegative, continuous, monotone on* $[r_0, +\infty)$, *and tends to* ρ *as* $r \to +\infty$*;*

(ii) $\lim_{r\to+\infty} \rho'(r) r \log r = 0,$*;*

(iii) $U_1(kr) = [k^\rho + o(1)]U_1(r)(r \to +\infty)$ *for every positive integer* k, *and* $U_1(r)$ *is an increasing function on* $r \geq r_0' > r_0$.

Recently, the authors [18] further investigated the relations between the error $E_n(f,\alpha)$ and the growth order of $f(s)$ when $f(s)$ has infinite order, by introducing the concept of β-order.

Theorem 1.2 (see [18]). *Let* $f(s) \in D$ *be of finite* β*-order* ρ_β, *then for any real number* $-\infty < \alpha < 0$, *we have*

$$\limsup_{n\to\infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log \log^+(E_{n-1}(f,\alpha)e^{-\alpha\lambda_n})} = \rho_\beta.$$

Remark 1.1 *In Theorem 1.2, the definitions of* β*-order and the function* $\beta(x)$ *will be introduced in Section 2.*

Thus, a question arises naturally: what will happen when $\rho_\beta = \infty$ *in Theorem1.2?*

In this paper, we will investigate the above question by using the type functions $U_2(x)$ to enlarge the growth of the denominator $-\log(-\sigma)$ and obtain the main results as follows.

Theorem 1.3 *If Dirichlet series* $f(s) \in D$ *of infinite* β*-order, then we have*

$$\limsup_{\sigma\to0^-} \frac{\beta(\log^+ M(\sigma,F))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T \Longleftrightarrow \limsup_{\sigma\to0^+} \frac{\beta(\log^+ m(\sigma,F))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T,$$

where $0 < T < \infty$ *and* $U_2(x) = x^{\rho(x)}$ *satisfies the following conditions*

(i) $\rho(x)$ *is monotone and* $\lim_{x\to\infty} \rho(x) = \infty$*;*

(ii) $\lim_{x\to\infty} \frac{\log U_2(x')}{\log U_2(x)} = 1$, *where* $x' = x\left(1 + \frac{1}{\log U_2(x)}\right)$.

Remark 1.2 *From Lemma 2.1 and Lemma 1.1 in Section 2, we can prove the conclusion of Theorem 1.3 easily.*

Remark 1.3 *This type function* $U_2(x)$ *is different from the type function* $U_1(x)$ *in Theorem 1.1.*

Remark 1.4 *If Dirichlet series* $f(s)$ *of infinite order has infinite* β*-order and satisfies*

$$\limsup_{\sigma\to0^-} \frac{\beta(\log^+ M(\sigma,f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T, \tag{6}$$

then T *is called the* β_U*-order of Dirichlet series* $f(s)$.

Theorem 1.4 *If Dirichlet series* $f(s) \in D$ *with infinite* β*-order, then for any fixed real number* $-\infty < \alpha < 0$, *we have*

$$\limsup_{\sigma\to0^-} \frac{\beta(\log^+ M(\sigma,f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T \Longleftrightarrow \limsup_{n\to\infty} \Psi_n(f,\alpha,\lambda_n) = T; \tag{7}$$

where

$$\Psi_n(f,\alpha,\lambda_n) = \frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]}\right)}.$$

Remark 1.5 *From Theorem 1.4, we can see that the type function $U_2(x)$ is more simple then the type function of Wang [16].*

Theorem 1.5 *Under the assumptions of Theorem 1.4, we have*

$$\lim_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2\left(\frac{1}{-\sigma}\right)}=T \Longleftrightarrow \ \textit{the right hand of (7) is verified},$$

and there exists a subsequence $\{\lambda_{n(p)}\}\subseteq\{\lambda_n\}$ satisfying

$$\lim_{p\to\infty}\Psi_{n(p)}(f,\alpha,\lambda_{n(p)})=T, \quad \textit{and} \quad \lim_{p\to\infty}\frac{\beta(\lambda_{n(p)})}{\beta(\lambda_{n(p+1)})}=1, \tag{8}$$

where

$$\Psi_{n(p)}(f,\alpha,\lambda_{n(p)})=\frac{\beta(\lambda_{n(p)})}{\log U_2\left(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}(f,\alpha)e^{-\alpha\lambda_{n(p)}}]}\right)}.$$

Remark 1.6 *From Theorem 1.5, we get the necessary and sufficient conditions for the limit about the regular growth of $f(s)$, however, Wang [16] only gave the necessary and sufficient conditions for the superior limit. Thus, our results of this paper are more accurate than the previous form [16].*

2 Some Lemmas and the concept of β-order

According to observations, we find that to study the growth of Dirichlet series better, many mathematicians proposed the type functions $U(x)$ to enlarge the growth of the denominator $\log\frac{1}{-\sigma}$ or $-\sigma$ (see [13, 4, 12]), or use some function to control the molecular $M(\sigma,f)$ or $m(\sigma,f)$ in the definition of order. In this paper, we will deal with the growth of Dirichlet series of infinite order by using a class of functions to reduce $M(\sigma,f)$ or $m(\sigma,f)$ which is better than the previous form. So, we firstly give the definition of β-order of Dirichlet series as follows, which is an extension of [10].

Let $\mathfrak{F}$ be the class of all functions $\beta(x)$ satisfies the following conditions:

(i) $\beta(x)$ is defined on $[a,+\infty), a>0$, and positive, strictly increasing, differential and tends to $+\infty$ as $x\to+\infty$;

(ii) $x\beta'(x)=o(1)$ as $x\to+\infty$.

Definition 2.1 *([18]). If Dirichlet series $f(s)$ of infinite order satisfies*

$$\limsup_{\sigma\to 0^+}\frac{\beta(\log^+ M(\sigma,f))}{\log\frac{1}{-\sigma}}=\rho^*,$$

where $\beta(x)\in\mathfrak{F}$, then ρ^* is called the β-order of $f(s)$.

Remark 2.1 *Obviously, the functions $h(x)=\log_p x, p\geq 2, p\in N_+$ satisfy the conditions (i) and (ii), where p is a positive integer, and $\log_1 x=\log x$ and $\log_p x=\log(\log_{p-1}x)$. Thus, p-order is regard as a special case of β-order of Dirichlet series.*

Remark 2.2 *Furthermore, β-order is more precise than p-order to some extent. In fact, for $p(\geq 2)$ is a positive integer, we can find function $\beta(x)\in\mathfrak{F}$ and a positive real function $M(x)$ satisfying*

$$\limsup_{x\to\infty}\frac{\beta(\log M(x))}{\log x}=t, \quad (0<t<\infty),$$

and

$$\limsup_{x\to\infty}\frac{\log_p(\log M(x))}{\log x}=\infty,\quad \text{and}\quad \limsup_{x\to\infty}\frac{\log_{p+1}(\log M(x))}{\log x}=0.$$

For example, let

$$M(x)=\exp_{p+1}\{(t\log x)^{1/d}\},\qquad \beta(x)=(\log_{p+1}x)^d,$$

where t is a finite positive real constant and $0<d<1$, we can get that $\rho_p(M)=\infty,\rho_{p+1}(M)=0$ and $\rho_\beta(M)=t$, where $\rho_p(f)$ denote the p-order of f, and $\rho_\beta(f)$ the β-order of f.

Remark 2.3 *If $\rho^*=\infty$ in Definition 2.1, then $f(s)$ is called a Dirichlet series of infinite β-order.*

Lemma 2.1 *(see [16]). Let $\beta(x)\in\mathfrak{F}$ and $\varphi(x)$ be the function satisfying*

$$\limsup_{x\to\infty}\frac{\log^+\varphi(x)}{\log x}=\varrho,\quad (0\le\varrho<\infty),$$

if $M(x)$ satisfies $\limsup_{x\to\infty}\frac{\beta(\log M(x))}{\log x}=\nu(>0)$. Then we have

$$\limsup_{x\to\infty}\frac{\beta(\varphi(x)\log M(x))}{\log x}=\nu.$$

Proof: To prove this lemma, two cases will be considered as follows.

Case 1. If $\varphi(x)$ is not a constant. From the assumptions of Lemma 2.1, we can get that $\varphi(x)\to\infty$ as $x\to\infty$. Then, for sufficiently large x, we have $\varphi(x)>1$. From $\beta(x)\in\mathfrak{F}$, we have $\lim_{x\to\infty}\log M(x)=\infty$. Then from the Cauchy mean value theorem, there exists $\xi(\log M(x)<\xi<\beta(x)\log M(x))$ satisfying

$$\frac{\beta(\varphi(x)\log M(x))-\beta(\log M(x))}{\log(\varphi(x)\log M(x))-\log\log M(x)}=\frac{\beta'(\xi)}{(\log\xi)'}=\xi\beta'(\xi),$$

that is,

$$\beta(\varphi(x)\log M(x))=\beta(\log M(x))+\log\varphi(x)\xi\beta'(\xi). \tag{9}$$

Since $x\beta'(x)=o(1)$ as $x\to+\infty$ and $\limsup_{x\to\infty}\frac{\log\varphi(x)}{\log x}=\varrho,(0\le\varrho<\infty)$, by (9), we can get the conclusion of Lemma 2.1.

Case 2. If $\varphi(x)$ is a constant. By using the same argument as in Case 1, we can prove that Lemma 2.1 is true.

Thus, this completes the proof of Lemma 2.1. □

The following lemma plays an important role to deal with the growth of Dirichlet series, which shows the relation between $M(\sigma,f)$ and $m(\sigma,f)$ of such functions.

Lemma 2.2 ([19]). If Dirichlet series (1) satisfies (2) (3), then for any given $\varepsilon\in(0,1)$ and for $\sigma(<0)$ sufficiently reaching 0, we have

$$m(\sigma,f)\le M(\sigma,f)\le K(\varepsilon)\frac{1}{-\sigma}m((1-\varepsilon)\sigma,f),$$

where $K(\varepsilon)$ is a constant depending on ε and (3).

Lemma 2.3 *If $f(s)\in\overline{D}_\alpha(-\infty<\alpha<+\infty)$, then for any positive integer $n\in\mathbb{N}_+:=\mathbb{N}\backslash\{0\}$, we have*

$$|a_n|e^{\alpha\lambda_n}\le K_2E_{n-1}(f,\alpha),$$

where $K_2>1$ is a real constant.

Proof: From the definition of $E_n(f,\alpha)$, there exists $p(s)\in\Pi_{n-1}$ such that

$$||f-p||_\alpha \le K_2 E_{n-1}(f,\alpha). \tag{10}$$

Since $f(s)\in\overline{D}_\alpha$ and from [19, P.16], for any real numbers $t_0,\vartheta(\neq 0)$, we have

$$\lim_{R\to+\infty}\frac{1}{R}\int_{t_0}^{R} e^{\vartheta it}dt=0 \tag{11}$$

and

$$a_n e^{\alpha\lambda_n}=\lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R} f(\alpha+it)e^{-\lambda_n it}dt. \tag{12}$$

From (11), for any real number $x\neq 0$, we have

$$\lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R} e^{x(\alpha+it)}dt=0. \tag{13}$$

Thus, from (12) and (13), for any $p_1(s)\in\Pi_{n-1}$, we have

$$a_n e^{\alpha\lambda_n}=\lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R}[f(\alpha+it)-p_1(\alpha+it)]e^{-\lambda_n it}dt,$$

that is,

$$|a_n|e^{\alpha\lambda_n}\le ||f-p_1||_\alpha. \tag{14}$$

From (10) and (14), we can prove the conclusion of Lemma 2.3. □

3 The proof of Theorem 1.4

We prove the conclusions of Theorem 1.4 by using the properties of two functions $\beta(x)$ and $U_2(x)$, this method is different from the previous method to some extent.

We first prove " $\Longleftarrow$ " of Theorem 1.4. Suppose that

$$\limsup_{n\to\infty}\Psi_n(f,\alpha,\lambda_n)=\limsup_{n\to\infty}\frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]}\right)}=T. \tag{15}$$

Let

$$A_n=E_{n-1}(f,\alpha)e^{-\alpha\lambda_n},\quad n=1,2,\ldots,$$

then for any positive real number $\tau>0$, for sufficiently large n, we have

$$\lambda_n<\gamma\left((T+\tau)\log U_2\left(\frac{\lambda_n}{\log^+ A_n}\right)\right),$$

where $\gamma(x)$ is the inverse functions of $\beta(x)$. Let $V_2(x)$ and $U_2(x)$ be two reciprocally inverse functions, then we have

$$V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)<\frac{\lambda_n}{\log^+ A_n},\quad \log^+ A_n\le\lambda_n\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1}.$$

Thus, we have

$$\log^+(A_n e^{\lambda_n\sigma})\le\lambda_n\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1}+\sigma\right). \tag{16}$$

For any fixed and sufficiently small $\sigma < 0$, set

$$G = \gamma\left((T+\tau)\log U_2\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)} = V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(G)\right\}\right). \tag{17}$$

If $\lambda_n \le G$, for sufficiently large n, let $V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right) \ge 1$, from $\sigma < 0$,(16),(17) and the definition of $U_2(x)$, we have

$$\begin{aligned}
\log^+ A_n e^{\lambda_n\sigma} &\le G\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1}+\sigma\right)\\
&\le G = \gamma\left((T+\tau)\log U_2\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right)\\
&\le \gamma\left((T+\tau)\log\left[(1+o(1))U_2\left(\frac{1}{-\sigma}\right)\right]\right). \qquad (18)
\end{aligned}$$

If $\lambda_n > G$, from (16) and (17), we have

$$\begin{aligned}
\log^+ A_n e^{\lambda_n\sigma} &\le \lambda_n\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(G)\right\}\right)\right)^{-1}+\sigma\right)\\
&\le \lambda_n\left(\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)^{-1}+\sigma\right) < 0. \qquad (19)
\end{aligned}$$

For sufficiently large n, from (18) and (19), we have

$$\log^+ A_n e^{\lambda_n\sigma} \le \gamma\left((T+\tau)\log\left[(1+o(1))U_2\left(\frac{1}{-\sigma}\right)\right]\right)$$

Since $A_n = E_{n-1}e^{-\alpha\lambda_n}$ and τ is arbitrary, by Lemma 2.1,Lemma 2.3 and Theorem 1.3, we can get

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})} \le T.$$

Suppose that

$$\limsup_{\sigma\to 0^+}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})} = \eta < T.$$

Thus, there exists any real number $\varepsilon(0<\varepsilon<\frac{\eta}{2})$, for any positive integer n and any sufficient small $\sigma < 0$, from Lemma 2.2, we have

$$\log^+|a_n|e^{\lambda_n\sigma} \le \log M(\sigma,f) \le \gamma\left((T-2\varepsilon)\log U_2(\frac{1}{-\sigma})\right). \tag{20}$$

From (15), there exists a subsequence $\{\lambda_{n(p)}\}$, for sufficiently large p, we have

$$\beta(\lambda_{n(p)}) > (T-\varepsilon)\log U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right). \tag{21}$$

Take a sequence $\{\sigma_p\}$ satisfying

$$\gamma\left((\eta-2\varepsilon)\log U_2(\frac{1}{-\sigma_p})\right)=\frac{\log^+ A_{n(p)}}{1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}. \tag{22}$$

From (20) and (22), we get

$$\log^+ A_{n(p)}+\lambda_{n(p)}\sigma_p\leq\gamma\left((\eta-2\varepsilon)\log U_2(\frac{1}{-\sigma_p})\right)=\frac{\log^+ A_{n(p)}}{1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})},$$

that is,

$$\frac{1}{-\sigma_p}\leq\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\frac{1}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}\right).$$

Thus, we have

$$U_2(\frac{1}{-\sigma_p})\leq U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\frac{1}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}\right)\right)\leq U_2^{1+o(1)}\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right). \tag{23}$$

From (22) and (23), we have

$$\begin{aligned}\lambda_{n(p)}&=\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\gamma\left((T-2\varepsilon)\log U_2(\frac{1}{\sigma_p})\right)\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\\&=\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\gamma\left((\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right).\end{aligned}$$

Thus, from the Cauchy mean value theorem, there exists a real number ξ between $\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\gamma(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})$ and $\gamma(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})$ such that

$$\begin{aligned}\beta\left(\lambda_{n(p)}\right)&=\beta\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\gamma\left((\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\\&=\beta\left(\gamma\left((T-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\\&\quad+\log\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\xi\beta'(\xi),\end{aligned}$$

Since

$$\lim_{p\to\infty}\frac{\log\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}=0,$$

then for sufficiently large p, we have

$$\beta\left(\lambda_{n(p)}\right)=(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})+K_2\xi\beta'(\xi)\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}), \tag{24}$$

where K_2 is a constant.

From (21),(24) and $\eta < T$, we can get a contradiction. Thus, we can get

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})}=T.$$

Hence, the sufficiency of Theorem 1.4 is completed.

We can prove the necessity of Theorem 1.4 by using the similar argument as in the proof of the sufficiency of Theorem 1.4.

Thus, the proof of Theorem 1.4 is completed.

4 The Proof of Theorem 1.5

We will consider two steps as follows:

Step one: We first prove the sufficiency of Theorem 1.5. From the conditions of Theorem 1.5, for any $\varepsilon(>0)$, there exists a subsequence $\{\lambda_{n(p)}\}$ such that

$$\lambda_{n(p)}\geq \gamma\left((T-\varepsilon)\log U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right)\right),\qquad \lim_{p\to\infty}\frac{\beta(\lambda_{n(p)})}{\beta(\lambda_{n(p+1)})}=1, \tag{25}$$

that is,

$$\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\leq V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right),\quad \log^+ A_{n(p)}\geq \lambda_{n(p)}V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right)^{-1}.$$

Take the sequence $\{\sigma_p\}$ satisfying

$$\begin{aligned}\lambda_{n(p)}&=\gamma\left((T-\varepsilon)\log U_2\left(\frac{1}{-\sigma_p}+\frac{1}{\sigma_p\log U_2(\frac{1}{-\sigma_p})}\right)\right),\\ \frac{1}{-\sigma_p}+\frac{1}{\sigma_p\log U_2(\frac{1}{-\sigma_p})}&=V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right).\end{aligned} \tag{26}$$

For any sufficiently small $\sigma<0$ and $-\infty<\alpha<\sigma<0$, we have

$$E_{n-1}(f,\alpha)\leq ||f-p_{n-1}||_\alpha\leq \sum_{k=n}^{\infty}|a_k|e^{\lambda_k\alpha}\leq M(\sigma,f)\sum_{k=n}^{\infty}e^{\lambda_n(\alpha-\sigma)}, \tag{27}$$

where $p_{n-1}(s)=\sum_{k=1}^{n-1}a_ke^{\lambda_k s}$. From (3), we take $0<h'<h$ satisfying $\lambda_{n+1}-\lambda_n\geq h'$ for any integer $n\geq 1$. Thus, for sufficiently small $\sigma<0$ such that $\sigma\geq\frac{\alpha}{2}$, from (27) we have

$$\begin{aligned}E_{n-1}(f,\alpha)&\leq M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}\sum_{k=n}^{\infty}e^{(\lambda_k-\lambda_n)(\alpha-\sigma)}\\ &\leq M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}e^{-\frac{\alpha}{2}h'n}\sum_{k=n}^{\infty}e^{\frac{\alpha}{2}h'k}\\ &=M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}\left(1-e^{\frac{\alpha}{2}h'}\right)^{-1}.\end{aligned}$$

Then for sufficiently small $\sigma<0$ and $-\infty<\alpha<\sigma<0$, we have

$$M(\sigma,f)\geq K_3E_{n-1}(f,\alpha)e^{-\lambda_n(\alpha-\sigma)}=K_3A_ne^{\lambda_n\sigma}, \tag{28}$$

where $K_3 = 1 - e^{\frac{\alpha}{2}h'}$. For sufficiently small $\sigma < 0$, we take $\sigma_p \le \sigma < \sigma_{p+1}$, from (25),(26) and (28), we have

$$\begin{aligned}\log^+ M(\sigma, f) &\ge \log^+ A_{n(p)} + \lambda_{n(p)}\sigma_p + O(1) \qquad (29)\\ &\ge \lambda_{n(p)}\left(V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right)^{-1} + \sigma_p\right) + O(1)\\ &\ge \gamma\left((T-\varepsilon)\log U_2\left(\frac{1}{-\sigma_p} + \frac{1}{\sigma_p \log U_2(\frac{1}{-\sigma_p})}\right)\right)\frac{-\sigma_p}{\log U_2(\frac{1}{-\sigma_p}) - 1} + O(1)\\ &\ge (1+o(1))\gamma\left((T-\varepsilon)\log U_2\left(\frac{1}{-\sigma_{p+1}} + \frac{1}{\sigma_{p+1}\log U_2(\frac{1}{-\sigma_{p+1}})}\right)\right)\frac{-\sigma_p}{\log U_2(\frac{1}{-\sigma_p}) - 1}\\ &\ge (1+o(1))\gamma\left((T-\varepsilon)\log U_2\left(\frac{1}{-\sigma} + \frac{1}{\sigma\log U_2(\frac{1}{-\sigma})}\right)\right)\frac{-\sigma}{\log U_2(\frac{1}{-\sigma}) - 1}.\end{aligned}$$

Set
$$\frac{1}{-\sigma} + \frac{1}{\sigma \log U_2(\frac{1}{-\sigma})} = r, \quad r\left(1 + \frac{1}{\log U_2(r)}\right) = R, \quad R\left(1 + \frac{1}{\log U_2(R)}\right) = R',$$

by using a simple calculation, we can get $R' \ge \frac{1}{-\sigma}$. Thus, from the definitions of $U_2(x)$ (ii), we can get
$$\limsup_{\sigma\to 0^-} \frac{\log U_2(r)}{\log U_2(\frac{1}{-\sigma})} = 1. \tag{30}$$

Since
$$\limsup_{\sigma\to 0^-} \frac{\log \frac{-\sigma}{\log U_2(\frac{1}{-\sigma})-1}}{\log U_2(\frac{1}{-\sigma})} = 0,$$

and from Lemma 2.1, (29) and (30), we have
$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2(\frac{1}{-\sigma})} = T.$$

Step two: The necessity of the Theorem 1.5 will be proved as follows. From Theorem 1.4, we can get that the right hand of (7) is verified. Next, we will prove that (8) also holds. We take a positive decreasing sequence $\{\varepsilon_i\}(0 < \varepsilon_i < T)$,$\varepsilon_i \to 0(i \to \infty)$.

Set
$$F_i = \left\{n : \Psi_n(f, \alpha, \lambda_n) = \frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+ A_n}\right)} > T - \varepsilon_i\right\}, \tag{31}$$

it follows that $\forall i, F_i \ne \Phi$ and $F_i \subset F_{i-1}$. For each i, we arrange the $n(\in F_i)$ in an increasing sequence $\{n^{(i)}(p)\}_{p=1}^{\infty}$, then we consider the two cases in the following.

Case 1. Suppose that $\lim_{\nu\to+\infty} \frac{\beta(\lambda_{n^{(i)}(p+1)})}{\beta(\lambda_{n^{(i)}(p)})} = 1$ for any i. Then there exists $N_i \in F_i(i \in N_+)$, when $n^{(i)}(p) \ge N_i$, we have
$$\frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)} \le 1 + \varepsilon_i. \tag{32}$$

Note $F_{i+1} \subset F_i$, take $N_{i+1} > N_i$, denote F_i' the subset of F_i
$$F_i' = \{n \in F_i : N_i \le n \le N_{i+1}\},$$

thus the elements of F_i' satisfy (31) and (32).

Therefore let $F = \bigcup_{i=1}^{\infty} F_i'$ and arrange the $n(\in E_i')$ in an increasing sequence $\{n_\nu\}$. Thus, the necessity of Theorem 1.5 is proved.

Case 2. If there exists $i \in N_+$ satisfying $\lim_{\nu\to+\infty} \frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)} \neq 1$, then since $\lambda_{n^{(i)}(p+1)} > \lambda_{n^{(i)}(p)}$, we get $\lim_{\nu\to+\infty} \frac{\beta(\lambda_{n^{(i)}(p+1)})}{\beta(\lambda_{n^{(i)}(p)})} > 1$. Hence there exists $\{n^{(i)}(p_k)\} \subseteq \{n^{(i)}(p)\}$ (still marked with $\{n^{(i)}(p)\}$) and positive real constant $\tau > 0$, it follows that

$$\frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)} \geq 1+\tau.$$

Let

$$n'(1) = n^{(i)}(1), n'(2) = n^{(i)}(3), \cdots, n'(p) = n^{(i)}(2p-1), \cdots$$
$$n''(1) = n^{(i)}(1), n''(2) = n^{(i)}(4), \cdots, n''(p) = n^{(i)}(2p), \cdots$$

where $\{n'(p)\}, \{n''(p)\}$ are two increasing positive integer sequences, and

$$n''(p) < n'(p+1), \quad \beta(\lambda_{n''(p)}) > (1+\tau)\beta(\lambda_{n'(p)}), \quad \nu = 1, 2, \cdots.$$

From (31), for any sufficiently large p, when $n \notin F_i$ satisfies $n'(p) < n < n''(p)$, there exists a positive real number $\delta > 0$ such that

$$\lambda_n \leq \gamma\left((T-\delta)\log U_2(\frac{\lambda_n}{\log^+ A_n})\right), \quad \frac{\lambda_n}{\log^+ A_n} \geq V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right). \tag{33}$$

Thus we have

$$\log^+ A_n e^{\sigma\lambda_n} < \lambda_n\left(\frac{1}{V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right)} + \sigma\right). \tag{34}$$

Set

$$G = \gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma} + \frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma} + \frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)} = V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(G)\right\}\right). \tag{35}$$

If $\lambda_n \geq G$, from (34) and (35), we have

$$\log^+ A_n e^{\sigma\lambda_n} \leq \lambda_n\left(\frac{1}{V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right)} + \sigma\right) < 0. \tag{36}$$

If $\lambda_n < G$, from (34) and (35), we have

$$\log^+ |a_n| e^{\sigma\lambda_n} < G = \gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma} + \frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right). \tag{37}$$

Choose the sequence $\{\sigma_p\}$ satisfying

$$\sigma_p = -\left[V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_{n''(p)})\right\}\right)\right]^{-1}, \tag{38}$$

from the assumptions of the necessity of Theorem 1.5, there exists an integer $N_2 \in N_+$ such that $V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_n)\right\}\right) \geq 1$. Then for $n \geq N_2$, we have

$$\log^+ A_n e^{\sigma_p \lambda_n} < \lambda_n \left(V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_n)\right\}\right)^{-1} + \sigma_p \right).$$

When $n \geq n''(p)$, it follows $\lambda_n \geq \lambda_{n''(p)}$, and from (38), we have

$$\log^+ A_n e^{\sigma_p \lambda_n} < \lambda_n \left(V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_{n''(p)})\right\}\right)^{-1} + \sigma_p \right) = 0. \tag{39}$$

For sufficiently large ν, we have $\lambda_{n'(p)} \geq \lambda_n$ as $N_2 \leq n \leq n'(p)$, and

$$\log^+ A_n e^{\sigma_p \lambda_n} \leq \lambda_{n'(p)} \left(V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right)^{-1} + \sigma_p \right).$$

Since $\lambda_{n'(p)} < \gamma\left(\frac{1}{1+\tau}\beta(\lambda_{n''(p)})\right)$ and $\sigma_p < 0$, from the definition of σ_p, N_2, we can get

$$\log^+ A_n e^{\sigma_\nu \lambda_n} \leq \gamma\left(\frac{1}{1+\tau}\beta(\lambda_{n''(p)})\right) \leq \gamma\left(\frac{T-\delta}{1+\tau}\log U_2\left(\frac{1}{-\sigma_p}\right)\right). \tag{40}$$

Thus, from (36), (37), (39) and (40), we have

$$\log^+ A_n e^{\sigma_p \lambda_n} \leq \gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma} + \frac{1}{-\sigma \log U_2\left(\frac{1}{-\sigma}\right)}\right)\right), \quad as \quad n > N_2.$$

By Lemma 2.2, we have

$$\lim_{\sigma_p \to 0^-} \frac{\beta(\log^+ m(\sigma_p, f))}{\log U_2\left(\frac{1}{-\sigma_p}\right)} \leq T - \delta < T. \tag{41}$$

From (41), Theorem 1.3, we can get a contradiction with the following equality

$$\lim_{\sigma \to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T.$$

Thus, the proof of Theorem 1.5 is completed by Step one and Step two.

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On special fuzzy differential subordinations using multiplier transformation

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Abstract

In the present paper we establish several fuzzy differential subordinations regardind the operator $I(m,\lambda,l)$, given by $I(m,\lambda,l):\mathcal{A}\to\mathcal{A}$, $I(m,\lambda,l)f(z)=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j z^j$ and $\mathcal{A}=\{f\in\mathcal{H}(U),\ f(z)=z+a_2z^2+\dots,\ z\in U\}$ is the class of normalized analytic functions. A certain fuzzy class, denoted by $SI_{\mathcal{F}}^{\delta}(m,\lambda,l)$, of analytic functions in the open unit disc is introduced by means of this operator. By making use of the concept of fuzzy differential subordination we will derive various properties and characteristics of the class $SI_{\mathcal{F}}^{\delta}(m,\lambda,l)$. Also, several fuzzy differential subordinations are established regarding the operator $I(m,\lambda,l)$.

Keywords: fuzzy differential subordination, convex function, fuzzy best dominant, differential operator.
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1 Introduction

S.S. Miller and P.T. Mocanu have introduced [10], [11] and developed [12] in the one complex variable functions theory the admissible functions method known as "the differential subordination method" . The application of this method allows to one obtain some special results and to prove easily some classical results from this domain.

G.I. Oros and Gh.Oros [13], [14] wanted to launch a new research direction in mathematics that combines the notions from the complex functions domain with the fuzzy sets theory.

In the same way as mentioned, we can justify that by knowing the properties of a differential expression on a fuzzy set for a function one can be determined the properties of that function on a given fuzzy set. We have analyzed the case of one complex functions, leaving as "open problem" the case of real functions. We are aware that this new research alternative can be realized only through the joint effort of researchers from both domains. The "open problem" statement leaves open the interpretation of some notions from the fuzzy sets theory such that each one interpret them personally according to their scientific concerns, making this theory more attractive.

The notion of fuzzy subordination was introduced in [13]. In [14] the authors have defined the notion of fuzzy differential subordination. In this paper we will study fuzzy differential subordinations obtained with the differential operator studied in [3] using the methods from [4], [5].

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ with $\mathcal{A}_1=\mathcal{A}$ and $\mathcal{H}[a,n]=\{f\in\mathcal{H}(U):f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ for $a\in\mathbb{C}$ and $n\in\mathbb{N}$.

Denote by $\mathcal{K}=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{zf''(z)}{f'(z)}+1>0,\ z\in U\right\}$, the class of normalized convex functions in U.

In order to use the concept of fuzzy differential subordination, we remember the following definitions:

Definition 1.1 *[9] A pair (A,F_A), where $F_A:X\to[0,1]$ and $A=\{x\in X:0<F_A(x)\le 1\}$ is called fuzzy subset of X. The set A is called the support of the fuzzy set (A,F_A) and F_A is called the membership function of the fuzzy set (A,F_A). One can also denote $A=\operatorname{supp}(A,F_A)$.*

Remark 1.1 *In the development work we use the following notations for fuzzy sets:*
$F_{f(D)}(f(z)) = supp\big(f(D), F_{f(D)}\cdot\big) = \{z \in D : 0 < F_{f(D)} f(z) \le 1\}$,
$F_{g(D)}(g(z)) = supp\big(g(D), F_{g(D)}\cdot\big) = \{z \in D : 0 < F_{g(D)} g(z) \le 1\}$,
$p(U) = supp\big(p(U), F_{p(U)}\cdot\big) = \{z \in U : 0 < F_{p(U)}(p(z)) \le 1\}$,
$q(U) = supp\big(q(U), F_{q(U)}\cdot\big) = \{z \in U : 0 < F_{q(U)}(q(z)) \le 1\}$,
$h(U) = supp\big(h(U), F_{h(U)}\cdot\big) = \{z \in U : 0 < F_{h(U)}(h(z)) \le 1\}$.

We give a new definition of membership function on complex numbers set using the module notion of a complex number $z = x + iy$, $x, y \in \mathbb{R}$, $|z| = \sqrt{x^2 + y^2} \ge 0$.

Example 1.1 *Let $F : \mathbb{C} \to \mathbb{R}_+$ a function such that $F_{\mathbb{C}}(z) = |F(z)|$, $\forall\, z \in \mathbb{C}$. Denote by $F_{\mathbb{C}}(\mathbb{C}) = \{z \in \mathbb{C} : 0 < F(z) \le 1\} = \{z \in \mathbb{C} : 0 < |F(z)| \le 1\} = supp(\mathbb{C}, F_{\mathbb{C}})$ the fuzzy subset of the complex numbers set.*

Remark 1.2 *We call the subset $F_{\mathbb{C}}(\mathbb{C}) = \{z \in \mathbb{C} : 0 < |F(z)| \le 1\} = U_{\mathcal{F}}(0,1)$ the fuzzy unit disk.*

Example 1.2 *Let $F : \mathbb{C} \to \mathbb{R}_+$, $F(z) = \frac{2-|z|}{2+|z|}$, where $|z| = \sqrt{x^2+y^2} \ge 0$. A fuzzy subset of the complex numbers set is $A = \{z \in \mathbb{C} : 0 < F_A(z) \le 1\} = supp(A, F_A) = \{z \in \mathbb{C} : |z| < 2\}$, where $F_A(z) = \begin{cases} F(z), & z \in \{|z| \le 2\} \\ 0, & z \in \mathbb{C} - \{|z| \le 2\}. \end{cases}$*

We show that the fuzzy subset is nonempty. Indeed, for $z = 0$, $F_A(0) = F(0) = 1$, so $z = 0 \in A$. More we see that the fuzzy subset A contains all the complex numbers with the properties $|z| < 2$ and all the complex numbers for which $|z| > 2$ not belong to A, i.e. $supp(A, F_A) = \{z \in \mathbb{C} : x^2 + y^2 < 4\}$.

Remark 1.3 *The membership functions can be defined otherwise and we propose that each choose how to define according to their research.*

Definition 1.2 *([13]) Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, if are satisfied the conditions:*
1) $f(z_0) = g(z_0)$,
2) $F_{f(D)} f(z) \le F_{g(D)} g(z)$, $z \in D$.

Definition 1.3 *([14, Definition 2.2]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h univalent in U, with $\psi(a, 0; 0) = h(0) = a$. If p is analytic in U, with $p(0) = a$ and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3 \times U)} \psi(p(z), zp'(z), z^2 p''(z); z) \le F_{h(U)} h(z), \quad z \in U, \tag{1.1}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $F_{p(U)} p(z) \le F_{q(U)} q(z)$, $z \in U$, for all p satisfying (1.1). A fuzzy dominant $\widetilde{q}$ that satisfies $F_{\widetilde{q}(U)} \widetilde{q}(z) \le F_{q(U)} q(z)$, $z \in U$, for all fuzzy dominants q of (1.1) is said to be the fuzzy best dominant of (1.1).

Lemma 1.1 *([12, Corollary 2.6g.2, p. 66]) Let $h \in \mathcal{A}$ and $L[f](z) = G(z) = \frac{1}{z}\int_0^z h(t)\,dt$, $z \in U$. If $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$, $z \in U$, then $L(f) = G \in \mathcal{K}$.*

Lemma 1.2 *([15]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\,\gamma \ge 0$. If $p \in \mathcal{H}[a, n]$ with $p(0) = a$, $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = p(z) + \frac{1}{\gamma} zp'(z)$ an analytic function in U and $F_{\psi(\mathbb{C}^2 \times U)}\left(p(z) + \frac{1}{\gamma} zp'(z)\right) \le F_{h(U)} h(z)$, i.e. $p(z) + \frac{1}{\gamma} zp'(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, then $F_{p(U)} p(z) \le F_{g(U)} g(z) \le F_{h(U)} h(z)$, i.e. $p(z) \prec_{\mathcal{F}} g(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt$, $z \in U$. The function q is convex and is the fuzzy best dominant.*

Lemma 1.3 *([15]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, $z \in U$, where $\alpha > 0$ and n is a positive integer. If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots$, $z \in U$, is holomorphic in U and $F_{p(U)}(p(z) + \alpha z p'(z)) \le F_{h(U)} h(z)$, i.e. $p(z) + \alpha z p'(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, then $F_{p(U)} p(z) \le F_{g(U)} g(z)$, i.e. $p(z) \prec_{\mathcal{F}} g(z), z \in U$, and this result is sharp.*

We will study the following differential operator, known as multiplier transformation.

Definition 1.4 *For $f \in \mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + \ldots, \ z \in U\}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I(m, \lambda, l) f(z)$ is defined by the following infinite series $I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{\lambda(j-1)+l+1}{l+1}\right)^m a_j z^j$.*

Remark 1.4 *It follows from the above definition that $(l+1) I(m+1, \lambda, l) f(z) = [l+1-\lambda] I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))'$, $z \in U$.*

Remark 1.5 *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [2], which is reduced to the Sălăgean differential operator [16] for $\lambda = 1$. The operator $I(m, 1, l)$ was studied by Cho and Srivastava [8] and Cho and Kim [7]. The operator $I(m, 1, 1)$ was studied by Uralegaddi and Somanatha [17] and the operator $I(\alpha, \lambda, 0)$ was introduced by Acu and Owa [1]. Cătaş [6] has studied the operator $I_p(m, \lambda, l)$ which generalizes the operator $I(m, \lambda, l)$.*

2 Main results

Using the operator $I(m, \lambda, l)$ we define the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ and we study fuzzy subordinations.

Definition 2.1 *Let $f(D) = supp\big(f(D), F_{f(D)}\big) = \{z \in D : 0 < F_{f(D)} f(z) \leq 1\}$, where $F_{f(D)}\cdot$ is the membership function of the fuzzy set $f(D)$ asociated to the function f.*

The membership function of the fuzzy set $(\mu f)(D)$ asociated to the function μf coincide with the membership function of the fuzzy set $f(D)$ asociated to the fuction f, i.e. $F_{(\mu f)(D)}((\mu f)(z)) = F_{f(D)} f(z)$, $z \in D$.

The membership function of the fuzzy set $(f+g)(D)$ asociated to the function $f+g$ coincide with the half of the sum of the membership functions of the fuzzy sets $f(D)$, respectively $g(D)$, asociated to the function f, respectively g, i.e. $F_{(f+g)(D)}((f+g)(z)) = \frac{F_{f(D)} f(z) + F_{g(D)} g(z)}{2}$, $z \in D$.

Remark 2.1 *$F_{(f+g)(D)}((f+g)(z))$ can be defined in other ways.*

Remark 2.2 *Since $0 < F_{f(D)} f(z) \leq 1$ and $0 < F_{g(D)} g(z) \leq 1$, it is evidently that $0 < F_{(f+g)(D)}((f+g)(z)) \leq 1$, $z \in D$.*

Definition 2.2 *Let $\delta \in (0, 1]$, $\lambda, l \geq 0$ and $m \in \mathbb{N}$. A function $f \in \mathcal{A}$ is said to be in the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ if it satisfies the inequality $F_{(I(m,\lambda,l)f)'(U)}(I(m, \lambda, l) f(z))' > \delta$, $z \in U$.*

Theorem 2.1 *The set $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ is convex.*

Proof. Let the functions $f_j(z) = z + \sum_{j=2}^{\infty} a_{jk} z^j$, $k = 1, 2$, $z \in U$, be in the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$. It is sufficient to show that the function $h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$ is in the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

We have $h'(z) = (\mu_1 f_1 + \mu_2 f_2)'(z) = \mu_1 f_1'(z) + \mu_2 f_2'(z)$, $z \in U$, and
$(I(m, \lambda, l) h(z))' = (I(m, \lambda, l)(\mu_1 f_1 + \mu_2 f_2)(z))' = \mu_1 (I(m, \lambda, l) f_1(z))' + \mu_2 (I(m, \lambda, l) f_2(z))'$.

From Definition 2.1 we obtain that

$F_{(I(m,\lambda,l)h)'(U)}(I(m, \lambda, l) h(z))' = F_{(I(m,\lambda,l)(\mu_1 f_1 + \mu_2 f_2))'(U)}(I(m, \lambda, l)(\mu_1 f_1 + \mu_2 f_2)(z))' =$
$F_{(I(m,\lambda,l)(\mu_1 f_1 + \mu_2 f_2))'(U)}\big(\mu_1 (I(m, \lambda, l) f_1(z))' + \mu_2 (I(m, \lambda, l) f_2(z))'\big) =$
$\frac{F_{(\mu_1 I(m,\lambda,l)f_1)'(U)}\left(\mu_1 (I(m,\lambda,l)f_1(z))'\right) + F_{(\mu_2 I(m,\lambda,l)f_2)'(U)}\left(\mu_2 (I(m,\lambda,l)f_2(z))'\right)}{2} =$
$\frac{F_{(I(m,\lambda,l)f_1)'(U)}(I(m,\lambda,l)f_1(z))' + F_{(I(m,\lambda,l)f_2)'(U)}(I(m,\lambda,l)f_2(z))'}{2}$.

Since $f_1, f_2 \in SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ we have $\delta < F_{(I(m,\lambda,l)f_1)'(U)}(I(m, \lambda, l) f_1(z))' \leq 1$ and
$\delta < F_{(I(m,\lambda,l)f_2)'(U)}(I(m, \lambda, l) f_2(z))' \leq 1$, $z \in U$.

Therefore $\delta < \frac{F_{(I(m,\lambda,l)f_1)'(U)}(I(m,\lambda,l)f_1(z))' + F_{(I(m,\lambda,l)f_2)'(U)}(I(m,\lambda,l)f_2(z))'}{2} \leq 1$ and we obtain that
$\delta < F_{(I(m,\lambda,l)h)'(U)}(I(m, \lambda, l) h(z))' \leq 1$, which means that $h \in SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ and $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ is convex. ■

We highlight a fuzzy subset obtained using a convex function. Let the function $h(z) = \frac{1+z}{1-z}$, $z \in U$. After a short calculation we obtain that $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\frac{1+z}{1-z} > 0$, so $h \in \mathcal{K}$ and $h(U) = \{z \in \mathbb{C} : Rez > 0\}$. We define the membership function for the set $h(U)$ as $F_{h(U)}(h(z)) = Reh(z)$, $z \in U$ and we have $F_{h(U)} h(z) = \text{supp}\big(h(U), F_{h(u)}\big) = \{z \in \mathbb{C} : 0 < F_{h(U)}(h(z)) \leq 1\} = \{z \in U : 0 < Rez \leq 1\}$.

Remark 2.3 *In this case the membership function can be defined otherwise too and we recommend that those interested to make it in accordance with their scientific concern.*

Theorem 2.2 *Let g be a convex function in U and let $h(z) = g(z) + \frac{1}{c+2} zg'(z)$, where $z \in U$, $c > 0$. If $f \in SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ and $G(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\, dt$, $z \in U$, then*

$$F_{(I(m,\lambda,l)f)'(U)} (I(m, \lambda, l) f(z))' \leq F_{h(U)} h(z), \quad i.e. \quad (I(m, \lambda, l) f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.1}$$

implies $F_{(I(m,\lambda,l)G)'(U)} (I(m, \lambda, l) G(z))' \leq F_{g(U)} g(z)$, i.e. $(I(m, \lambda, l) G(z))' \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. We obtain that

$$z^{c+1} G(z) = (c+2) \int_0^z t^c f(t)\, dt. \tag{2.2}$$

Differentiating (2.2), with respect to z, we have $(c+1) G(z) + zG'(z) = (c+2) f(z)$ and

$$(c+1) I(m, \lambda, l) G(z) + z (I(m, \lambda, l) G(z))' = (c+2) I(m, \lambda, l) f(z), \quad z \in U. \tag{2.3}$$

Differentiating (2.3) we have

$$(I(m, \lambda, l) G(z))' + \frac{1}{c+2} z (I(m, \lambda, l) G(z))'' = (I(m, \lambda, l) f(z))', \quad z \in U. \tag{2.4}$$

Using (2.4), the fuzzy differential subordination (2.1) becomes

$$F_{I(m,\lambda,l)G(U)} \left((I(m, \lambda, l) G(z))' + \frac{1}{c+2} z (I(m, \lambda, l) G(z))'' \right) \leq F_{g(U)} \left(g(z) + \frac{1}{c+2} zg'(z) \right). \tag{2.5}$$

If we denote

$$p(z) = (I(m, \lambda, l) G(z))', \quad z \in U, \tag{2.6}$$

then $p \in \mathcal{H}[1, 1]$.

Replacing (2.6) in (2.5) we obtain $F_{p(U)} \left(p(z) + \frac{1}{c+2} zp'(z) \right) \leq F_{g(U)} \left(g(z) + \frac{1}{c+2} zg'(z) \right)$, $z \in U$.

Using Lemma 1.3 we have $F_{p(U)} p(z) \leq F_{g(U)} g(z)$, $z \in U$, i.e. $F_{(I(m,\lambda,l)G)'(U)} (I(m, \lambda, l) G(z))' \leq F_{g(U)} g(z)$, $z \in U$, and g is the fuzzy best dominant. We have obtained that $(L_{\alpha}^m G(z))' \prec_{\mathcal{F}} g(z)$, $z \in U$. ■

Example 2.1 *If $f \in SI_{\mathcal{F}}^{1}\left(1, \frac{1}{2}, \frac{1}{2}\right)$, then $f'(z) + \frac{1}{3} zf''(z) \prec_{\mathcal{F}} \frac{3-2z}{3(1-z)^2}$ implies $G'(z) + \frac{1}{3} zG''(z) \prec_{\mathcal{F}} \frac{1}{1-z}$, where $G(z) = \frac{3}{z^2} \int_0^z tf(t)\, dt$.*

Theorem 2.3 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$, $\beta \in [0, 1)$ and $c > 0$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$ and $I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\, dt$, $z \in U$, then*

$$I_c \left[SI_{\mathcal{F}}^{\beta}(m, \lambda, l) \right] \subset SI_{\mathcal{F}}^{\beta^*}(m, \lambda, l), \tag{2.7}$$

where $\beta^* = 2\beta - 1 + (c+2)(2 - 2\beta) \int_0^1 \frac{t^{c+1}}{t+1} dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.2 we get from the hypothesis of Theorem 2.3 that $F_{p(U)} \left(p(z) + \frac{1}{c+2} zp'(z) \right) \leq f_{h(U)} h(z)$, where $p(z)$ is defined in (2.6). Using Lemma 1.2 we deduce that $F_{p(U)} p(z) \leq F_{g(U)} g(z) \leq F_{h(U)} h(z)$, i.e. $F_{(I(m,\lambda,l)G)'(U)} (I(m, \lambda, l) G(z))' \leq F_{g(U)} g(z) \leq F_{h(U)} h(z)$, where $g(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} \frac{1+(2\beta-1)t}{1+t} dt = 2\beta - 1 + \frac{(c+2)(2-2\beta)}{z^{c+2}} \int_0^z \frac{t^{c+1}}{t+1} dt$. Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$F_{I(m,\lambda,l)G(U)} (I(m, \lambda, l) G(z))' \geq \min_{|z|=1} F_{g(U)} g(z) = F_{g(U)} g(1) \tag{2.8}$$

and $\beta^* = g(1) = 2\beta - 1 + (c+2)(2 - 2\beta) \int_0^1 \frac{t^{c+1}}{t+1} dt$.

From (2.8) we deduce inclusion (2.7). ■

Theorem 2.4 *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)} (I(m, \lambda, l) f(z))' \leq F_{h(U)} h(z), \quad i.e. \quad (I(m, \lambda, l) f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.9}$$

then $F_{I(m,\lambda,l)f(U)} \frac{I(m,\lambda,l)f(z)}{z} \leq F_{g(U)} g(z)$, i.e. $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. Consider $p(z) = \frac{I(m,\lambda,l)f(z)}{z} = \frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j z^j}{z} = 1 + p_1 z + p_2 z^2 + ...,\quad z \in U$. We deduce that $p \in \mathcal{H}[1,1]$.

Let $I(m,\lambda,l) f(z) = zp(z)$, for $z \in U$. Differentiating we obtain $(I(m,\lambda,l) f(z))' = p(z) + zp'(z)$, $z \in U$. Then (2.9) becomes $F_{p(U)}(p(z) + zp'(z)) \le F_{h(U)} h(z) = F_{g(U)}(g(z) + zg'(z))$, $z \in U$.

By using Lemma 1.3, we have $F_{p(U)} p(z) \le F_{g(U)} g(z)$, $z \in U$, i.e. $F_{(I(m,\lambda,l)f)'(U)} \frac{I(m,\lambda,l)f(z)}{z} \le F_{g(U)} g(z)$, $z \in U$.We obtain that $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp. ■

Theorem 2.5 *Let h be an holomorphic function which satisfies the inequality* Re $\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *and $h(0) = 1$. If $\lambda, l \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)}\left(I(m,\lambda,l) f(z)\right)' \le F_{h(U)} h(z), \quad \text{i.e.} \quad (I(m,\lambda,l) f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.10}$$

then $F_{I(m,\lambda,l)f(U)} \frac{I(m,\lambda,l)f(z)}{z} \le F_{q(U)} q(z)$, i.e. $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$, where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the fuzzy best dominant.

Proof. Let $p(z) = \frac{I(m,\lambda,l)f(z)}{z}$, $z \in U$, $p \in \mathcal{H}[1,1]$. Since Re $\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, from Lemma 1.1, we obtain that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.10) $q(z) + zq'(z) = h(z)$, therefore it is the fuzzy best dominant.

Differentiating, we obtain $(I(m,\lambda,l) f(z))' = p(z) + zp'(z)$, $z \in U$ and (2.10) becomes $F_{p(U)}(p(z) + zp'(z)) \le F_{h(U)} h(z)$, $z \in U$.

Using Lemma 1.3, we have $F_{p(U)} p(z) \le F_{q(U)} q(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)} \frac{I(m,\lambda,l)f(z)}{z} \le F_{q(U)} q(z)$, $z \in U$. We have obtained that $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$. ■

Corollary 2.6 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ a convex function in U, $0 \le \beta < 1$. If $\lambda, l \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)}\left(I(m,\lambda,l) f(z)\right)' \le F_{h(U)} h(z), \text{ i.e. } (I(m,\lambda,l) f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.11}$$

then $F_{I(m,\lambda,l)f(U)} \frac{I(m,\lambda,l)f(z)}{z} \le F_{q(U)} q(z)$, i.e. $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$, where q is given by $q(z) = 2\beta - 1 + \frac{2(1-\beta)}{z}\ln(1+z)$, $z \in U$. The function q is convex and it is the fuzzy best dominant.

Proof. We have $h(z) = \frac{1+(2\beta-1)z}{1+z}$ with $h(0) = 1$, $h'(z) = \frac{-2(1-\beta)}{(1+z)^2}$ and $h''(z) = \frac{4(1-\beta)}{(1+z)^3}$, therefore $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta - i\rho\sin\theta}{1+\rho\cos\theta + i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$.

Following the same steps as in the proof of Theorem 2.5 and considering $p(z) = \frac{I(m,\lambda,l)f(z)}{z}$, the fuzzy differential subordination (2.11) becomes $F_{I(m,\lambda,l)f(U)}(p(z) + zp'(z)) \le F_{h(U)} h(z)$, $z \in U$.

By using Lemma 1.2 for $\gamma = 1$ and $n = 1$, we have $F_{p(U)} p(z) \le F_{q(U)} q(z)$, i.e., $F_{I(m,\lambda,l)f(U)} \frac{I(m,\lambda,l)f(z)}{z} \le F_{q(U)} q(z)$ and $q(z) = \frac{1}{z}\int_0^z h(t)\,dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t} dt = 2\beta - 1 + \frac{2(1-\beta)}{z}\ln(1+z)$, $z \in U$. ■

Example 2.2 *Let $h(z) = \frac{1-z}{1+z}$ with $h(0) = 1$, $h'(z) = \frac{-2}{(1+z)^2}$ and $h''(z) = \frac{4}{(1+z)^3}$.*

Since $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta - i\rho\sin\theta}{1+\rho\cos\theta + i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$, the function h is convex in U.

Let $f(z) = z + z^2$, $z \in U$. For $n = 1$, $m = 1$, $l = 2$, $\lambda = 1$, we obtain $I(1,1,2) f(z) = \frac{2}{3} f(z) + \frac{1}{3} z f'(z) = z + \frac{4}{3} z^2$. Then $(I(1,1,2) f(z))' = 1 + \frac{8}{3} z$ and $\frac{I(1,1,2)f(z)}{z} = 1 + \frac{4}{3} z$. We have $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.5 we obtain $1 + \frac{8}{3} z \prec_{\mathcal{F}} \frac{1-z}{1+z}$, $z \in U$, induce $1 + \frac{4}{3} z \prec_{\mathcal{F}} -1 + \frac{2\ln(1+z)}{z}$, $z \in U$.

Theorem 2.7 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$. If $\lambda, l \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and the fuzzy differential subordination*

$$F_{I(m,\lambda,l)f(U)}\left(\frac{zI(m+1,\lambda,l) f(z)}{I(m,\lambda,l) f(z)}\right)' \le F_{h(U)} h(z), \quad \text{i.e.} \quad \left(\frac{zI(m+1,\lambda,l) f(z)}{I(m,\lambda,l) f(z)}\right)' \prec_{\mathcal{F}} h(z), \quad z \in U \tag{2.12}$$

holds, then $F_{I(m,\lambda,l)f(U)} \frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \le F_{g(U)} g(z)$, i.e. $\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. Consider $p(z) = \frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)}$. We have $p'(z) = \frac{(I(m+1,\lambda,l)f(z))'}{I(m,\lambda,l)f(z)} - p(z) \cdot \frac{(I(m+1,\lambda,l)f(z))'}{I(m,\lambda,l)f(z)}$ and we obtain $p(z) + z \cdot p'(z) = \left(\frac{zI(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)}\right)'$.

Relation (2.12) becomes $F_{p(U)}(p(z)+zp'(z)) \leq F_{h(U)}h(z) = F_{g(U)}(g(z)+zg'(z))$, $z \in U$. By using Lemma 1.3, we have $F_{p(U)}p(z) \leq F_{g(U)}g(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)}\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \leq F_{g(U)}g(z)$, $z \in U$. We obtain that $\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \prec_{\mathcal{F}} g(z)$, $z \in U$. ■

Theorem 2.8 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and the fuzzy differential subordination* $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right) \leq F_{h(U)}h(z)$, *i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z) \prec_{\mathcal{F}} h(z), \quad z \in U \tag{2.13}$$

holds, then $F_{I(m,\lambda,l)f(U)}[I(m,\lambda,l)f(z)]' \leq F_{g(U)}g(z)$, i.e. $[I(m,\lambda,l)f(z)]' \prec_{\mathcal{F}} g(z)$, $z \in U$. This result is sharp.

Proof. Let $p(z) = (I(m,\lambda,l)f(z))'$. We deduce that $p \in \mathcal{H}[1,1]$. We obtain $p(z) + z \cdot p'(z) = I(m,\lambda,l)f(z) + z(I(m,\lambda,l)f(z))' = I(m,\lambda,l)f(z) + \frac{(l+1)I(m+1,\lambda,l)f(z)-(l+1-\lambda)I(m,\lambda,l)f(z)}{\lambda} = \frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)$.

The fuzzy differential subordination becomes $F_{p(U)}(p(z)+zp'(z)) \leq F_{h(U)}h(z) = F_{g(U)}(g(z)+zg'(z))$. By using Lemma 1.3, we have $F_{p(U)}p(z) \leq F_{g(U)}g(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{g(U)}g(z)$, $z \in U$, and this result is sharp. ■

Theorem 2.9 *Let h be an holomorphic function which satisfies the inequality* $\operatorname{Re}\left[1+\frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}$, $z \in U$, *and $h(0) = 1$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the fuzzy differential subordination* $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right) \leq F_{h(U)}h(z)$, *i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z) \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.14}$$

then $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{q(U)}q(z)$, i.e. $(I(m,\lambda,l)f(z))' \prec_{\mathcal{F}} q(z)$, $z \in U$, where q is given by $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the fuzzy best dominant.

Proof. Since $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, from Lemma 1.1, we obtain that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.14) $q(z) + zq'(z) = h(z)$, therefore it is the fuzzy best dominant.
Considering $p(z) = (I(m,\lambda,l)f(z))'$, we obtain $p(z)+zp'(z) = \frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)$, $z \in U$. Then (2.14) becomes $F_{p(U)}(p(z)+zp'(z)) \leq F_{h(U)}h(z)$, $z \in U$.
Since $p \in \mathcal{H}[1,1]$, using Lemma 1.3, we deduce $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{q(U)}q(z)$, $z \in U$. We have obtained that $(I(m,\lambda,l)f(z))' \prec_{\mathcal{F}} q(z)$, $z \in U$. ■

Corollary 2.10 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \leq \beta < 1$.If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination* $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right) \leq F_{h(U)}h(z)$, *i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z) \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.15}$$

then $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{q(U)}q(z)$, i.e. $(I(m,\lambda,l)f(z))' \prec_{\mathcal{F}} q(z)$, $z \in U$, where q is given by $q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}$, for $z \in U$. The function q is convex and it is the fuzzy best dominant.

Proof. Following the same steps as in the proof of Theorem 2.8 and considering $p(z) = (I(m,\lambda,l)f(z))'$, the fuzzy differential subordination (2.15) becomes $F_{p(U)}(p(z)+zp'(z)) \leq F_{h(U)}h(z)$, $z \in U$.

By using Lemma 1.2 for $\gamma = 1$ and $n = 1$, we have $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, i.e., $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{q(U)}q(z)$, i.e. $(I(m,\lambda,l)f(z))' \prec_{\mathcal{F}} q(z)$, $z \in U$, and $q(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = 2\beta - 1 + 2(1-\beta)\frac{1}{z}\ln(z+1)$, $z \in U$. ■

Example 2.3 *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$ *and* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$ *(see Example 2.2).*

Let $f(z) = z + z^2$, $z \in U$. *For* $n = 1$, $m = 1$, $l = 2$, $\lambda = 1$, *we obtain* $I(1,1,2) f(z) = \frac{2}{3} f(z) + \frac{1}{3} z f'(z) = z + \frac{4}{3} z^2$ *and* $(I(1,1,2) f(z))' = 1 + \frac{8}{3} z$. *We obtain also* $\frac{l+1}{\lambda} I(m+1,\lambda,l) f(z) + \left(2 - \frac{l+1}{\lambda}\right) I(m,\lambda,l) f(z) = 3I(2,1,2) f(z) - I(1,1,2) f(z) = 2z + 4z^2$, *where* $I(2,1,2) f(z) = \frac{2}{3} I(1,1,2) f(z) + \frac{z}{3}(I(1,1,2) f(z))' = 3z + \frac{16}{3} z^2$. *We have* $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.9 we obtain $2z + 4z^2 \prec_{\mathcal{F}} \frac{1-z}{1+z}$, $z \in U$, *induce* $1 + \frac{8}{3} z \prec_{\mathcal{F}} -1 + \frac{2\ln(1+z)}{z}$, $z \in U$.

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On some differential sandwich theorems involving a multiplier transformation and Ruscheweyh derivative

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Abstract

In this paper we obtain some subordination and superordination results for the operator $IR_{\lambda,l}^{m,n}$ and we establish differential sandwich-type theorems. The operator $IR_{\lambda,l}^{m,n}$ is defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative R^n.

Keywords: analytic functions, differential operator, differential subordination, differential superordination.
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1 Introduction

Consider $\mathcal{H}(U)$ the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{H}(a,n)$ the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$ with $\mathcal{A} = \mathcal{A}_1$.

Next we remind the definition of differential subordination and superordination.

Let the functions f and g be analytic in U. The function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z), zp'(z), z^2 p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \qquad z \in U, \tag{1.2}$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [6] obtained conditions h, q and ψ for which the following implication holds $h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z)$.

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by $f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j$.

We need the following differential operators.

Definition 1.1 *[5] For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the multiplier transformation $I(m,\lambda,l) f(z)$ is defined by the following infinite series $I(m,\lambda,l) f(z) := z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m a_j z^j$.*

Remark 1.1 *We have* $(l+1)\, I(m+1,\lambda,l)\, f(z) = (l+1-\lambda)\, I(m,\lambda,l)\, f(z) + \lambda z\, (I(m,\lambda,l)\, f(z))'$, $\quad z \in U$.

Remark 1.2 *For* $l=0$, $\lambda \geq 0$, *the operator* $D_\lambda^m = I(m,\lambda,0)$ *was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator* $S^m = I(m,1,0)$ *for* $\lambda = 1$.

Definition 1.2 *(Ruscheweyh [8]) For* $f \in \mathcal{A}$ *and* $n \in \mathbb{N}$, *the Ruscheweyh derivative* R^n *is defined by* $R^n : \mathcal{A} \to \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z), \ R^1 f(z) = z f'(z), \ ... \\ (n+1)\, R^{n+1} f(z) &= z\,(R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.3 *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ *for* $z \in U$.

Definition 1.3 *([2]) Let* $\lambda, l \geq 0$ *and* $n, m \in \mathbb{N}$. *Denote by* $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ *the operator given by the Hadamard product of the multiplier transformation* $I(m,\lambda,l)$ *and the Ruscheweyh derivative* R^n, $IR_{\lambda,l}^{m,n} f(z) = (I(m,\lambda,l) * R^n)\, f(z)$, *for any* $z \in U$ *and each nonnegative integers* m, n.

Remark 1.4 *If* $f \in \mathcal{A}$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

Using simple computation we obtain the following relation.

Proposition 1.1 *[1]For* $m, n \in \mathbb{N}$ *and* $\lambda \geq 0$ *we have*

$$IR_{\lambda,l}^{m+1,n} f(z) = \frac{1+l-\lambda}{l+1} IR_{\lambda,l}^{m,n} f(z) + \frac{\lambda}{l+1} z \left(IR_{\lambda,l}^{m,n} f(z)\right)' \tag{1.3}$$

Definition 1.4 *[7] Denote by* Q *the set of all functions* f *that are analytic and injective on* $\overline{U} \backslash E(f)$, *where* $E(f) = \{\zeta \in \partial U : \lim\limits_{z \to \zeta} f(z) = \infty\}$, *and are such that* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U \backslash E(f)$.

Lemma 1.1 *[7] Let the function* q *be univalent in the unit disc* U *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$ *with* $\phi(w) \neq 0$ *when* $w \in q(U)$. *Set* $Q(z) = z q'(z)\, \phi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that* Q *is starlike univalent in* U *and* $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ *for* $z \in U$. *If* p *is analytic with* $p(0) = q(0)$, $p(U) \subseteq D$ *and* $\theta(p(z)) + z p'(z)\, \phi(p(z)) \prec \theta(q(z)) + z q'(z)\, \phi(q(z))$, *then* $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 1.2 *[4] Let the function* q *be convex univalent in the open unit disc* U *and* ν *and* ϕ *be analytic in a domain* D *containing* $q(U)$. *Suppose that* $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ *for* $z \in U$ *and 2.* $\psi(z) = z q'(z)\, \phi(q(z))$ *is starlike univalent in* U. *If* $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, *with* $p(U) \subseteq D$ *and* $\nu(p(z)) + z p'(z)\, \phi(p(z))$ *is univalent in* U *and* $\nu(q(z)) + z q'(z)\, \phi(q(z)) \prec \nu(p(z)) + z p'(z)\, \phi(p(z))$, *then* $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2 Main results

We intend to find sufficient conditions for certain normalized analytic functions f such that $q_1(z) \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, $z \in U$, $0 < \delta \leq 1$, where q_1 and q_2 are given univalent functions.

Theorem 2.1 *Let* $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}(U)$ *and let the function* $q(z)$ *be analytic and univalent in* U *such that* $q(z) \neq 0$, *for all* $z \in U$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U. *Let*

$$Re\left(\frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0, \tag{2.1}$$

for $\alpha, \xi, \beta, \mu \in \mathbb{C}$, $\beta \neq 0$, $z \in U$ *and*

$$\psi_{\lambda,l}^{m,n}(\alpha, \xi, \mu, \beta; z) := \alpha + \beta\frac{(l+1)}{\lambda} + \beta\frac{(l+1)}{\lambda}\frac{IR_{\lambda,l}^{m+2,n} f(z)}{IR_{\lambda,l}^{m+1,n} f(z)} - \tag{2.2}$$

$$\beta\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}+\xi\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\mu\frac{z^{2\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+2\delta}}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)},\tag{2.3}$$

for $\alpha,\xi,\beta,\mu\in\mathbb{C}$, $\beta\neq0$, then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, and q is the best dominant.

Proof. Consider $p(z):=\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, $z\neq0$, $f\in\mathcal{A}$. Differentiating we obtain $p'(z)=\frac{\delta(1+l)}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\frac{l+1}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{(l+1)(1+\delta)}{\lambda}\frac{z^{\delta-1}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$.

By using the identity (1.3), we obtain

$$\frac{zp'(z)}{p(z)}=\frac{\delta(l+1)}{\lambda}+\frac{l+1}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}-\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}.\tag{2.4}$$

By setting $\theta(w):=\alpha+\xi w+\mu w^{2}$ and $\phi(w):=\frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\beta\frac{zq'(z)}{q(z)}$ and $h(z)=\theta(q(z))+Q(z)=\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

We get $h'(z)=\xi q'(z)+2\mu q(z)q'(z)+\beta\frac{q'(z)}{q(z)}+\beta z\frac{q''(z)}{q(z)}-\beta z\left(\frac{q'(z)}{q(z)}\right)^{2}$ and $\frac{zh'(z)}{Q(z)}=\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^{2}(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}$.

So we deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^{2}(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}\right)>0$.

By using (2.4), we obtain $\alpha+\xi p(z)+\mu(p(z))^{2}+\beta\frac{zp'(z)}{p(z)}=$
$\alpha+\beta\frac{(l+1)}{\lambda}+\beta\frac{(l+1)}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}-\beta\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}+\xi\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\mu\frac{z^{2\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+2\delta}}$.

By using (2.3), we have $\alpha+\xi p(z)+\mu(p(z))^{2}+\beta\frac{zp'(z)}{p(z)}\prec\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)}$.

Appying Lemma 1.1, we obtain $p(z)\prec q(z)$, $z\in U$, i.e. $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, $z\in U$ and q is the best dominant. ■

Corollary 2.2 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq0$. Assume that (2.1) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^{2}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq0$, $-1\leq B<A\leq1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq1$ in Theorem 2.1 we get the corollary. ■

Corollary 2.3 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq0$. Assume that (2.1) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^{2}}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $0<\gamma\leq1$, $\beta\neq0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.*

Proof. Corollary follows by using Theorem 2.1 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq1$. ■

Theorem 2.4 *Let q be analytic and univalent in U such that $q(z)\neq0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$Re\left(\frac{\xi}{\beta}q(z)q'(z)+\frac{2\mu}{\beta}q^{2}(z)q'(z)\right)>0,\ \textit{for}\ \xi,\beta,\mu\in\mathbb{C},\ \beta\neq0.\tag{2.5}$$

If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is univalent in* U*, where* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2), then*

$$\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \tag{2.6}$$

implies $q(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, *and* q *is the best subordinant.*

Proof. Consider $p(z) := \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)q(z)[\xi + 2\mu q(z)]}{\beta}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$.

By using (2.4) and (2.6) we get $\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \alpha + \xi p(z) + \mu (p(z))^2 + \frac{\beta z p'(z)}{p(z)}$. Applying Lemma 1.2, we obtain $q(z) \prec p(z) = \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, and q is the best subordinant. ■

Corollary 2.5 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.5) holds. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \frac{1+Az}{1+Bz} + \mu \left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\beta,\xi,\mu \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+Az}{1+Bz} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, *and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.4 we get the corollary. ■

Corollary 2.6 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.5) holds. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. For $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$ in Theorem 2.4 we get the corollary. ■

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7 *Let* q_1 *and* q_2 *be analytic and univalent in* U *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$, *for all* $z \in U$, *with* $\frac{zq_1'(z)}{q_1(z)}$ *and* $\frac{zq_2'(z)}{q_2(z)}$ *being starlike univalent. Suppose that* q_1 *satisfies (2.1) and* q_2 *satisfies (2.5). If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2) univalent in* U*, then* $\alpha + \xi q_1(z) + \mu (q_1(z))^2 + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi q_2(z) + \mu (q_2(z))^2 + \frac{\beta z q_2'(z)}{q_2(z)}$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, *implies* $q_1(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, *and* q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.8 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.1) and (2.5) hold. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \frac{1+A_1 z}{1+B_1 z} + \mu \left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \frac{\beta(A_1-B_1)z}{(1+A_1 z)(1+B_1 z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi \frac{1+A_2 z}{1+B_2 z} + \mu \left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \frac{\beta(A_2-B_2)z}{(1+A_2 z)(1+B_2 z)}$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z}$, *hence* $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.9 *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) and (2.5) hold. If $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1}+\frac{2\beta\gamma_1 z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2}+\frac{2\beta\gamma_2 z}{1-z^2}$, for $\alpha, \beta, \mu, \xi \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

Changing the functions θ and ϕ we obtain the following results.

Theorem 2.10 *Let $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}(U)$, $f \in \mathcal{A}$, $z \in U$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$ and let the function $q(z)$ be convex and univalent in U such that $q(0) = 1$, $z \in U$. Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta} + z\frac{q''(z)}{q'(z)}\right) > 0, \tag{2.7}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, and

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) := \frac{\beta(l+1)}{\lambda}\frac{z^\delta IR_{\lambda,l}^{m+2,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} + \left(\alpha + \frac{\beta\delta(l+1)}{\lambda}\right)\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \tag{2.8}$$

$$-\frac{\beta(1+\delta)(l+1)}{\lambda}\frac{z^\delta\left(IR_{\lambda,l}^{m+1,n} f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{2+\delta}}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha q(z) + \beta z q'(z), \tag{2.9}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, then $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q(z)$, $z \in U$, and q is the best dominant.

Proof. Consider $p(z) := \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$. Differentiating we get $p'(z) = \frac{\delta(1+l)}{\lambda}\frac{z^{\delta-1} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} + \frac{l+1}{\lambda}\frac{z^{\delta-1} IR_{\lambda,l}^{m+2,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} - \frac{(l+1)(1+\delta)}{\lambda}\frac{z^{\delta-1}\left(IR_{\lambda,l}^{m+1,n} f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{2+\delta}}$.
By using the identity (1.3), we get

$$zp'(z) = \frac{l+1}{\lambda}\frac{z^\delta IR_{\lambda,l}^{m+2,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} + \frac{\delta(1+l)}{\lambda}\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} - \frac{(l+1)(1+\delta)}{\lambda}\frac{z^\delta\left(IR_{\lambda,l}^{m+1,n} f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{2+\delta}}. \tag{2.10}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \beta zq'(z)$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta zq'(z)$. We have $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\alpha+\beta}{\beta} + z\frac{q''(z)}{q'(z)}\right) > 0$.

By using (2.10), we obtain $\alpha p(z) + \beta zp'(z) = \frac{\beta(l+1)}{\lambda}\frac{z^\delta IR_{\lambda,l}^{m+2,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} + \left(\alpha + \frac{\beta\delta(l+1)}{\lambda}\right)\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} - \frac{\beta(1+\delta)(l+1)}{\lambda}\frac{z^\delta\left(IR_{\lambda,l}^{m+1,n} f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{2+\delta}}$. By using (2.9), we have $\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z)$. From Lemma 1.1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q(z)$, $z \in U$, and q is the best dominant. ■

Corollary 2.11 *Let* $q(z)=\frac{1+Az}{1+Bz}$, $z\in U$, $-1\le B<A\le 1$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. *Assume that (2.7) holds. If* $f\in\mathcal{A}$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec\alpha\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Bz)^2}$, *for* $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, $-1\le B<A\le 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\frac{1+Az}{1+Bz}$, *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, in Theorem 2.10 we get the corollary. ■

Corollary 2.12 *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. *Assume that (2.7) holds. If* $f\in\mathcal{A}$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma}$, *for* $\alpha,\beta\in\mathbb{C}$, $0<\gamma\le 1$, $\beta\ne 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.10 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\le 1$. ■

Theorem 2.13 *Let* q *be convex and univalent in* U *such that* $q(0)=1$. *Assume that*

$$Re\left(\frac{\alpha}{\beta}q'(z)\right)>0,\ \text{for } \alpha,\beta\in\mathbb{C},\ \beta\ne 0. \tag{2.11}$$

If $f\in\mathcal{A}$, $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}[q(0),1]\cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ *is univalent in* U, *where* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ *is as defined in (2.8), then*

$$\alpha q(z)+\beta zq'(z)\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \tag{2.12}$$

implies $q(z)\prec\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $\delta\in\mathbb{C}$, $\delta\ne 0$, $z\in U$, *and* q *is the best subordinant.*

Proof. Consider $p(z):=\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, $z\ne 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p(0)=1$.

By setting $\nu(w):=\alpha w$ and $\phi(w):=\beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\ne 0$, $w\in\mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))}=\frac{\alpha}{\beta}q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)=Re\left(\frac{\alpha}{\beta}q'(z)\right)>0$, for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$.

Now, by using (2.12) we obtain $\alpha q(z)+\beta zq'(z)\prec\alpha p(z)+\beta zp'(z)$, $z\in U$. From Lemma 1.2, we have $q(z)\prec p(z)=\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, and q is the best subordinant. ■

Corollary 2.14 *Let* $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. *Assume that (2.11) holds. If* $f\in\mathcal{A}$, $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}[q(0),1]\cap Q$, *and* $\alpha\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Bz)^2}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$, *for* $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, $-1\le B<A\le 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\frac{1+Az}{1+Bz}\prec\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, *and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, in Theorem 2.13 we get the corollary. ■

Corollary 2.15 *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. *Assume that (2.11) holds. If* $f\in\mathcal{A}$, $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}[q(0),1]\cap Q$ *and* $\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$, *for* $\alpha,\beta\in\mathbb{C}$, $0<\gamma\le 1$, $\beta\ne 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.13 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\le 1$. ■

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 *Let q_1 and q_2 be convex and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$. Suppose that q_1 satisfies (2.7) and q_2 satisfies (2.11). If $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$, and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is as defined in (2.8) univalent in U, then $\alpha q_1(z) + \beta z q_1'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha q_2(z) + \beta z q_2'(z)$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, implies $q_1(z) \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, $z \in U$, and q_1 and q_2 are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.17 *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) and (2.11) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha \frac{1+A_1 z}{1+B_1 z} + \frac{\beta(A_1-B_1)z}{(1+B_1 z)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ $\prec \alpha \frac{1+A_2 z}{1+B_2 z} + \frac{\beta(A_2-B_2)z}{(1+B_2 z)^2}$, $z \in U$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z}$, $z \in U$, hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.18 *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) and (2.11) hold for $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha \left(\frac{1+z}{1-z}\right)^{\gamma_1} + \frac{2\beta\gamma_1 z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ $\prec \alpha \left(\frac{1+z}{1-z}\right)^{\gamma_2} + \frac{2\beta\gamma_2 z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

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FUZZY STABILITY OF A CLASS OF ADDITIVE-QUADRATIC FUNCTIONAL EQUATIONS

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Abstract. In this paper, we consider the following functional equation

$$\begin{aligned} &af(x+y)+bf(x-y)+cf(y-x)\\ &=(a+b)f(x)+cf(-x)+(a+c)f(y)+bf(-y) \end{aligned}$$

for a fixed real numbers a, b, c with $a = b + c$ and $a \neq 0$. We study the fuzzy version of the generalized Hyers-Ulam stability for it in the sense of Mirmostafaee and Moslehian.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [20]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exists a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [1] for additive mappings, and by Rassias [19] for linear mappings, to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians ([5], [6], [7], [10], [18]).

Recently, the stability in fuzzy spaces has been extensively studied ([3], [12], [15], [16], [17]). The concept of fuzzy norm on a linear space was introduced by Katsaras [14] in 1984. Later, Cheng and Mordeson [4] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. In 2008, for the first time, Mirmostafaee and Moslehian [16], [17] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of stability for the Cauchy functional equation

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.2}$$

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We call a solution of (1.1) *an additive mapping* and a solution of (1.2) is called *a quadratic mapping*. Also,

$$f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)=0$$

is called *Drygas functional equation*(see [8], [9] for detail.). It is easy to see that the function $f(x)=px^2+qx$ is a solution of Drygas functional equation and so we can expect that a solution of Drygas functional equation is an additive-quadratic mapping.

Now, we consider the following functional equation

$$\begin{aligned} &af(x+y)+bf(x-y)+cf(y-x)\\ &=(a+b)f(x)+cf(-x)+(a+c)f(y)+bf(-y) \end{aligned} \tag{1.3}$$

for fixed real numbers a, b, c with $a=b+c$ and $a\neq 0$ and show the generalized Hyers-Ulam stability of (1.3) in a fuzzy sense [18].

Definition 1.1. Let X be a real vector space. A function $N : X\times\mathbb{R}\longrightarrow[0,1]$ is called *a fuzzy norm on* X if for all $x, y\in X$ and all $s, t\in\mathbb{R}$,

(N1) $N(x,t)=0$ for $t\leq 0$;

(N2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;

(N3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;

(N4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;

(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;

(N6) for any $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X,N) is called *a fuzzy normed space.*

Let (X,N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* in (X,N) if there exists an $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X,N) and one denotes it by $N-\lim_{n\to\infty}x_n=x$. A sequence $\{x_n\}$ in X is said to be *Cauchy* if for any $\epsilon>0$, there is an $m\in N$ such that for any $n\geq m$ and any positive integer p, $N(x_{n+p}-x_n,t)>1-\epsilon$ for all $t>0$.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space.*

2. Solutions and the Generalized Hyers-Ulam stability of (1.3)

In this section, we investigate solutions of (1.3) and prove the generalized Hyers-Ulam stability of (1.3) in fuzzy Banach spaces. Throughout this section, we assume that (X,N) is a fuzzy normed space and (Y,N') is a fuzzy Banach space. In Theorem 2.3, it can be concluded that any solution of (1.3) is additive-quadratic. We start with the following lemma.

Lemma 2.1. *Let $f : X\longrightarrow Y$ be an odd mapping satisfying (1.3). Then f is an additive mapping.*

Proof. Since $a\neq 0$, $f(0)=0$. Since f is an odd mapping, the functional equation (1.3) can be written by

$$af(x+y)+(b-c)f(x-y)=(a+b-c)f(x)+(a-b+c)f(y) \tag{2.1}$$

for all $x, y \in X$. Interchanging x and y in (2.1), we have

$$af(x+y) - (b-c)f(x-y) = (a+b-c)f(y) + (a-b+c)f(x) \tag{2.2}$$

for all $x, y \in X$. By (2.1) and (2.2),

$$af(x+y) = af(x) + af(y)$$

for all $x, y \in X$ and since $a \neq 0$, f is additive. $\square$

Lemma 2.2. *Let $f : X \longrightarrow Y$ be an even mapping satisfying (1.3). Then f is a quadratic mapping.*

Proof. Since $a \neq 0$, $f(0) = 0$. Since f is an even mapping, the functional equation (1.3) can be written by

$$af(x+y) + (b+c)f(x-y) = (a+b+c)f(x) + (a+b+c)f(y) \tag{2.3}$$

for all $x, y \in X$. Letting $y = -y$ in (2.3), we have

$$af(x-y) + (b+c)f(x+y) = (a+b+c)f(x) + (a+b+c)f(y) \tag{2.4}$$

for all $x, y \in X$. Since $a = b + c$, by (2.3) and (2.4), we have

$$2af(x-y) + 2af(x+y) = 4af(x) + 4af(y)$$

for all $x, y \in X$ and since $a \neq 0$, f is a quadratic mapping. $\square$

Combining Lemma 2.1 and Lemma 2.2, we have the following theorem.

Theorem 2.3. *Let $f : X \longrightarrow Y$ be a mapping. If f satisfies (1.3), then f is an additive-quadratic mapping.*

For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x,y) = af(x+y)+bf(x-y)+cf(y-x)-(a+b)f(x)-cf(-x)-(a+c)f(y)-bf(-y)$$

for all $x, y \in X$. For a given $q > 0$, the mapping f is said to be *a fuzzy q-almost additive-quadratic mapping if*

$$N'(Df(x,y), t+s) \geq \min\{N(x, t^q), N(y, s^q)\} \tag{2.5}$$

for all $x, y \in X$ and all positive real numbers t, s.

Theorem 2.4. *Let q be a positive real number with $q \neq 1, \frac{1}{2}$ and $f : X \longrightarrow Y$ a fuzzy q-almost additive-quadratic mapping. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

(2.6)

$$N(F(x)-f(x), t) \geq \begin{cases} \sup_{s<t}\{N(x, (1-2^{p-1})^q|a|^q s^q)\}, & \text{if } q > 1 \\ \sup_{s<t}\{N(x, (2^{p-1}-1)^q(2-2^{(p-1)})^q|a|^q s^q)\}, & \text{if } \frac{1}{2} < q < 1 \\ \sup_{s<t}\{N(x, (2^{p-1}-2)^q|a|^q s^q\}, & \text{if } 0 < q < \frac{1}{2} \end{cases}$$

holds for all $x \in X$ and all $t > 0$, where $p = \frac{1}{q}$.

Proof. By (2.5), (N2), and (N4), since $a = b + c$, we have

$$N'(Df(0,0),\ t) = N'(f(0),\ \frac{t}{2|a|}) \geq N'(0,\ t^q) = 1$$

for all $t > 0$ and by (N2), $f(0) = 0$.

Case 1. Let $q > 1$ and define a mapping $J_n f : X \longrightarrow Y$ by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2 \cdot 2^n}$$

for all $x \in X$ and all positive integer n. Then we have

$$\begin{aligned} &J_n f(x) - J_{n+1} f(x) \\ &= \frac{2^{n+1} - 1}{a \cdot 2 \cdot 4^{n+1}} Df(-2^n x, -2^n x) - \frac{2^{n+1} + 1}{a \cdot 2 \cdot 4^{n+1}} Df(2^n x, 2^n x) \end{aligned} \tag{2.7}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.7), (N3), and (N4), we have

(2.8)

$$\begin{aligned} &N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t^p) \\ &= N'(\sum_{i=m}^{m+n-1} [J_i f(x) - J_{i+1} f(x)], \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t^p) \\ &\geq \min\{N'(J_i f(x) - J_{i+1} f(x), \frac{2^{pi}}{|a| \cdot 2^i} t^p) \mid m \leq i \leq m+n-1\} \\ &\geq \min\{N'(\frac{2^{i+1} - 1}{a \cdot 2 \cdot 4^{i+1}} Df(-2^i x, -2^i x) - \frac{2^{i+1} + 1}{a \cdot 2 \cdot 4^{i+1}} Df(2^i x, 2^i x), \frac{2^{pi}}{|a| \cdot 2^i} t^p) \mid \\ &\quad m \leq i \leq m+n-1\} \\ &\geq \min\{\min\{N'(\frac{2^{i+1} + 1}{a \cdot 2 \cdot 4^{i+1}} Df(2^i x, 2^i x), \frac{(2^{i+1} + 1) 2^{pi}}{|a| \cdot 4^{i+1}} t^p), \\ &\quad N'(\frac{2^{i+1} - 1}{a \cdot 2 \cdot 4^{i+1}} Df(-2^i x, -2^i x), \frac{(2^{i+1} - 1) 2^{pi}}{|a| \cdot 4^{i+1}} t^p)\} \mid m \leq i \leq m+n-1\} \\ &\geq \min\{\min\{N'(Df(2^i x, 2^i x), 2^{pi+1} t^p), N'(Df(-2^i x, -2^i x), 2^{pi+1} t^p)\} | m \leq i \leq m+n-1\} \\ &\geq \min\{\min\{N(2^i x, 2^i t), N(-2^i x, 2^i t)\} \mid m \leq i \leq m+n-1\} \\ &= N(x, t) \end{aligned}$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Let $\epsilon > 0$ be given. Since $\lim_{t \longrightarrow \infty} N(x,t) = 1$, there is a t_1 such that $N(x, t_1) > 1 - \epsilon$. Let $t_2 > t_1$. Since $p < 1$, $\sum_{n=0}^{\infty} \frac{2^{pn}}{|a| \cdot 2^n} t_2^p$ is convergent. Let $s > 0$. Then there is a positive integer k such that $\sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t_2^p < s$ for $m, n > k$ and so by (2.8), we have

$$\begin{aligned}
&N'(J_m f(x) - J_{m+n} f(x), s)\\
&\geq N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t_2^p)\\
&\geq N(x, t_2)\\
&\geq 1 - \epsilon
\end{aligned}$$

for all $x \in X$. Hence $\{J_n f(x)\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, we can define a mapping $F : X \longrightarrow Y$ by

$$F(x) = N' - \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Letting $m = 0$ in (2.8), we have

$$N'(f(x) - J_n f(x), t) \geq N(x, \frac{t^q}{[\sum_{i=0}^{n-1} \frac{2^{pi}}{|a|\cdot 2^i}]^q}) \tag{2.9}$$

for all $x \in X$, all positive integer n, and all $t > 0$. By (N4), we have

$$\begin{aligned}
&N'(DF(x, y), t)\\
&\geq \min\{N'(a[F - J_n f](x + y), \frac{t}{14}), N'(b[F - J_n f](x - y), \frac{t}{14}),\\
&N'(c[F - J_n f](y - x), \frac{t}{14}), N'((a + b)[F - J_n f](x), \frac{t}{14})\\
&- N'(c[F - J_n f](-x), \frac{t}{14}), N'((a + c)[F - J_n f](y), \frac{t}{14})\\
&- N'(b[F - J_n f](-y), \frac{t}{14}), N'(J_n Df(x, y), \frac{t}{2})\}
\end{aligned} \tag{2.10}$$

for all $x, y \in X$ and all positive integer n. The first seven terms on the right-hand of (2.10) tend to 1 as $n \to \infty$ and by (N4), we have

$$\begin{aligned}
&N'(J_n Df(x, y), \frac{t}{2})\\
&\geq \min\{N'(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8}), N'(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8}),\\
&N'(\frac{Df(-2^n x, -2^n y)}{2 \cdot 2^n}, \frac{t}{8}), N'(\frac{Df(2^n x, 2^n y)}{2 \cdot 2^n}, \frac{t}{8})\}
\end{aligned} \tag{2.11}$$

for all $x, y \in X$, all positive integer n and all $t > 0$. By (N3) and (2.5), we have

$$\begin{aligned}
&N'(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 4^n}, \frac{t}{8})\\
&= N'(Df(\pm 2^n x, \pm 2^n y, 4^{n-1} t))\\
&\geq \min\{N(2^n x, 2^{q(2n-3)} t^q), N(2^n y, 2^{q(2n-3)} t^q)\}\\
&\geq \min\{N(x, 2^{(2q-1)n-3q} t^q), N(y, 2^{(2q-1)n-3q} t^q)\}
\end{aligned} \tag{2.12}$$

for all $x, y \in X$, all positive integer n, and all $t > 0$. Since $q > 1$, by (2.11) and (2.12), we have

$$\lim_{n\to\infty} N'(J_n Df(x, y), \frac{t}{2}) = 1$$

and so by (2.10), $N'(DF(x,y),t) = 0$ for all $x, y \in X$ and all $t > 0$. By (N2), $DF(x,y) = 0$ for all $x, y \in X$ and by Theorem 2.3, F is additive-quadraic.

Now we will show that (2.6) holds. Let $x \in X$, $t > 0$, $s > 0$ with $0 < s < t$ and $0 < \epsilon < 1$. Since $F(x) = N' - \lim_{n\to\infty} J_n f(x)$, there is a positive integer n such that

$$N'(F(x) - J_n f(x), t - s) \geq 1 - \epsilon$$

and so by (2.9),

$$\begin{aligned}
&N'(F(x) - f(x), t)\\
&\geq \min\{N'(F(x) - J_n f(x), t - s), N'(J_n f(x) - f(x), s)\}\\
&\geq \min\{1 - \epsilon, N(x, \frac{s^q}{[\sum_{i=0}^{n-1} \frac{2^{pi}}{|a|\cdot 2^i}]^q})\}\\
&\geq \min\{1 - \epsilon, N(x, (1 - 2^{p-1})^q s^q |a|^q)\}.
\end{aligned}$$

and so we have (2.6).

To prove the uniqueness of F, let $F_1 : X \longrightarrow Y$ be another additive-quadratic mapping satisfying (2.6). Then

$$F(x) - F_1(x) = J_n F(x) - J_n F_1(x)$$

for all $x \in X$ and all positive integer n. Hence by (N4), (N5), and (2.6), we have

$$\begin{aligned}
&N'(F(x) - F_1(x), t)\\
&= N'(J_n F(x) - J_n F_1(x), t)\\
&\geq \min\{N'(J_n F(x) - J_n f(x), \frac{t}{2}), N'(J_n F_1(x) - J_n f(x), \frac{t}{2})\}\\
&\geq \min\{N'(\frac{F(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(\frac{F(2^n x) - f(2^n x)}{2\cdot 2^n}, \frac{t}{8}), N'(\frac{F(-2^n x) - f(-2^n x)}{2\cdot 2^n}, \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F_1(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^n x) - f(2^n x)}{2\cdot 2^n}, \frac{t}{8}), N'(\frac{F_1(-2^n x) - f(-2^n x)}{2\cdot 2^n}, \frac{t}{8})\}\\
&\geq \sup_{s<t}\{N(2^n x, (1 - 2^{p-1})^q 2^{(n-3)q} s^q |a|^q)\}\\
&\geq \sup_{s<t}\{N(x, (1 - 2^{p-1})^q |a|^q s^q 2^{(q-1)n-3q})\}
\end{aligned}$$

for all $x, y \in X$, all positive integer n and all $0 < s < t$. Since $q > 1$,

$$\lim_{n\to\infty} \sup_{s<t}\{N(x, (1 - 2^{p-1})^q |a|^q s^q 2^{(q-1)n-3q}\} = 1$$

and so $N'(F(x) - F_1(x), t) = 1$ for all $t > 0$. Hence $F = F_1$.

Case 2. Let $\frac{1}{2} < q < 1$ and define a mapping $J_n f : X \longrightarrow Y$ by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x)}{2\cdot 4^n} + \frac{2^n}{2}[f(2^{-n}x) - f(-2^{-n}x)]$$

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for all $x \in X$ and all positive integer n. Then we have

$$\begin{aligned}
&J_nf(x) - J_{n+1}f(x)\\
&= \frac{2^n}{2\cdot a}Df(2^{-(n+1)}x, 2^{-(n+1)}x) - \frac{2^n}{2\cdot a}Df(-2^{-(n+1)}x, -2^{-(n+1)}x)\\
&\quad - \frac{1}{a\cdot 2\cdot 4^{n+1}}Df(2^nx, 2^nx) - \frac{1}{a\cdot 2\cdot 4^{n+1}}Df(-2^nx, -2^nx)
\end{aligned} \tag{2.13}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.13), (N3), and (N4), we have

(2.14)

$$\begin{aligned}
&N'(J_mf(x) - J_{m+n}f(x), \sum_{i=m}^{m+n-1}[\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p)\\
&= N'(\sum_{i=m}^{m+n-1}[J_if(x) - J_{i+1}f(x)], \sum_{i=m}^{m+n-1}[\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p)\\
&\geq \min\{N'(J_if(x) - J_{i+1}f(x), [\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p) \mid m \leq i \leq m+n-1\}\\
&\geq \min\{N'(\frac{1}{a\cdot 2\cdot 4^{i+1}}Df(2^ix, 2^ix) + \frac{1}{a\cdot 2\cdot 4^{i+1}}Df(-2^ix, -2^ix)\\
&\quad - \frac{2^i}{2\cdot a}Df(2^{-(i+1)}x, 2^{-(i+1)}x) + \frac{2^i}{2\cdot a}Df(-2^{-(i+1)}x, -2^{-(i+1)}x),\\
&\quad \frac{2^{pi+1}}{|a|\cdot 4^{i+1}}t^p + \frac{2^{1-p(i+1)+i}}{|a|}t^p) \mid m \leq i \leq m+n-1\}\\
&\geq \min\{\min\{N'(\frac{1}{a\cdot 2\cdot 4^{i+1}}Df(2^ix, 2^ix), \frac{2^{pi+1}}{|a|\cdot 2\cdot 4^{i+1}}t^p),\\
&\quad N'(\frac{1}{a\cdot 2\cdot 4^{i+1}}Df(-2^ix, -2^ix), \frac{2^{pi+1}}{|a|\cdot 2\cdot 4^{i+1}}t^p),\\
&\quad N'(\frac{2^i}{2\cdot a}Df(2^{-(i+1)}x, 2^{-(i+1)}x), \frac{2^{1-p(i+1)+i}}{2\cdot |a|}t^p),\\
&\quad N'(\frac{2^i}{2\cdot a}Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{1-p(i+1)+i}}{2\cdot |a|}t^p)\} \mid m \leq i \leq m+n-1\}\\
&\quad \geq \min\{\min\{N'(Df(2^ix, 2^ix), 2^{pi+1}t^p), N'(Df(-2^ix, -2^ix), 2^{pi+1}t^p),\\
&\quad N'(Df(2^{-(i+1)}x, 2^{-(i+1)}x), 2^{1-p(i+1)}t^p), N'(Df(-2^{-(i+1)}x, -2^{-(i+1)}x), 2^{1-p(i+1)}t^p)\} \mid\\
&\quad m \leq i \leq m+n-1\}]\\
&\geq \min\{\min\{N(2^ix, 2^it), N(-2^ix, 2^it), N(2^{-(i+1)}x, 2^{-(i+1)}t),\\
&\quad N(-2^{-(i+1)}x, 2^{-(i+1)}t)\} \mid m \leq i \leq m+n-1\}\\
&= N(x,t)
\end{aligned}$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Let $\epsilon > 0$ be given. Since $\lim_{t\longrightarrow\infty} N(x,t) = 1$, there is a t_1 such that $N(x, t_1) > 1 - \epsilon$. Let $t_2 > t_1$. Since $1 < p < 2$, $\sum_{n=0}^{\infty}[\frac{2^{pn+1}}{|a|\cdot 4^{n+1}} + \frac{2^{1-p(n+1)+n}}{|a|}]t_2^p$ is convergent. Let $s > 0$. Then there is a positive integer n such that $\sum_{i=m}^{m+n-1}[\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t_2^p < s$ for $m, n > k$ and

so by (2.14), we have

$$\begin{aligned}
&N'(J_m f(x) - J_{m+n} f(x), s)\\
&\geq N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} [\frac{2^{pi+1}}{|a| \cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}] t_2^p)\\
&\geq N(x, t_2)\\
&\geq 1 - \epsilon
\end{aligned}$$

for all $x \in X$. Hence $\{J_n f(x)\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, we can define a mapping $F : X \longrightarrow Y$ by

$$F(x) = N' - \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Letting $m = 0$ in (2.14), we have

$$N'(f(x) - J_n f(x), t) \geq N(x, \frac{t^q}{[\sum_{i=0}^{n-1} (\frac{2^{pi+1}}{|a| \cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|})]^q}) \tag{2.15}$$

for all $x \in X$, all positive integer n, and all $t > 0$. By (N4), we have

$$\begin{aligned}
&N'(DF(x, y), t)\\
&\geq \min\{N'(a[F - J_n f](x + y), \frac{t}{14}), N'(b[F - J_n f](x - y), \frac{t}{14}),\\
&N'(c[F - J_n f](y - x), \frac{t}{14}), N'((a + b)[F - J_n f](x), \frac{t}{14})\\
&- N'(c[F - J_n f](-x), \frac{t}{14}), N'((a + c)[F - J_n f](y), \frac{t}{14})\\
&- N'(b[F - J_n f](-y), \frac{t}{14}), N'(J_n Df(x, y), \frac{t}{2})\}
\end{aligned} \tag{2.16}$$

for all $x, y \in X$ and all positive integer n. The first seven terms on the right-hand of (2.16) tend to 1 as $n \to \infty$ and by (N4), we have

$$\begin{aligned}
&N'(J_n Df(x, y), \frac{t}{2})\\
&\geq \min\{N'(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8}), N'(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8}),\\
&N'(2^{n-1} Df(2^{-n} x, 2^{-n} y), \frac{t}{8}), N'(2^{n-1} Df(-2^{-n} x, -2^{-n} y), \frac{t}{8})\}
\end{aligned} \tag{2.17}$$

for all $x, y \in X$, all positive integer n and all $t > 0$. By (N3) and (2.5), we have

$$\begin{aligned}
&N'(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 4^n}, \frac{t}{8})\\
&\geq \min\{N(x, 2^{(2q-1)n-3q} t^q), N(y, 2^{(2q-1)n-3q} t^q)\}
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&N'(2^{n-1} Df(\pm 2^{-n} x, \pm 2^{-n} y), \frac{t}{8})\\
&\geq \min\{N(x, 2^{(1-q)n-3q)} t^q), N(y, 2^{(1-q)n-3q)} t^q)\}
\end{aligned} \tag{2.19}$$

for all $x, y \in X$, all positive integer n, and all $t > 0$. Since $\frac{1}{2} < q < 1$, by (2.17), (2.18), and (2.19), we have

$$\lim_{n\to\infty} N'(J_n Df(x, y), \frac{t}{2}) = 1$$

and so by (2.16), $N'(DF(x,y),t) = 0$ for all $x, y \in X$ and all $t > 0$. By (N2), $DF(x,y) = 0$ for all $x, y \in X$ and by Theorem 2.3, F is additive-quadratic.

Now we will show that (2.6) holds. Let $x \in X$, $t > 0$, $s > 0$ with $0 < s < t$ and $0 < \epsilon < 1$. Since $F(x) = N' - \lim_{n\to\infty} J_n f(x)$, there is a positive integer n such that

$$N'(F(x) - J_n f(x), t-s) \geq 1 - \epsilon$$

and so by (2.15),

$$\begin{aligned}
&N'(F(x) - f(x), t)\\
&\geq \min\{N'(F(x) - J_n f(x), t-s), N'(J_n f(x) - f(x), s)\}\\
&\geq \min\{1-\epsilon, N(x, \frac{s^q}{[\sum_{i=0}^{n-1}(\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|})]^q})\}\\
&\geq \min\{1-\epsilon, N(x, (2^{p-1}-1)^q(2-2^{p-1})^q|a|^q s^q)\}.
\end{aligned}$$

and so we have (2.6).

To prove the uniqueness of F, let $F_1 : X \longrightarrow Y$ be another additive-quadratic mapping satisfying (2.6). Then

$$F(x) - J_n F(x) = F_1(x) - J_n F_1(x)$$

for all $x \in X$ and all positive integer n. Hence by (N4), (N5), and (2.6), we have

$$\begin{aligned}
&N'(F(x) - F_1(x), t)\\
&= N'(J_n F(x) - J_n F_1(x), t)\\
&\geq \min\{N'(J_n F(x) - J_n f(x), \frac{t}{2}), N'(J_n F_1(x) - J_n f(x), \frac{t}{2})\}\\
&\geq \min\{N'(\frac{F(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(2^{n-1}[F(2^{-n}x) - f(2^{-n}x)], \frac{t}{8}), N'(2^{n-1}[F(-2^{-n}x) - f(-2^{-n}x)], \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F_1(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(2^{n-1}[F_1(2^{-n}x) - f(2^{-n}x)], \frac{t}{8}), N'(2^{n-1}[F_1(-2^{-n}x) - f(-2^{-n}x)], \frac{t}{8})\}\\
&\geq \sup_{s<t}\{N(\pm 2^n x, (2^{p-1}-1)^q(2-2^{p-1})^q 4^{(n-1)q}|a|^q s^q)\}\\
&\geq \sup_{s<t}\{N(x, (2^{p-1}-1)^q(2-2^{p-1})^q 2^{(2q-1)n-2q}|a|^q s^q)\}
\end{aligned}$$

for all $x, y \in X$, all positive integer n and all $t > 0$. Since $\frac{1}{2} < q < 1$, $N'(F(x) - F_1(x), t) = 1$ for all $t > 0$. Hence $F = F_1$.

Case 3. Let $0 < q < \frac{1}{2}$ and define a mapping $J_n f : X \longrightarrow Y$ by

$$J_n f(x) = 2^{2n-1}[f(2^{-n}x) + f(-2^{-n}x)] + 2^{n-1}[f(2^{-n}x) - f(-2^{-n}x)]$$

for all $x \in X$ and all positive integer n. Then we have

(2.20)

$$\begin{aligned}
&J_n f(x) - J_{n+1} f(x)\\
&= \frac{2^{2n-1} + 2^{n-1}}{a} Df(2^{-(n+1)}x, 2^{-(n+1)}x) + \frac{2^{2n-1} - 2^{n-1}}{a} Df(-2^{-(n+1)}x, -2^{-(n+1)}x)
\end{aligned}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.20), (N3), and (N4), we have

$$
\begin{aligned}
&N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \\
&= N'(\sum_{i=m}^{m+n-1} [J_i f(x) - J_{i+1} f(x)], \sum_{i=m}^{m+n-1} \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \\
&\geq \min\{N'(J_i f(x) - J_{i+1} f(x), \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \mid m \leq i \leq m+n-1\} \\
&\geq \min\{N'(\frac{2^{2i-1}+2^{i-1}}{a} Df(2^{-(i+1)}x, 2^{-(i+1)}x) \\
&\quad + \frac{2^{2i-1}-2^{i-1}}{a} Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{1-p(i+1)+2i}}{|a|} t^p)\} \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N'(\frac{2^{2i-1}+2^{i-1}}{a} Df(2^{-(i+1)}x, 2^{-(i+1)}x), \frac{2^{2i-1}+2^{i-1}}{|a|} 2^{1-p(i+1)} t^p), \\
&\quad N'(\frac{2^{2i-1}-2^{i-1}}{a} Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{2i-1}-2^{i-1}}{|a|} 2^{1-p(i+1)} t^p)\} \\
&\quad \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N'(Df(2^{-(i+1)}x, 2^{-(i+1)}x), 2^{1-p(i+1)} t^p), \\
&\quad N'(Df(-2^{-(i+1)}x, -2^{-(i+1)}x), 2^{1-p(i+1)} t^p)\} \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N(2^{-(i+1)}x, 2^{-(i+1)}t), N(-2^{-(i+1)}x, 2^{-(i+1)}t)\} \mid m \leq i \leq m+n-1\} \\
&= N(x, t)
\end{aligned}
$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Similar to **Case 1.** and **Case 2.**, there is a unique cubic mapping $C : X \longrightarrow Y$ with (2.6). □

We can use Theorem 2.4 to get a classical result in the framework of normed spaces. For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0,1]$, defined by

$$
N_X(x,t) = \begin{cases} 0, & \text{if } t < \|x\| \\ 1, & \text{if } t \geq \|x\| \end{cases}
$$

a fuzzy norm on X. In [15], [16] *and* [17]*, some examples are provided for the fuzzy norm N_X. Here using the fuzzy norm N_X, we have the following corollary.*

Corollary 2.5. *Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$
\|Df(x,y)\| \leq \|x\|^p + \|y\|^p \tag{2.21}
$$

for a fixed positive number p such that $p \neq 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that the inequality

$$
\|F(x) - f(x)\| \leq \begin{cases} \frac{1}{(1-2^{p-1})|a|}\|x\|^p, & \text{if } 1 < p \\ \frac{1}{(2^{p-1}-1)(2-2^{(p-1)})|a|}\|x\|^p, & \text{if } 1 < p < 2 \\ \frac{1}{(2^{p-1}-2)|a|}\|x\|^p, & \text{if } 2 < p \end{cases}
$$

holds for all $x \in X$.

Proof. By the definition of N_Y, we have

$$N_Y(Df(x,y), s+t) = \begin{cases} 0, & \text{if } s+t \le \|Df(x,y)\| \\ 1, & \text{if } s+t \ge \|Df(x,y)\|. \end{cases}$$

for all $x, y \in X$ and all $s, t \in \mathbb{R}$. Now, we claim that

$$N_Y(Df(x,y), s+t) \ge \min\{N_X(x, s^q), N_X(y, t^q)\}$$

for all $x, y \in X$ and $s, t > 0$. If $N_Y(Df(x,y), s+t) = 1$, then it is trivial. Suppose that $N_Y(Df(x,y), s+t) = 0$. Then $s+t \le \|Df(x,y)\|$ and by (2.21), either $s \le \|x\|^p$ or $t \le \|y\|^p$. Hence either $N_X(x, s^q) = 0$ or $N_X(y, t^q) = 0$ and thus f is a fuzzy q-almost additive-quadratic mapping. By Theorem 2.4, we have the results. □

The condition $p \ne 1, 2$ in Corollary 2.5 is indispensable. The following example shows that the inequality (2.21) is not stable for $p = 1$, 2, especially in the case of $b = 2$ and $c = -1$. We will give the proof when $p = 1$, and the proof when $p = 2$ is similar. For any $f : X \longrightarrow Y$, let $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$.

Example 2.6. Define mappings $t, s : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$t(x) = \begin{cases} x, & \text{if } |x| < 1 \\ -1, & \text{if } x \le -1 \\ 1, & \text{if } 1 \le x, \end{cases}$$

$$s(x) = \begin{cases} x^2, & \text{if } |x| < 1 \\ 1, & \text{ortherwise} \end{cases}$$

and a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} [\frac{t(2^n x)}{2^n} + \frac{s(2^n x)}{4^n}]$$

We will show that there is a positive integer M such that

$$|D_2 f(x,y)| \le M(|x| + |y|) \tag{2.22}$$

for all $x, y \in \mathbb{R}$, where

$$D_2 g(x,y) = g(x+y) + 2g(x-y) - g(y-x) - 3g(x) + g(-x) - 2g(-y).$$

But there do not exist an additive-quadratic mapping $F : \mathbb{R} \longrightarrow \mathbb{R}$ and a non-negative constant K such that

$$|F(x) - f(x)| \le K|x|^2 \tag{2.23}$$

for all $x \in \mathbb{R}$.

Proof. Note that $s_o(x) = 0$, $t_o(x) = t(x)$, and $|f_o(x)| \le 2$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{2} \le |x| + |y|$. Then $|D_2 f_o(x,y)| \le 40(|x| + |y|)$. Now suppose that $\frac{1}{2} > |x| + |y|$. Then there is a non-negative integer m such that

$$\frac{1}{2^{m+2}} \le |x| + |y| < \frac{1}{2^{m+1}}$$

and so $2^m|x| < \frac{1}{2}$, $2^m|y| < \frac{1}{2}$. Hence $\{2^m(x \pm y),\ 2^m x,\ 2^m y\} \subseteq (-1,\ 1)$ and so for any $n = 0, 1, 2, \cdots, m$, $D_2 t_0(2^n x, 2^n y) = 0$ for all $x, y \in X$. Thus

$$D_2 f_o(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n} D_2 t(2^n x, 2^n y) = \sum_{n=m+1}^{\infty} \frac{1}{2^n} D_2 t(2^n x, 2^n y) \le \frac{40}{2^{m+2}} \le 40(|x|+|y|).$$

Note that $t_e(x) = 0$, $s_e(x) = s(x)$, and $|f_e(x)| \le \frac{4}{3}$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \le |x| + |y|$. Then $|D_2 f_e(x,y)| \le \frac{128}{3}(|x| + |y|)$ for all $x, y \in \mathbb{R}$. Now suppose that $\frac{1}{4} > |x| + |y|$. Then there is a non-negative integer k such that

$$\frac{1}{2^{k+2}} \le \Big(|x| + |y|\Big)^{\frac{1}{2}} < \frac{1}{2^{k+1}}.$$

Hence $\{2^k(x \pm y),\ 2^k x,\ 2^k y\} \subseteq (-1,\ 1)$ and so for any $n = 0, 1, 2, \cdots, m$, $D_2 s_e(2^n x, 2^n y) = 0$. Hence

$$D_2 f_e(x,y) = \sum_{n=0}^{\infty} \frac{1}{4^n} D_2 s_e(2^n x, 2^n y) = \sum_{n=k+1}^{\infty} \frac{1}{4^n} D_2 s_e(2^n x, 2^n y) \le \frac{8}{3} \cdot \frac{1}{2^{2k}}.$$

and so we have

$$\Big(D_2 f_e(x,y)\Big)^{\frac{1}{2}} \le 4\Big(\frac{8}{3}\Big)^{\frac{1}{2}} \Big(|x| + |y|\Big)^{\frac{1}{2}}.$$

Thus we have

$$D_2 f_e(x,y) \le \frac{128}{3}(|x| + |y|).$$

and so we have (2.22).

Suppose that there exist an additive mapping $A : \mathbb{R} \longrightarrow \mathbb{R}$, a quadratic mapping $Q : \mathbb{R} \longrightarrow \mathbb{R}$, and a non-negative constant K such that $A + Q$ satisfies (2.23). Since $|f(x)| \le \frac{10}{3}$, by (2.23), we have

$$\frac{10}{3n} - K|x|^2 \le \frac{A(x)}{n} + Q(x) \le \frac{10}{3n} + K|x|^2$$

for all $x \in X$ and all positive integers n and so

$$|Q(x)| \le K|x|^2$$

for all $x \in X$. Hence by (2.23), we have

$$|f - A(x)| \le 2K|x|^2$$

for all $x \in X$.

Since f_o, A are odd and f_e is even,

$$(2.24)\quad |f_e(x)| \le \frac{1}{2}\Big[|f_e(x) + f_o(x) - A(x)| + |f_e(-x) + f_o(-x) - A(-x)|\Big] \le 4K|x|^2$$

for all $x \in X$. Take a positive integer l such that $l > 4K$, and pick $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Then

$$f_e(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{4^n} \ge \sum_{n=0}^{l-1} \frac{s(2^n x)}{4^n} \ge l x^2 > 4K x^2$$

which contradicts to (2.24). □

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Exact controllability for fuzzy differential equations using extremal solutions

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Abstract

In this paper, we devoted study exact controllability for fuzzy differential equations with the control function in credibility spaces. Moreover we study exact controllability for every solutions of fuzzy differential equations. The result is obtained by using extremal solutions.

1 Introduction

The theory of controlled processes is one of the most recent mathematical concepts to enable very important applications in modern engineering. However, actual systems subject to control do not admit a strictly deterministic analysis in view of various random factors that influence their behavior. The theory of controlled processes takes the random nature of a systems behavior into account. Many researchers have studied controlled processes in a credibility space. Arapostathis et al. [1] studied the controllability properties of the class of stochastic differential systems characterized by a linear controlled diffusion perturbed by a

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smooth, bounded, and uniformly Lipschitz nonlinearity. Kwun et al. [8] proved the approximate controllability for fuzzy differential equations driven by Liu process. Lee et al. [10] examined the exact controllability for abstract fuzzy differential equations in a credibility space.

Recently, Kwun et al. [14] studied the existence of extremal solutions for fuzzy differential equations driven by Liu process. Kwun et al. [6, 7] have studied the existence of extremal solutions for fuzzy differential equations in a n-dimensional fuzzy vector space. In this paper, using the extremal solutions, we study the exact controllability for every solutions of fuzzy differential equations in credibility space. We consider the following fuzzy differential equation:

$$\begin{cases} dx(t,\theta) = f(t, x(t,\theta))dC_t + Bu(t), \ t \in [0,T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{1}$$

where the state function $x(t,\theta)$ takes values in $X(\subset E_N)$ and another bounded space $Y(\subset E_N)$. E_N is the set of all upper semi-continuously convex fuzzy numbers on R, $(\Theta, \mathcal{P}, Cr)$ is credibility space, the state function $x : [0,T] \times (\Theta, \mathcal{P}, Cr) \to X$ is a fuzzy process, $f : [0,T] \times X \to X$ is a regular fuzzy function, $u : [0,T] \times (\Theta, \mathcal{P}, Cr) \to Y$ is a control function, B is a linear bounded operator from Y to X. C_t is a standard Liu process, $x_0 \in E_N$ is an initial value.

2 Preliminaries

In this section, we give basic definitions, terminologies, notations and lemmas which are most relevant to our investigated and are needed in later section. All undefined concepts and notions used here are standard.

A fuzzy set of R^n is a function $u : R^n \to [0,1]$. For each fuzzy set u, we denote by $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$ for any $\alpha \in [0,1]$, its α-level set. Let u, v be fuzzy sets of R^n. It is well known that $[u]^\alpha = [v]^\alpha$ for each $\alpha \in [0,1]$ implies $u = v$. Let E^n denote the collection of all fuzzy sets of R^n that satisfies the following conditions:

(1) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$, $0 \leq \lambda \leq 1$;

(3) $u(x)$ is upper semi-continuous, i.e., $u(x_0) \geq \overline{\lim}_{k\to\infty} u(x_k)$ for any $x_k \in R^n$ $(k = 0, 1, 2, \ldots)$, $x_k \to x_0$;

(4) $[u]^0$ is compact.

Definition 2.1. [17] The complete metric D_L on E_N is defined by

$$\begin{aligned} D_L(u,v) &= \sup_{0<\alpha\leq 1} d_L([u]^\alpha, [v]^\alpha) \\ &= \sup_{0<\alpha\leq 1} \max\{|u_l^\alpha - v_l^\alpha|, |u_r^\alpha - v_r^\alpha|\}, \end{aligned}$$

for any $u, v \in E_N$, which satisfies $d_L(u+w, v+w) = d_L(u,v)$.

Definition 2.2. [5] Let $u, v \in C([0,T], E_N)$. The metric H_1 on $C([0,T], E_N)$ is defined by

$$H_1(u,v) = \sup_{0<t\leq T} D_L(u(t), v(t)).$$

Let Θ be a nonempty set, and let $\mathcal{P}$ the power set of Θ. Each element in $\mathcal{P}$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign to each event A a number $Cr\{A\}$ which indicates the credibility that A will occur. In order to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following four axioms:

1. (Normality) $Cr\{A\} = 1$.
2. (Monotonicity) Cr is increasing, i.e., $Cr\{A\} \leq Cr\{B\}$ whenever $A \subset B$.
3. (Self-Duality) Cr is self-dual, i.e., $Cr\{A\}+Cr\{A^c\} = 1$ for any $A \in \mathcal{P}(\Theta)$.
4. (Maximality) $Cr\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any $\{A_i\}$ with $Cr\{A_i\} \leq 0.5$.

Definition 2.3. [11] Let ξ be a fuzzy variable with the possibility distribution function $\mu : R \to [0,1]$. A fuzzy variable ξ is said to be normal if there exists a real number r such that $\mu(r) = 1$. It is well known that the possibility of $\{\xi \leq r\}$ is defined by

$$\text{Pos}\{\xi \leq r\} = \sup_{u\leq r} \mu(u)$$

while the necessity of $\{\xi \leq r\}$ is defined by

$$\text{Nec}\{\xi \leq r\} = 1 - \text{Pos}\{\xi < r\} = 1 - \sup_{u<r} \mu(u).$$

Definition 2.4. [11] The set function Cr is called a credibility measure if it satisfies above four axioms, and defined as follows:

$$Cr\{A\} = \frac{1}{2}(\text{Pos}\{A\} + \text{Nec}\{A\}),$$

where $\text{Pos}\{A\} = 1 - \text{Nec}\{A^c\}$ with A^c is the complement of A.

Definition 2.5. [12] Let Θ be a nonempty set, $\mathcal{P}$ be the power set of Θ, and let Cr be a credibility measure. Then the triplet $(\Theta, \mathcal{P}, C_r)$ is called a credibility space.

Definition 2.6. [13] A fuzzy variable is a function from a credibility space $(\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

Definition 2.7. [13] Let T be an index set and let $(\Theta, \mathcal{P}, C_r)$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

That is, a fuzzy process $x(t,\theta)$ is a function of two variables such that the function $x(t^*,\theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $x(t,\theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $x(t,\theta)$ is said to be sample-continuous if the sample path is continuous for almost all θ.

Definition 2.8. Let $(\Theta,\mathcal{P},C_r)$ be a credibility space. For fuzzy random variable $x(t,\theta)$ in a credibility space, for each $\alpha\in(0,1]$, the α-level set $[x(t,\theta)]^\alpha=[x_l^\alpha(t,\theta),x_r^\alpha(t,\theta)]$ is defined by

$$x_l^\alpha(t,\theta)=\inf x^\alpha(t,\theta)=\inf\{a\in R|x(t,\theta)(a)\geq\alpha\},$$
$$x_r^\alpha(t,\theta)=\sup x^\alpha(t,\theta)=\sup\{a\in R|x(t,\theta)(a)\geq\alpha\}.$$

Definition 2.9. [11] Let ξ be a fuzzy variable and r is a real number. Then the expected value of ξ is defined by

$$E\xi=\int_0^{+\infty}Cr\{\xi\geq r\}dr-\int_{-\infty}^0 Cr\{\xi\leq r\}dr$$

provided that at least one of the integrals is finite.

Definition 2.10. [13] A fuzzy process C_t is said to be a Liu process if
(1) $C_0=0$;
(2) C_t has stationary and independent increments;
(3) every increment $C_{t+s}-C_s$ is a normally distributed fuzzy variable with expected value et and variance σ^2t^2, whose membership function is

$$\mu(x)=2\Big(1+\exp\Big(\frac{\pi|x-et|}{\sqrt{6}\sigma t}\Big)\Big)^{-1},\ x\in R.$$

The parameters e and σ are called the *drift* and *diffusion* coefficients, respectively. Liu process is said to be standard if $e=0$ and $\sigma=1$.

Definition 2.11. [3] Let $x(t)$ be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[c,d]$ with $c=t_0<\cdots<t_n=d$, the mesh is written as $\triangle=\max_{1\leq i\leq n}(t_i-t_{i-1})$. Then the fuzzy integral of $x(t)$ with respect to C_t is

$$\int_c^d x(t)dC_t=\lim_{\triangle\to 0}\sum_{i=1}^n x(t_{i-1})(C_{t_i}-C_{t_{i-1}})$$

provided that the limit exists almost surely and is a fuzzy variable.

Lemma 2.1. [3] Let C_t be a standard Liu process. For any given θ with $Cr\{\theta\}>0$, the path C_t is Lipschitz continuous, that is, the following inequality holds

$$|C_{t_1}-C_{t_2}|<K(\theta)|t_1-t_2|,$$

where K is a fuzzy variable called the Lipschitz constant of a Liu process with

$$K(\theta)=\begin{cases}\sup_{0\leq s<t}\frac{|C_t-C_s|}{t-s}, & Cr\{\theta\}>0,\\ \infty, & \text{otherwise},\end{cases}$$

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and $E[K^p] < \infty$, $\forall p > 0$.

Lemma 2.2. [3] Let C_t be a standard Liu process, and let $h(t;c)$ be a continuously differentiable function. Define $x_t = h(t;C_t)$. Then we have the following chain rule

$$dx_t = \frac{\partial h(t;C_t)}{\partial t}dt + \frac{\partial h(t;C_t)}{\partial C}dC_t.$$

Lemma 2.3. [3] Let $f(t)$ be continuous fuzzy process, the following inequality of fuzzy integral holds

$$\Big|\int_c^d f(t)dC_t\Big| \le K\int_c^d |f(t)|dt,$$

where $K = K(\theta)$ is defined in Lemma 2.1.

Definition 2.12. [14] For the partial ordering $\le_T$, a function $a \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$ is a $\le_T$-lower solution for equation (1)($u \equiv 0$) if

$$\begin{cases} a(t,\theta) \le_T U(t)x_0 + \int_0^t U(t-s)G(s,a(s,\theta))dC(s), \ t \in [0,T], \\ a(0) \le_T x_0 \in E_N \end{cases} \tag{2}$$

and a function $b \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$ is a $\le_T$-upper solution for equation (1)($u \equiv 0$) if

$$\begin{cases} b(t,\theta) \ge_T S(t)x_0 + \int_0^t S(t-s)F(s,b(s,\theta))dC(s), \ t \in [0,T], \\ b(0) \ge_T x_0 \in E_N. \end{cases} \tag{3}$$

Theorem 2.1. [14] Let $a, b \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$ be, respectively, $\le_T$-lower and $\le_T$-upper solutions for equation (1)($u \equiv 0$) on $[0,T]$. Then, there exist monotone sequences $\{a_n\} \uparrow \rho$, $\{b_n\} \downarrow \gamma$ in $C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$, where ρ, γ are extremal solutions to equation (1) in the stochastic fuzzy functional interval $[a,b] := \{x \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N) | a \le_T x \le_T b \text{ on } [0,T]\}$.

3 Exact controllability for fuzzy differential equation using extremal solutions

In this section, we study exact controllability for fuzzy differential equation using extremal solutions (1). In [14], Park et al. proved the existence of extremal solutions for the equation (1). Hence we consider extremal solutions for the equation (1), for each u in Y.

$$\begin{cases} x_t = U(t)x_0 + \int_0^t U(t-s)G(s,x_s)dC_s + \int_0^t U(t-s)Bu_s ds, \\ x(0) = x_0 \in E_N, \end{cases} \tag{4}$$

where $U(t) = e^{-Mt}$ is continuous with $U(0) = I,\ |U(t)| \le c,\ c > 0$, for all $t \in [0,T]$. And

$$\begin{cases} x_t = S(t)x_0 + \int_0^t S(t-s)F(s,x_s)dC_s + \int_0^t S(t-s)Bu_s ds, \\ x(0) = x_0 \in E_N, \end{cases} \tag{5}$$

where $S(t) = e^{Mt}$ is continuous with $S(0) = I,\ |S(t)| \le d,\ d > 0$, for all $t \in [0,T]$.

Now we assume the following hypotheses:

(**H1**) For $L_1, L_2 > 0,\ x_0 \in E_N$,

$$d_L\Big([U(t)x_0]^\alpha, [x_0]^\alpha\Big) \le L_1,\quad d_L\Big([S(t)x_0]^\alpha, [x_0]^\alpha\Big) \le L_2.$$

(**H2**) For $x(\cdot), y(\cdot) \in C([0,T]\times(\Theta,\mathcal{P},C_r),E_N),\ t\in[0,T]$, there exist positive numbers m_1, m_2 such that

$$d_L\Big([G(t,x)]^\alpha, [G(t,y)]^\alpha\Big) \le m_1 d_L([x]^\alpha,[y]^\alpha),$$
$$d_L\Big([F(t,x)]^\alpha, [F(t,y)]^\alpha\Big) \le m_2 d_L([x]^\alpha,[y]^\alpha)$$

and $F(0,\mathcal{X}_{\{0\}}(0)) \equiv 0,\ G(0,\mathcal{X}_{\{0\}}(0)) \equiv 0$.

(**H3**) For $L_3 > 0,\ x_0 \in E_N,\ d_L\Big([x_0]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big) \le L_3$.

(**H4**) For $\varepsilon > 0,\ (L_1 + cm_1KL_3T)e^{cm_1KT} \le \varepsilon$.

(**H5**) For $\varepsilon > 0,\ (L_2 + dm_2KL_3T)e^{dm_2KT} \le \varepsilon$.

(**H6**) Let a, b be, respectively, lower solution and upper solution of equation (1)($u \equiv 0$), then $[a,b]$ is convex.

We define the controllability concept for a fuzzy differential equation.

Definition 3.1. The equation (1) is said to be controllable on [0,T], if for every $x_0 \in E_N$ there exists a control $u_t \in Y$ such that every solutions $x(\cdot)$ of (1) satisfies a.s. θ, $x_T = x^1 \in X$ (i.e., $[x_T]^\alpha = [x^1]^\alpha$).

Definition 3.2. Define the fuzzy mappings $P_1 : \widetilde{P}(R) \to X$ and $P_2 : \widetilde{P}(R) \to X$ by

$$P_1^\alpha(v) = \begin{cases} \int_0^T U^\alpha(T-s)Bv_s ds, & v \subset \overline{\Gamma}_u, \\ 0, & \text{otherwise}, \end{cases}$$
$$P_2^\alpha(v) = \begin{cases} \int_0^T S^\alpha(T-s)Bv_s ds, & v \subset \overline{\Gamma}_u, \\ 0, & \text{otherwise}, \end{cases}$$

where $\widetilde{P}(R)$ is a nonempty fuzzy subset of R and $\overline{\Gamma}_u$ is the closure of support u. Then there exist $P_{1i}^\alpha, P_{2i}^\alpha (i = l, r)$ such that

$$P_{1l}^\alpha(v_l) = \int_0^T U_l^\alpha(T-s)B(v_s)_l ds,\ (v_s)_l \in [(u_s)_l^\alpha, (u_s)^1],$$
$$P_{1r}^\alpha(v_r) = \int_0^T U_r^\alpha(T-s)B(v_s)_r ds,\ (v_s)_r \in [(u_s)^1, (u_s)_r^\alpha],$$

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$$\begin{aligned}
P_{2l}^{\alpha}(v_l) &= \int_0^T S_l^{\alpha}(T-s)B(v_s)_l ds,\ (v_s)_l \in [(u_s)_l^{\alpha}, (u_s)^1],\\
P_{2r}^{\alpha}(v_r) &= \int_0^T S_r^{\alpha}(T-s)B(v_s)_r ds,\ (v_s)_r \in [(u_s)^1, (u_s)_r^{\alpha}].
\end{aligned}$$

We assume that $\widetilde{P}_{1l}^{\alpha}, \widetilde{P}_{1r}^{\alpha}, \widetilde{P}_{2l}^{\alpha}$ and $\widetilde{P}_{2r}^{\alpha}$ are bijective mappings.

By Definition 3.2, we can introduce α-level set of u_s is

$$\begin{aligned}
[u_s]^{\alpha} &= [(u_s)_l^{\alpha}, (u_s)_r^{\alpha}]\\
&= \frac{1}{2}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - U_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\\
&\quad +(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - S_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\},\\
&\quad (\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - U_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\\
&\quad +(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - S_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big].
\end{aligned}$$

Theorem 3.1. If Lemma 2.3 and hypotheses (H1)-(H5) are satisfied, then the equation (4) is controllable on $[0,T]$.

Proof By Definition 3.2 and above u_s, substitute the control into the equation (4) yields α-level of $\underline{x}_T$.

$$\begin{aligned}
[\underline{x}_T]^{\alpha} &= \Big[U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_s ds\Big]^{\alpha}\\
&= \Big[U_l^{\alpha}(T)(x_0)_l^{\alpha} + \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s + \int_0^T U_l^{\alpha}(T-s)B\\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - U_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\\
&\qquad +(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - S_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\Big]ds,\\
&\quad U_r^{\alpha}(T)(x_0)_r^{\alpha} + \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s + \int_0^T U_r^{\alpha}(T-s)B\\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - U_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\\
&\qquad +(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - S_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big]ds\Big]\\
&= \Big[U_l^{\alpha}(T)(x_0)_l^{\alpha} + \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s
\end{aligned}$$

$$+\frac{1}{2}P_{1l}^{\alpha}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - U_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}$$
$$+(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - S_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\Big],$$
$$U_r^{\alpha}(T)(x_0)_r^{\alpha} + \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s$$
$$+\frac{1}{2}P_{1r}^{\alpha}\Big[(\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - U_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}$$
$$+(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - S_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big]\Big]$$
$$= [(x^1)_l^{\alpha}, (x^1)_r^{\alpha}] = [x^1]^{\alpha}.$$

Hence this control u_t satisfy a.s. θ, $x_T = x^1$.

Also, using this control, we shall show that the nonlinear operator Φ_1 defined by

$$(\Phi_1 x)_t = U(t)x_0 + \int_0^t U(t-s)G(s,x_s)dC_s + \int_0^t U(t-s)B$$
$$\times\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds,$$

where the fuzzy mappings $(\widetilde{P}_1)^{-1}$ satisfy above statements.

Form hypothesis (H2) and Lemma 2.3, for any given θ with $Cr\{\theta\} > 0$, $x(\cdot),\ y(\cdot) \in C([0,T]\times(\Theta,\mathcal{P},Cr),E_N)$, we have

$$d_L\Big([(\Phi_1 x)_t]^{\alpha}, [(\Phi_1 y)_t]^{\alpha}\Big)$$
$$= d_L\Big(\Big[U(t)x_0 + \int_0^t U(t-s)G(s,x_s)dC_s$$
$$+\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^{\alpha},$$
$$\Big[U(t)x_0 + \int_0^t U(t-s)G(s,y_s)dC_s$$
$$+\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^{\alpha}\Big)$$
$$\le d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^{\alpha}, \Big[\int_0^t U(t-s)G(s,y_s)dC_s\Big]^{\alpha}\Big)$$

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$$+d_L\Big(\Big[\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,$$
$$\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)$$
$$\leq d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t U(t-s)G(s,y_s)dC_s\Big]^\alpha\Big)$$
$$+d_L\Big(\Big[\frac{1}{2}P_1\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\frac{1}{2}P_1\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]^\alpha,$$
$$\Big[\frac{1}{2}P_1\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}$$
$$+\frac{1}{2}P_1\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]^\alpha\Big)$$
$$\leq d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^T U(t-s)G(s,y_s)dC_s\Big]^\alpha\Big)$$
$$+d_L\Big(\Big[\int_0^T U(T-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t U(T-s)G(s,y_s)dC_s\Big]^\alpha\Big)$$
$$\leq cm_1K\int_0^t d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds+cm_1K\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds.$$

Therefore, by Lemma 2.1, we get

$$E\Big(H_1(\Phi_1x,\Phi_1y)\Big)$$
$$=E\Big(\sup_{t\in[0,T]}D_L\Big((\Phi_1x)_t,(\Phi_1y)_t\Big)\Big)$$
$$=E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\leq 1}d_L\Big([(\Phi_1x)_t]^\alpha,[(\Phi_1y)_t]^\alpha\Big)\Big)$$
$$\leq E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\leq 1}cm_1K\Big(\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds+\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds\Big)\Big)$$
$$\leq E\Big(\sup_{t\in[0,T]}cm_1K\Big(\int_0^t D_L(x_s,y_s)ds+\int_0^T D_L(x_s,y_s)ds\Big)\Big)$$
$$\leq 2cm_1KTE\Big(H_1(x,y)\Big).$$

We take sufficiently small T, $2cm_1KT<1$. Hence Φ_1 is contraction mapping. By the Banach fixed point theorem, (4) has a unique fixed point. Thus

the equation (1) is controllable in $[0, T]$.

Theorem 3.2. If Lemma 2.3 and hypotheses (H1)-(H5) are satisfied, then the equation (5) is controllable on $[0, T]$.

Proof By Definition 3.2 and above u_s, substitute the control into the equation (5) yields α-level of $\overline{x}_T$.

$$
\begin{aligned}
[\overline{x}_T]^\alpha &= \Big[S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_s ds\Big]^\alpha \\
&= \Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s + \int_0^T S_l^\alpha(T-s)B \\
&\quad \times \frac{1}{2}\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big]ds, \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s + \int_0^T S_r^\alpha(T-s)B \\
&\quad \times \frac{1}{2}\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]ds\Big] \\
&= \Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&= [(x^1)_l^\alpha, (x^1)_r^\alpha] = [x^1]^\alpha.
\end{aligned}
$$

Hence this control u_t satisfy a.s. θ, $x_T = x^1$.

Also, using this control, we shall show that the nonlinear operator Φ_2 defined by

$$
(\Phi_2 x)_t = S(t)x_0 + \int_0^t S(t-s)F(s,x_s)dC_s + \int_0^t S(t-s)B
$$

$$\times\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds,$$

where the fuzzy mappings $(\widetilde{P}_2)^{-1}$ satisfy above statements.

Form hypothesis (H2) and Lemma 2.3, for any given θ with $Cr\{\theta\} > 0$, $x(\cdot),\ y(\cdot) \in C([0,T]\times(\Theta,\mathcal{P},Cr),E_N)$, we have

$$\begin{aligned}
&d_L\Big([(\Phi_2 x)_t]^\alpha, [(\Phi_2 y)_t]^\alpha\Big)\\
&= d_L\Big(\Big[S(t)x_0 + \int_0^t S(t-s)F(s,x_s)dC_s\\
&\quad + \int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\quad +\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,\\
&\quad \Big[S(t)x_0 + \int_0^t S(t-s)F(s,y_s)dC_s\\
&\quad + \int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\quad +\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha, \Big[\int_0^t S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big)\\
&\quad +d_L\Big(\Big[\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\quad +\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,\\
&\quad \int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\quad +\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha, \Big[\int_0^t S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big)\\
&\quad +d_L\Big(\Big[\frac{1}{2}P_2\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\quad +\frac{1}{2}P_2\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]^\alpha,\\
&\quad \Big[\frac{1}{2}P_2\widetilde{P}_1^{-1}\Big\{x^1 - U(T)x_0 - \int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}
\end{aligned}$$

11

$$
\begin{aligned}
&\qquad +\frac{1}{2}P_2\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau, y_\tau)dC_\tau\Big\}\Big]^\alpha\Big) \\
&\le d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha, \Big[\int_0^T S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big) \\
&\quad +d_L\Big(\Big[\int_0^T S(T-s)F(s,x_s)dC_s\Big]^\alpha, \Big[\int_0^t S(T-s)F(s,y_s)dC_s\Big]^\alpha\Big) \\
&\le dm_2K\int_0^t d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds + dm_2K\int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds.
\end{aligned}
$$

Therefore, by Lemma 2.1, we get

$$
\begin{aligned}
&E\Big(H_1(\Phi_2 x, \Phi_2 y)\Big) \\
&\quad = E\Big(\sup_{t\in[0,T]} D_L\Big((\Phi_2 x)_t, (\Phi_2 y)_t\Big)\Big) \\
&\quad = E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le 1} d_L\Big([(\Phi_2 x)_t]^\alpha, [(\Phi_2 y)_t]^\alpha\Big)\Big) \\
&\quad \le E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le 1} dm_2K\Big(\int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds + \int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds\Big)\Big) \\
&\quad \le E\Big(\sup_{t\in[0,T]} 3m_2K\Big(\int_0^t D_L(x_s, y_s)ds + \int_0^T D_L(x_s, y_s)ds\Big)\Big) \\
&\quad \le 2dm_2KTE\Big(H_1(x,y)\Big).
\end{aligned}
$$

We take sufficiently small T and $2dm_2KT < 1$. Hence Φ_2 is contraction mapping. By the Banach fixed point theorem, (5) has a unique fixed point. Thus the equation (1) is controllable in $[0, T]$.

Theorem 3.3. If Theorems 3.1 and 3.2 and hypotheses (H1)-(H6) are satisfied, then the equation (1) is controllable on $[0, T]$.

Proof For $x_T \in [\underline{x}_T, \overline{x}_T]$, if $[\underline{x}_T, \overline{x}_T]$ is convex, then $x_T = \lambda\underline{x}_T + (1-\lambda)\overline{x}_T, 0 \le \lambda \le 1$, we can obtain the following result.

$$
\begin{aligned}
&[x_T]^\alpha = [\lambda\underline{x}_T + (1-\lambda)\overline{x}_T]^\alpha \\
&\quad = \Big[\lambda\Big\{U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_s ds\Big\} \\
&\qquad +(1-\lambda)\Big\{S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_s ds\Big\}\Big]^\alpha \\
&\quad = \lambda\Big[U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_s ds\Big]^\alpha \\
&\qquad +(1-\lambda)\Big[S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_s ds\Big]^\alpha
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda\Big[U_l^\alpha(T)(x_0)_l^\alpha + \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad + \frac{1}{2}P_{1l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad + (\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad U_r^\alpha(T)(x_0)_r^\alpha + \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad + \frac{1}{2}P_{1r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad + (\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&\quad + (1-\lambda)\Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad + \frac{1}{2}P_{2l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad + (\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad + \frac{1}{2}P_{2r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad + (\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&= [(x^1)_l^\alpha, (x^1)_r^\alpha] = [x^1]^\alpha.
\end{aligned}
$$

Hence this control u_t satisfy a.s. θ, $x_T = x^1, x_T \in [\underline{x}_T, \overline{x}_T]$. Therefore every solutions of the equation (1) are controllable in $[0, T]$.

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Generalized interval-valued intuitionistic fuzzy soft rough set and its application

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Abstract

In this paper, by integrating interval-valued intuitionistic fuzzy soft set with rough set theory, the concept of generalized interval-valued intuitionistic fuzzy soft rough sets is proposed, which is an extension of generalized intuitionistic fuzzy soft rough sets. Then the properties of this model are investigated. Furthermore, classical representations of generalized interval-valued intuitionistic fuzzy soft rough approximation operators are also introduced. Finally, an approach based on generalized interval-valued intuitionistic fuzzy soft rough sets in decision making is developed, and we provide a practical example to illustrate the validity of this approach.

Key words: Interval-valued intuitionistic fuzzy soft set; Rough set; Generalized interval-valued intuitionistic fuzzy soft rough set; Decision making

1 Introduction

As a framework for the construction of approximations of concepts, rough sets proposed by Pawlak [21,22], is a formal tool for modeling and processing insufficient and incomplete information. In Pawlak's rough set model, the equivalence relation plays an important role, which seems very stringent in daily life. Therefore many researchers have generalized the notion of Pawlak rough set by replacing the equivalence relation with other binary relations. Since the appearance of Pawlak rough set, lots of fruitful results have been achieved [5, 10–12, 15, 16, 25, 28, 29, 31–40, 42, 44–46].

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Soft set theory is presented by Molodtsov [17], which is different from the existing uncertainty theories, such as fuzzy set theory [43], intuitionistic fuzzy set theory [1, 2], interval-valued fuzzy set theory [9, 13, 24], interval-valued intuitionistic fuzzy set theory [3,4], rough set theory [21,22], and so on. In [17], the author pointed out that these theories mentioned above have their inherent difficulties, but soft set has enough parameters so that it is free from inherent difficulties. Therefore, in recent years more and more researchers have joined the ranks of soft set research. For example, Maji et al. [18] initiated the study on hybrid structures involving fuzzy sets and soft sets, and introduced the concept of fuzzy soft sets, which can be viewed as a generalization of soft sets. Subsequently, Maji et al [19] modified the concept of fuzzy soft sets, and proposed a generalized fuzzy soft set theory. Furthermore, Yang et al. [30] extended soft sets to interval-valued fuzzy environment, and first presented the concept of interval-valued fuzzy soft sets by combining interval-valued fuzzy set and soft set. By integrating the intuitionistic fuzzy set with soft set theory, Maji et al. [20] presented the concept of the intuitionistic fuzzy soft set theory. Jiang et al. [14] initiated the concept of interval-valued intuitionistic fuzzy soft sets by the combination of the interval-valued intuitionistic fuzzy sets and soft sets. On the basis of [14], Zhang [46] presented an adjustable approach to interval-valued intuitionistic fuzzy soft sets based decision making by mean of level soft sets of interval-valued intuitionistic fuzzy soft sets. Recently, soft set theory has been developed into hesitant fuzzy environment, and the result is called hesitant fuzzy soft sets [6, 26, 27]. Because it is unreasonable to use hesitant fuzzy soft sets to handle some decision making problems, Zhang et al. [41] extended hesitant fuzzy soft sets to interval-valued hesitant fuzzy environment, and introduced the concept of interval-valued hesitant fuzzy soft sets by combining the interval-valued hesitant fuzzy set and soft set theory. More recently, by combining intuitionistic fuzzy soft set and rough set theory, Zhang et al. [38] introduced the concept of intuitionistic fuzzy soft rough sets, and gave an approach to decision making based on this model. Furthermore, in [42], they pointed out the drawback of the intuitionistic fuzzy soft rough sets, proposed a generalized intuitionistic fuzzy soft rough set model, and then illustrated the validity of this model by a practical example.

As a generalization of fuzzy soft sets, interval-valued fuzzy soft sets and intuitionistic fuzzy soft sets, interval-valued intuitionistic fuzzy soft set is more flexible and effective than other soft set theories to cope with imperfect and imprecise information. Meanwhile, we can note that the final decision results for the decision approach presented by Zhang [46] may be different based on different types of thresholds. That is to say, there actually does not exist a unique or uniform criterion for the evaluation of decision alternatives. That is its limitations and disadvantages. In order to overcome these limitations, we need to define a new interval-valued intuitionistic fuzzy soft set model such that the decision approach based on this model is less affected by subjective factors. In this paper, we mainly devote to the generalization of intuitionistic fuzzy soft rough sets [42] and propose the concept of generalized interval-valued intuitionistic fuzzy soft rough sets by integrating interval-

valued intuitionistic fuzzy soft set with rough set. Also its decision making method is given. The most advantage of the decision making method is that it will only use the data information provided by the decision making problem without any additional available information provided by decision makers. Thus it can avoid the effect of subjective factors provided by different experts.

The rest of this paper is organized as follows. Section 2 briefly reviews some preliminaries. In Section 3, an interval-valued intuitionistic fuzzy soft relation is first defined by us. By combining the interval-valued intuitionistic fuzzy soft set and rough sets, then the concept of generalized interval-valued intuitionistic fuzzy soft rough approximation operators is presented and the properties of generalized upper and lower interval-valued intuitionistic fuzzy soft rough approximation operators are examined. Furthermore, classical representations of generalized interval-valued intuitionistic fuzzy soft rough approximation operators are presented. Section 4 is devoted to studying the application of generalized interval-valued intuitionistic fuzzy soft rough sets. Some conclusions and outlooks for further research are given in Section 5.

2 Preliminaries

In this section, we shall briefly recall some basic notions being used in the study.

Before introducing the notion of interval-valued intuitionistic fuzzy soft relation, we first give the concept of soft sets [17] and fuzzy soft sets [18].

Definition 2.1 ([17]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a soft set over U if $F : E \to P(U)$, where $P(U)$ is the set of all subsets of U.*

Definition 2.2 ([18]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a fuzzy soft set over U if $F : E \to F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.*

By using the concepts of soft set and fuzzy soft set, Cagman et al. [7,8] introduced the definitions of crisp soft relation and fuzzy soft relation, respectively.

Definition 2.3 ([7]) *Let (F, E) be a soft set over U. Then a subset of $U \times E$ called a crisp soft relation from U to E is uniquely defined by*

$$R = \{< (u, x), \mu_R(u, x) > |(u, x) \in U \times E\},$$

where $\mu_R : U \times E \to \{0, 1\},\ \mu_R(u, x) = \begin{cases} 1, & (u, x) \in R \\ 0, & (u, x) \notin R. \end{cases}$

Definition 2.4 ([8]) *Let (F, E) be a fuzzy soft set over U. Then a fuzzy subset of $U \times E$ called a fuzzy soft relation from U to E is uniquely defined by*

$R = \{< (u,x), \mu_R(u,x) > |(u,x) \in U \times E\}$,
where $\mu_R : U \times E \to [0,1]$, $\mu_R(u,x) = \mu_{F(x)}(u)$.

Based on the crisp soft relation proposed by Cagman, Zhang et al. [42] constructed the following crisp soft rough sets.

Definition 2.5 ([42]) *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary crisp soft relation R over $U \times E$, we can define a set-valued function $R_s : U \to P(E)$ by $R_s(u) = \{x \in E|(u,x) \in R\}, u \in U$.*

R is referred to as serial if for all $u \in U, R_s(u) \neq \emptyset$. The pair (U, E, R) is called a crisp soft approximation space. For any $A \subseteq E$, the upper and lower soft approximations of A with respect to (U, E, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are defined, respectively, as follows:

$$\overline{R}(A) = \{u \in U|R_s(u) \cap A \neq \emptyset\},\ \underline{R}(A) = \{u \in U|R_s(u) \subseteq A\}.$$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a crisp soft rough set, and $\overline{R}, \underline{R} : P(E) \to P(U)$ are, referred to as upper and lower crisp soft rough approximation operators, respectively.

Definition 2.6 ([3, 4]) *Denote $L = \{(\alpha,\beta)|\alpha = [\alpha_1,\alpha_2] \in Int[0,1], \beta = [\beta_1,\beta_2] \in Int[0,1], \alpha_2 + \beta_2 \leq 1\}$, where $Int[0,1]$ denotes the set of all closed subintervals of $[0,1]$. We define a relation $\leq_L$ on L as follows: $\forall(\alpha,\beta),(\xi,\eta) \in L$,*

$$\begin{aligned}(\alpha,\beta) \leq_L (\xi,\eta) &\Leftrightarrow [\alpha_1,\alpha_2] \leq_{L^I} [\xi_1,\xi_2] \text{ and } [\beta_1,\beta_2] \geq_{L^I} [\eta_1,\eta_2] \\ &\Leftrightarrow \alpha_1 \leq \xi_1, \alpha_2 \leq \xi_2, \beta_1 \geq \eta_1, \text{ and } \beta_2 \geq \eta_2.\end{aligned}$$

Then the relation $\leq_L$ is a partial ordering on L and the pair $(L, \leq_L)$ is a complete lattice with the smallest element $0_L = ([0,0],[1,1])$ and the greatest element $1_L = ([1,1],[0,0])$. The meet operator $\wedge$ and the join operator $\vee$ on $(L, \leq_L)$ which are linked to the ordering $\leq_L$ are, respectively, defined as follows: $\forall(\alpha,\beta),(\xi,\eta) \in L$,

$(\alpha,\beta) \wedge (\xi,\eta) = ([\alpha_1 \wedge \xi_1, \alpha_2 \wedge \xi_2], [\beta_1 \vee \eta_1, \beta_2 \vee \eta_2])$,
$(\alpha,\beta) \vee (\xi,\eta) = ([\alpha_1 \vee \xi_1, \alpha_2 \vee \xi_2], [\beta_1 \wedge \eta_1, \beta_2 \wedge \eta_2])$.

Definition 2.7 ([3, 4]) *Let a set U be fixed. The mapping $A : U \to L$ is called an interval-valued intuitionistic fuzzy (IVIF, for short) set on U. An interval-valued intuitionistic fuzzy set A on U can also be denoted by*

$A = \{< x, \mu_A(x), \gamma_A(x) > |x \in U\} = \{< x, [\mu_A^-(x), \mu_A^+(x)], [\gamma_A^-(x), \gamma_A^+(x)] > |x \in U\}$,
where $\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ and $\gamma_A(x) = [\gamma_A^-(x), \gamma_A^+(x)]$ satisfy $0 \leq \mu_A^+(x) + \gamma_A^+(x) \leq 1$ for all $x \in U$, and are, respectively, called the degree of membership and the degree of non-membership of the element $x \in U$ to A.

Let $IVIF(U)$ denotes the family of all interval-valued intuitionistic fuzzy sets on U.

3 Construction of generalized interval-valued intuitionistic fuzzy soft rough sets

In this section, we will present the concept of generalized IVIF soft rough sets by using the IVIF soft relation defined by us.

Definition 3.1 ([14]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called an IVIF soft set over U if $F : E \to IVIF(U)$, where $IVIF(U)$ is the set of all IVIF subsets of U.*

In the following, an IVIF soft relation will be presented, which is important for us to construct generalized IVIF soft rough sets.

Definition 3.2 *Let (F, E) be an IVIF soft set over U. Then an IVIF subset of $U \times E$ called an IVIF soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x), \gamma_R(u, x) > |(u, x) \in U \times E\}$,

where $\mu_R : U \times E \to Int[0, 1]$ and $\gamma_R : U \times E \to Int[0, 1]$, for all $(u, x) \in U \times E$ such that $\mu_R(u, x) = [\mu_R^-(u, x), \mu_R^+(u, x)]$ and $\gamma_R(u, x) = [\gamma_R^-(u, x), \gamma_R^+(u, x)]$, which satisfy the condition $0 \leq \mu_R^+(u, x) + \gamma_R^+(u, x) \leq 1$.

Remark 3.3 *In Definition 3.2, if $\mu_R^-(u, x) = \mu_R^+(u, x)$ and $\gamma_R^-(u, x) = \gamma_R^+(u, x)$, namely, $\mu_R : U \times E \to [0, 1]$ and $\gamma_R : U \times E \to [0, 1]$, for all $(u, x) \in U \times E$ such that $0 \leq \mu_R(u, x) + \gamma_R(u, x) \leq 1$, then R is referred to as an intuitionistic fuzzy soft relation on $U \times E$. If R is an intuitionistic fuzzy soft relation on $U \times E$ and $\mu_R(u, x) + \gamma_R(u, x) = 1$, then R is degenerated to a fuzzy soft relation [8] in Definition 2.4. Hence, among fuzzy soft relation, intuitionistic fuzzy soft relation [42] and IVIF soft relation, the IVIF soft relation is the most generalized one. That is, the IVIF soft relation has included fuzzy soft relation and intuitionistic fuzzy soft relation.*

Let $U = \{u_1, u_2, \cdots, u_m\}$ and $E = \{x_1, x_2, \cdots, x_n\}$. Then the IVIF soft relation R from U to E can be presented by a table as in the following form

R	x_1	x_2	$\cdots$	x_n
u_1	$(\mu_R(u_1, x_1), \gamma_R(u_1, x_1))$	$(\mu_R(u_1, x_2), \gamma_R(u_1, x_2))$	$\cdots$	$(\mu_R(u_1, x_n), \gamma_R(u_1, x_n))$
u_2	$(\mu_R(u_2, x_1), \gamma_R(u_2, x_1))$	$(\mu_R(u_2, x_2), \gamma_R(u_2, x_2))$	$\cdots$	$(\mu_R(u_2, x_n), \gamma_R(u_2, x_n))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$(\mu_R(u_m, x_1), \gamma_R(u_m, x_1))$	$(\mu_R(u_m, x_2), \gamma_R(u_m, x_2))$	$\cdots$	$(\mu_R(u_m, x_n), \gamma_R(u_m, x_n))$

From the above form and the definition of IVIF soft set, we know that every IVIF soft set (F, E) is uniquely characterized by the IVIF soft relation, namely they are mutual determined. It means that an IVIF soft set (F, E) is formally equal to IVIF soft relation.

Therefore, we shall identify any IVIF soft set with IVIF soft relation and view these two concepts as interchangeable. Now, any discussion regard to IVIF soft set could be converted into analysis about IVIF soft relation, which will bring great convenience for our future researches.

In this case, according to the definition of IVIF soft relation, we can construct generalized IVIF soft rough sets as follows.

Definition 3.4 *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary IVIF soft relation R over $U \times E$, the pair (U, E, R) is called an IVIF soft approximation space. For any $A \in IVIF(E)$, we define the upper and lower soft approximations of A with respect to (U, E, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, as follows:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\}, \tag{1}$$

$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}. \tag{2}$$

where

$$\mu_{\overline{R}(A)}(u) = [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge \mu_A^-(x)), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge \mu_A^+(x))],$$
$$\gamma_{\overline{R}(A)}(u) = [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee \gamma_A^-(x)), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee \gamma_A^+(x))],$$
$$\mu_{\underline{R}(A)}(u) = [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee \mu_A^-(x)), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee \mu_A^+(x))],$$
$$\gamma_{\underline{R}(A)}(u) = [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge \gamma_A^-(x)), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge \gamma_A^+(x))].$$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a generalized IVIF soft rough set of A with respect to (U, E, R).

By $\mu_R^+(u,x) + \gamma_R^+(u,x) \leq 1$ and $\mu_A^+(x) + \gamma_A^+(x) \leq 1$, it can be easily verified that $\overline{R}(A)$ and $\underline{R}(A) \in IVIF(U)$. So we call $\overline{R}, \underline{R} : IVIF(E) \to IVIF(U)$ generalized upper and lower IVIF soft rough approximation operators, respectively.

Remark 3.5 *If R is an intuitionistic fuzzy soft relation on $U \times E$, then generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ in Definition 3.4 degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\},$$
$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}.$$

where

$$\mu_{\overline{R}(A)}(u) = \bigvee_{x\in E} (\mu_R(u,x) \wedge \mu_A(x)),\ \gamma_{\overline{R}(A)}(u) = \bigwedge_{x\in E} (\gamma_R(u,x) \vee \gamma_A(x)),$$

$\mu_{\underline{R}(A)}(u) = \bigwedge_{x \in E} (\gamma_R(u,x) \vee \mu_A(x)),\ \gamma_{\underline{R}(A)}(u) = \bigvee_{x \in E} (\mu_R(u,x) \wedge \gamma_A(x)).$

In that case, the pair $(\overline{R}(A), \underline{R}(A))$ *is generated into a generalized IF soft rough set of* A *with respect to* (U, E, R) *proposed by Zhang et al. [42]. That is, generalized IVIF soft rough set in Definition 4.4 includes generalized IF soft rough set [42] as a special case.*

Remark 3.6 *If* R *is a fuzzy soft relation on* $U \times E$ *and* $A \in F(E)$, *then generalized IVIF soft rough approximation operators* $\overline{R}(A)$ *and* $\underline{R}(A)$ *degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u) > | u \in U\}, \quad \underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u) > | u \in U\}.$$

where $\mu_{\overline{R}(A)}(u) = \bigvee_{x \in E} [\mu_R(u,x) \wedge \mu_A(x)]$, $\mu_{\underline{R}(A)}(u) = \bigwedge_{x \in E} [(1 - \mu_R(u,x)) \vee \mu_A(x)]$.

In that case, generalized IVIF soft rough approximation operators $\overline{R}(A)$ *and* $\underline{R}(A)$ *are identical with the soft fuzzy rough approximation operators defined by Sun [23]. That is, generalized IVIF soft rough approximation operators in Definition 4.4 are an extension of the soft fuzzy rough approximation operators defined by Sun [23].*

In order to better understand the concept of generalized IVIF soft rough approximation operators, let us consider the following example.

Example 3.7 *Suppose that* $U = \{u_1, u_2, u_3, u_4, u_5\}$ *is the set of five houses under consideration of a decision maker to purchase. Let* E *be a parameter set, where* $E = \{e_1, e_2, e_3, e_4\}$=\{*expensive; beautiful; size; location*\}. *Mr. X wants to buy the house which qualifies with the parameters of* E *to the utmost extent from available houses in* U. *Assume that Mr. X describes the "attractiveness of the houses" by constructing an IVIF soft relation* R *from* U *to* E. *And it is presented by a table as in the following form.*

R	e_1	e_2	e_3	e_4
u_1	$([0.7, 0.8], [0.2, 0.2])$	$([0.3, 0.4], [0.2, 0.5])$	$([0.1, 0.1], [0.7, 0.8])$	$([0.3, 0.4], [0.1, 0.3])$
u_2	$([0.1, 0.2], [0.4, 0.6])$	$([0.6, 0.7], [0.1, 0.2])$	$([0.2, 0.3], [0.5, 0.7])$	$([0.3, 0.6], [0.2, 0.3])$
u_3	$([0.5, 0.6], [0.2, 0.4])$	$([0.3, 0.6], [0.2, 0.3])$	$([0.5, 0.7], [0.1, 0.3])$	$([0.1, 0.8], [0.1, 0.2])$
u_4	$([0.1, 0.3], [0.2, 0.6])$	$([0.5, 0.7], [0.1, 0.2])$	$([0.1, 0.4], [0.3, 0.5])$	$([0.2, 0.3], [0.5, 0.7])$
u_5	$([0.8, 0.9], [0.0, 0.1])$	$([0.3, 0.5], [0.4, 0.5])$	$([0.6, 0.8], [0.1, 0.2])$	$([0.4, 0.6], [0.1, 0.4])$

We can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation are given. For example, we can not present the precise membership degree and non-membership degree of how beautiful house u_2 *is, however, house* u_2 *is at least beautiful on the membership degree of 0.6 and it is at most beautiful on the membership degree of 0.7; house* u_2 *is not at least beautiful on*

the non-membership degree of 0.1 and it is not at most beautiful on the non-membership degree of 0.2.

Now give an IVIF subset A over the parameter set E as follows:

$$A = \{ < e_1, [0.7, 0.8], [0.1, 0.2] >, < e_2, [0.5, 0.7], [0.2, 0.3] >,$$
$$< e_3, [0.4, 0.6], [0.1, 0.3] >, < e_4, [0.2, 0.6], [0.3, 0.4] > \}.$$

By Equations (1) and (2), we have

$\mu_{\overline{R}(A)}(u_1) = [0.7, 0.8]$, $\gamma_{\overline{R}(A)}(u_1) = [0.2, 0.2]$, $\mu_{\overline{R}(A)}(u_2) = [0.5, 0.7]$, $\gamma_{\overline{R}(A)}(u_2) = [0.2, 0.3]$, $\mu_{\overline{R}(A)}(u_3) = [0.5, 0.6]$, $\gamma_{\overline{R}(A)}(u_3) = [0.1, 0.3]$, $\mu_{\overline{R}(A)}(u_4) = [0.5, 0.7]$, $\gamma_{\overline{R}(A)}(u_4) = [0.2, 0.3]$, $\mu_{\overline{R}(A)}(u_5) = [0.7, 0.8]$, $\gamma_{\overline{R}(A)}(u_5) = [0.1, 0.2]$; $\mu_{\underline{R}(A)}(u_1) = [0.2, 0.6]$, $\gamma_{\underline{R}(A)}(u_1) = [0.3, 0.4]$, $\mu_{\underline{R}(A)}(u_2) = [0.2, 0.6]$, $\gamma_{\underline{R}(A)}(u_2) = [0.3, 0.4]$, $\mu_{\underline{R}(A)}(u_3) = [0.2, 0.6]$, $\gamma_{\underline{R}(A)}(u_3) = [0.2, 0.4]$, $\mu_{\underline{R}(A)}(u_4) = [0.4, 0.6]$, $\gamma_{\underline{R}(A)}(u_4) = [0.2, 0.3]$, $\mu_{\underline{R}(A)}(u_5) = [0.2, 0.6]$, $\gamma_{\underline{R}(A)}(u_5) = [0.3, 0.4]$.

Thus

$$\overline{R}(A) = \{ < u_1, [0.7, 0.8], [0.2, 0.2] >, < u_2, [0.5, 0.7], [0.2, 0.3] >, < u_3, [0.5, 0.6], [0.1, 0.3] >,$$
$$< u_4, [0.5, 0.7], [0.2, 0.3] >, < u_5, [0.7, 0.8], [0.1, 0.2] > \}$$

and

$$\underline{R}(A) = \{ < u_1, [0.2, 0.6], [0.3, 0.4] >, < u_2, [0.2, 0.6], [0.3, 0.4] >, < u_3, [0.2, 0.6], [0.2, 0.4] >,$$
$$< u_4, [0.4, 0.6], [0.2, 0.3] >, < u_5, [0.2, 0.6], [0.3, 0.4] > \}.$$

In what follows, we investigate the properties of generalized IVIF soft rough approximation operators.

Theorem 3.8 *Let (U, E, R) be an IVIF soft approximation space. Then the generalized upper and lower IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ satisfy the following properties: $\forall A, B \in IVIF(E)$,*

(IVIFSL1) $\underline{R}(A) = \sim \overline{R}(\sim A)$,
(IVIFSU1) $\overline{R}(A) = \sim \underline{R}(\sim A)$;
(IVIFSL2) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$,
(IVIFSU2) $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(IVIFSL3) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$,
(IVIFSU3) $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$;
(IVIFSL4) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$,
(IVIFSU4) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$;

Proof. We only prove the properties of the lower IVIF soft rough approximation operator $\underline{R}(A)$. The upper IVIF soft rough approximation operator $\overline{R}(A)$ can be proved similarly.
(IVIFSL1) By Definition 3.4, then we have

$$\begin{aligned}
\sim \underline{R}(\sim A) &= \{< u, \gamma_{\underline{R}(\sim A)}(u), \mu_{\underline{R}(\sim A)}(u) > | u \in U\} \\
&= \{< u, [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge \gamma_{\sim A}^-(x)), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge \gamma_{\sim A}^+(x))], \\
&\quad [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee \mu_{\sim A}^-(x)), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee \mu_{\sim A}^+(x))] > | u \in U\} \\
&= \{< u, [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge \mu_A^-(x)), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge \mu_A^+(x))], \\
&\quad [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee \gamma_A^-(x)), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee \gamma_A^+(x))] > | u \in U\} \\
&= \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\} = \overline{R}(A).
\end{aligned}$$

(IVIFSL2) By virtue of Equation (2), we have

$$\begin{aligned}
\underline{R}(A \cap B) &= \{< u, \mu_{\underline{R}(A\cap B)}(u), \gamma_{\underline{R}(A\cap B)}(u) > | u \in U\} \\
&= \{< u, \bigwedge_{x\in E} (\gamma_R(u,x) \vee \mu_{A\cap B}(x)), \bigvee_{x\in E} (\mu_R(u,x) \wedge \gamma_{A\cap B}(x)) > | u \in U\} \\
&= \{< u, [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee (\mu_A^-(x) \wedge \mu_B^-(x))), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee (\mu_A^+(x) \wedge \mu_B^+(x)))], \\
&\quad [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge (\gamma_A^-(x) \vee \gamma_B^-(x))), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge (\gamma_A^+(x) \vee \gamma_B^+(x)))] > | u \in U\} \\
&= \{< u, [\mu_{\underline{R}(A)}^-(u) \wedge \mu_{\underline{R}(B)}^-(u), \mu_{\underline{R}(A)}^+(u) \wedge \mu_{\underline{R}(B)}^+(u)], \\
&\quad [\gamma_{\underline{R}(A)}^-(u) \vee \gamma_{\underline{R}(B)}^-(u), \gamma_{\underline{R}(A)}^+(u) \vee \gamma_{\underline{R}(B)}^+(u)] > | u \in U\} \\
&= \{< u, \mu_{\underline{R}(A)}(u) \wedge \mu_{\underline{R}(B)}(u), \gamma_{\underline{R}(A)}(u) \vee \gamma_{\underline{R}(B)}(u) > | u \in U\} = \underline{R}(A) \cap \underline{R}(B).
\end{aligned}$$

(IVIFSL3) It can be easily verified by Definition 3.4.
(IVIFSL4) By (IVIFSL3), it is straightforward. □

In Theorem 3.8, properties (IVIFSL1) and (IVIFSU1) show that the generalized upper lower IVIF soft rough approximation operators $\overline{R}$ and $\underline{R}$ are dual to each other.

Inspired by the concept of cut sets of IF sets in [44, 45], we first present the concept of cut sets of IVIF sets before investigating the representing method of the generalized IVIF soft rough approximation operators.

Definition 3.9 *Let $A = \{< x, \mu_A(x), \gamma_A(x) > | x \in U\} \in IVIF(U)$, and $(\alpha, \beta) \in L$, where $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2] \in Int[0, 1]$ with $\alpha_2 + \beta_2 \leq 1$. The (α, β)-level cut set of A,*

denoted by A_α^β *, is defined as follows:*

$$\begin{aligned} A_\alpha^\beta &= \{x \in U | \mu_A(x) \geq_{L^I} \alpha, \gamma_A(x) \leq_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2, \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}. \end{aligned}$$

$$A_\alpha = \{x \in U | \mu_A(x) \geq_{L^I} \alpha\} = \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2\},$$

and

$$A_{\alpha+} = \{x \in U | \mu_A(x) >_{L^I} \alpha\} = \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2\}$$

are, respectively, called the α*-level cut set and the strong* α*-level cut set of membership generated by* A*. Meanwhile,*

$$A^\beta = \{x \in U | \gamma_A(x) \leq_{L^I} \beta\} = \{x \in U | \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}$$

and

$$A^{\beta+} = \{x \in U | \gamma_A(x) <_{L^I} \beta\} = \{x \in U | \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}$$

are, respectively, referred to as the β*-level cut set and the strong* β*-level cut set of non-membership generated by* A*.*

At the same time, other types of cut sets of the IVIF set A *are denoted as follows:*

$$\begin{aligned} A_{\alpha+}^\beta &= \{x \in U | \mu_A(x) >_{L^I} \alpha, \gamma_A(x) \leq_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2, \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}, \end{aligned}$$

which is called the $(\alpha+, \beta)$*-level cut set of* A*;*

$$\begin{aligned} A_\alpha^{\beta+} &= \{x \in U | \mu_A(x) \geq_{L^I} \alpha, \gamma_A(x) <_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2, \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}, \end{aligned}$$

which is called the $(\alpha, \beta+)$*-level cut set of* A*;*

$$\begin{aligned} A_{\alpha+}^{\beta+} &= \{x \in U | \mu_A(x) >_{L^I} \alpha, \gamma_A(x) <_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2, \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}, \end{aligned}$$

which is called the $(\alpha+, \beta+)$*-level cut set of* A*.*

Theorem 3.10 *The cut sets of IVIF sets satisfy the following properties:* $\forall A \in IVIF(U)$, $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2] \in Int[0,1]$ *with* $\alpha_2 + \beta_2 \leq 1$,

(1) $A_\alpha^\beta = A_\alpha \cap A^\beta$,

(2) $A \subseteq B \Rightarrow A_\alpha^\beta \subseteq B_\alpha^\beta$,

(3) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha,\ (A \cap B)^\beta = A^\beta \cap B^\beta$,

(4) $\alpha \geq_{L^I} \beta, \xi \leq_{L^I} \eta \Rightarrow A_\alpha \subseteq A_\beta, A^\xi \subseteq A^\eta, A_\alpha^\xi \subseteq A_\beta^\eta$.

Proof. By Definition 3.9, (1), (2) and (4) are straightforward.

(3) Since

$$\begin{aligned}A\cap B &= \{< x, \mu_{A\cap B}(x), \gamma_{A\cap B}(x) > | x\in U\}\\ &= \{< x, [\mu_A^-(x)\wedge\mu_B^-(x), \mu_A^+(x)\wedge\mu_B^+(x)],\\ &\qquad [\gamma_A^-(x)\vee\gamma_B^-(x), \gamma_A^+(x)\vee\gamma_B^+(x)] > | x\in U\},\end{aligned}$$

we have

$$\begin{aligned}(A\cap B)_\alpha &= \{x\in U|\mu_A^-(x)\wedge\mu_B^-(x)\geq\alpha_1, \mu_A^+(x)\wedge\mu_B^+(x)\geq\alpha_2\}\\ &= \{x\in U|\mu_A^-(x)\geq\alpha_1, \mu_B^-(x)\geq\alpha_1, \mu_A^+(x)\geq\alpha_2, \mu_B^+(x)\geq\alpha_2\}\\ &= \{x\in U|\mu_A(x)\geq_{L^I}\alpha, \mu_B(x)\geq_{L^I}\alpha\} = A_\alpha\cap B_\alpha,\end{aligned}$$

and

$$\begin{aligned}(A\cap B)^\beta &= \{x\in U|\gamma_A^-(x)\vee\gamma_B^-(x)\leq\beta_1, \gamma_A^+(x)\vee\gamma_B^+(x)\leq\beta_2\}\\ &= \{x\in U|\gamma_A^-(x)\leq\beta_1, \gamma_B^-(x)\leq\beta_1, \gamma_A^+(x)\leq\beta_2, \gamma_B^+(x)\leq\beta_2\}\\ &= \{x\in U|\gamma_A(x)\leq_{L^I}\beta, \gamma_B(x)\leq_{L^I}\beta\} = A^\beta\cap B^\beta.\end{aligned}$$

Meanwhile, according to (1), we can obtain

$$\begin{aligned}(A\cap B)_\alpha^\beta &= (A\cap B)_\alpha\cap(A\cap B)^\beta\\ &= (A_\alpha\cap A^\beta)\cap(B_\alpha\cap B^\beta) = A_\alpha^\beta\cap B_\alpha^\beta.\end{aligned}$$

□

Assume that R is an IVIF soft relation from U to E, denote

$$R_\alpha = \{(u,x)\in U\times E|\mu_R(u,x)\geq_{L^I}\alpha\} = \{(u,x)\in U\times E|\mu_R^-(u,x)\geq\alpha_1, \mu_R^+(u,x)\geq\alpha_2\},$$

$$R_\alpha(u) = \{x\in E|\mu_R(u,x)\geq_{L^I}\alpha\} = \{x\in E|\mu_R^-(u,x)\geq\alpha_1, \mu_R^+(u,x)\geq\alpha_2\}, \alpha_1,\alpha_2\in[0,1];$$

$$R_{\alpha+} = \{(u,x)\in U\times E|\mu_R(u,x)>_{L^I}\alpha\} = \{(u,x)\in U\times E|\mu_R^-(u,x)>\alpha_1, \mu_R^+(u,x)>\alpha_2\},$$

$$R_{\alpha+}(u) = \{x\in E|\mu_R(u,x)>_{L^I}\alpha\} = \{x\in E|\mu_R^-(u,x)>\alpha_1, \mu_R^+(u,x)>\alpha_2\}, \alpha_1,\alpha_2\in[0,1);$$

$$R^\beta = \{(u,x)\in U\times E|\gamma_R(u,x)\leq_{L^I}\beta\} = \{(u,x)\in U\times E|\gamma_R^-(u,x)\leq\beta_1, \gamma_R^+(u,x)\leq\beta_2\},$$

$$R^\beta(u) = \{x\in E|\gamma_R(u,x)\leq_{L^I}\beta\} = \{x\in E|\gamma_R^-(u,x)\leq\beta_1, \gamma_R^+(u,x)\leq\beta_2\}, \beta_1,\beta_2\in[0,1];$$

$$R^{\beta+} = \{(u,x)\in U\times E|\gamma_R(u,x)<_{L^I}\beta\} = \{(u,x)\in U\times E|\gamma_R^-(u,x)<\beta_1, \gamma_R^+(u,x)<\beta_2\},$$

$$R^{\beta+}(u) = \{x\in E|\gamma_R(u,x)<_{L^I}\beta\} = \{x\in E|\gamma_R^-(u,x)<\beta_1, \gamma_R^+(u,x)<\beta_2\}, \beta_1,\beta_2\in(0,1].$$

Then R_α, $R_{\alpha+}$, R^β and $R^{\beta+}$ are crisp soft relations on $U\times E$.

The following Theorems 3.12 and 3.13 show that the generalized IVIF soft rough approximation operators can be represented by crisp soft rough approximation operators proposed by Zhang et al. [42].

Theorem 3.11 *Let (U, E, R) be an IVIF soft approximation space, and $A \in IVIF(E)$. Then the generalized upper IVIF soft rough approximation operator can be represented as follows: $\forall u \in U$, $\overline{a} = [a, a] \in L^I$,*

(1)

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_\alpha)(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_{\alpha+})(u)}]$$
$$= \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_\alpha)(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_{\alpha+})(u)}],$$

(2)

$$\gamma_{\overline{R}(A)}(u) = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^\alpha}(A^\alpha)(u)}] = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^\alpha}(A^{\alpha+})(u)}]$$
$$= \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^{\alpha+}}(A^\alpha)(u)}] = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^{\alpha+}}(A^{\alpha+})(u)}]$$

and moreover, for any $\alpha \in L^I$,

(3) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha+}}(A_\alpha) \subseteq \overline{R_\alpha}(A_\alpha) \subseteq [\overline{R}(A)]_\alpha$,

(4) $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^{\alpha+}}(A^\alpha) \subseteq \overline{R^\alpha}(A^\alpha) \subseteq [\overline{R}(A)]^\alpha$.

Proof. (1) For any $u \in U$, we have

$$\bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_\alpha)(u)}] = sup\{\alpha \in L^I | u \in \overline{R_\alpha}(A_\alpha)\} = sup\{\alpha \in L^I | R_\alpha(u) \cap A_\alpha \neq \emptyset\}$$
$$= sup\{\alpha \in L^I | \exists x \in E[x \in R_\alpha(u), x \in A_\alpha]\}$$
$$= sup\{\alpha \in L^I | \exists x \in E[\mu_R(u,x) \geq_{L^I} \alpha, \mu_A(x) \geq_{L^I} \alpha]\}$$
$$= sup\{[\alpha_1, \alpha_2] \in L^I | \exists x \in E[\mu_R^-(u,x) \geq \alpha_1, \mu_R^+(u,x) \geq \alpha_2, \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2]\}$$
$$= sup\{[\alpha_1, \alpha_2] \in L^I | \exists x \in E[\mu_R^-(u,x) \wedge \mu_A^-(x) \geq \alpha_1, \mu_R^+(u,x) \wedge \mu_A^+(x) \geq \alpha_2]\}$$
$$= [\bigvee_{x \in E} (\mu_R^-(u,x) \wedge \mu_A^-(x)), \bigvee_{x \in E} (\mu_R^+(u,x) \wedge \mu_A^+(x))] = \mu_{\overline{R}(A)}(u).$$

Likewise, we can conclude that

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_{\alpha+})(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_\alpha)(u)}]$$
$$= \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_{\alpha+})(u)}].$$

(2) In terms of Definition 2.5 and notations above, we have

$$\begin{aligned}&\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^\alpha}(A^\alpha)(u)}]=inf\{\alpha\in L^I]|u\in\overline{R^\alpha}(A^\alpha)\}=inf\{\alpha\in L^I|R^\alpha(u)\cap A^\alpha\neq\emptyset\}\\&=inf\{\alpha\in L^I|\exists x\in E[x\in R^\alpha(u),x\in A^\alpha]\}\\&=inf\{\alpha\in L^I|\exists x\in E[\gamma_R(u,x)\leq_{L^I}\alpha,\gamma_A(x)\leq_{L^I}\alpha]\}\\&=inf\{[\alpha_1,\alpha_2]\in L^I|\exists x\in E[\gamma_R^-(u,x)\leq\alpha_1,\gamma_R^+(u,x)\leq\alpha_2,\gamma_A^-(x)\leq\alpha_1,\gamma_A^+(x)\leq\alpha_2]\}\\&=inf\{[\alpha_1,\alpha_2]\in L^I|\exists x\in E[\gamma_R^-(u,x)\vee\gamma_A^-(x)\leq\alpha_1,\gamma_R^+(u,x)\vee\gamma_A^+(x)\leq\alpha_2]\}\\&=[\bigwedge_{x\in E}(\gamma_R^-(u,x)\vee\gamma_A^-(x)),\bigwedge_{x\in E}(\gamma_R^+(u,x)\vee\gamma_A^+(x))]=\gamma_{\overline{R}(A)}(u).\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}\gamma_{\overline{R}(A)}(u)&=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^\alpha}(A^{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^{\alpha+}}(A^\alpha)(u)}]\\&=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^{\alpha+}}(A^{\alpha+})(u)}].\end{aligned}$$

(3) It is easily verified that $\overline{R_{\alpha+}}(A_{\alpha+})\subseteq\overline{R_{\alpha+}}(A_\alpha)\subseteq\overline{R_\alpha}(A_\alpha)$. We only need to prove that $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$ and $\overline{R_\alpha}(A_\alpha)\subseteq[\overline{R}(A)]_\alpha$.

In fact, $\forall u\in[\overline{R}(A)]_{\alpha+}$, we have $\mu_{\overline{R}(A)}(u)>_{L^I}\alpha$. According to Definition 3.4, $\bigvee_{x\in E}[\mu_R^-(u,x)\wedge\mu_A^-(x)]>\alpha_1$ and $\bigvee_{x\in E}[\mu_R^+(u,x)\wedge\mu_A^+(x)]>\alpha_2$. Then $\exists x_0\in E$, such that $\mu_R^-(u,x_0)\wedge\mu_A^-(x_0)>\alpha_1$ and $\mu_R^+(u,x_0)\wedge\mu_A^+(x_0)>\alpha_2$, that is, $\mu_R^-(u,x_0)>\alpha_1,\mu_A^-(x_0)>\alpha_1,\mu_R^+(u,x_0)>\alpha_2$, and $\mu_A^+(x_0)>\alpha_2$. Thus $\mu_R(u,x_0)>_{L^I}\alpha$ and $\mu_A(x_0)>_{L^I}\alpha$, which imply that $x_0\in R_{\alpha+}(u)$ and $x_0\in A_{\alpha+}$. Namely, $R_{\alpha+}(u)\cap A_{\alpha+}\neq\emptyset$. By Definition 2.5, we have $u\in\overline{R_{\alpha+}}(A_{\alpha+})$. Hence $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$.

On the other hand, for any $u\in\overline{R_\alpha}(A_\alpha)$, we have $\overline{R_\alpha}(A_\alpha)(u)=1$. Since $\mu_{\overline{R}(A)}(u)=\bigvee_{\beta\in L^I}[\beta\wedge\overline{\overline{R_\beta}(A_\beta)(u)}]\geq_{L^I}\alpha\wedge\overline{\overline{R_\alpha}(A_\alpha)(u)}=\alpha$, we obtain $u\in[\overline{R}(A)]_\alpha$. Hence, $\overline{R_\alpha}(A_\alpha)\subseteq[\overline{R}(A)]_\alpha$.

(4) Similar to the proof of (3), it can be easily verified. □

Theorem 3.12 *Let (U,E,R) be an IVIF soft approximation space, and $A\in IVIF(E)$. Then the generalized lower IVIF soft rough approximation operator can be represented as follows: $\forall u\in U$*

(1)

$$\begin{aligned}\mu_{\underline{R}(A)}(u)&=\bigwedge_{\alpha\in L^I}[\alpha\vee(\overline{1}-\overline{\underline{R^\alpha}(A_{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee(\overline{1}-\overline{\underline{R^\alpha}(A_\alpha)(u)}]\\&=\bigwedge_{\alpha\in L^I}[\alpha\vee(\overline{1}-\overline{\underline{R^{\alpha+}}(A_{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee(\overline{1}-\overline{\underline{R^{\alpha+}}(A_\alpha)(u)}],\end{aligned}$$

(2)

$$\gamma_{\underline{R}(A)}(u) = \bigvee_{\alpha\in L^I}[\alpha\wedge(\bar{1}-\overline{\underline{R_\alpha}(A^{\alpha+})(u)}] = \bigvee_{\alpha\in L^I}[\alpha\wedge(\bar{1}-\overline{\underline{R_\alpha}(A^{\alpha})(u)}]$$
$$= \bigvee_{\alpha\in L^I}[\alpha\wedge(\bar{1}-\overline{\underline{R_{\alpha+}}(A^{\alpha+})(u)}] = \bigvee_{\alpha\in L^I}[\alpha\wedge(\bar{1}-\overline{\underline{R_{\alpha+}}(A^{\alpha})(u)}]$$

and moreover, for any $\alpha\in L^I$*,*
(3) $[\underline{R}(A)]_{\alpha+}\subseteq \underline{R^\alpha}(A_{\alpha+})\subseteq \underline{R^{\alpha+}}(A_{\alpha+})\subseteq \underline{R^{\alpha+}}(A_\alpha)\subseteq[\underline{R}(A)]_\alpha$,
(4) $[\underline{R}(A)]^{\alpha+}\subseteq \underline{R_\alpha}(A^{\alpha+})\subseteq \underline{R_{\alpha+}}(A^{\alpha+})\subseteq \underline{R_{\alpha+}}(A^\alpha)\subseteq[\underline{R}(A)]^\alpha$.

Proof. The proof is similar to Theorem 3.12. □

4 Application of IVIF soft rough sets in decision making

In [46], Zhang et al. gave a decision method based on IVIF soft set theory. However, we note that the decision method need to choose the thresholds in advance by decision makers. Thus the decision results will be depend on the threshold values at some degree. Since the thresholds have different kind of subjective preference information, different experts can obtain the different decision results for the same decision problem. So, in order to avoid the effect of the subjective information for the decision results, we only use the data information provided by the decision making problem and don't need any additional available information provided by decision makers. Thus the decision results are more objectively.

Next, we shall develop a new approach to decision making problem based on the generalized IVIF soft rough sets proposed in this paper.

Let (U,E,R) be an IVIF soft approximation space, where U is the universe of the discourse, E is the parameter set, and R is an IVIF soft relation on $U\times E$. Then we can give this decision-making approach based on generalized IVIF soft rough sets with five steps.

First, according to their own needs, the decision makers can construct an IVIF soft relation R from U to E, or IVIF soft set (F,E) over U.

Second, for a ceratin decision evaluation problem, we suppose that one wants to find out the decision alternative in universe with the evaluation value as larger as possible on every evaluate index. On the basis of the assumption, we construct an optimum normal decision object A which is an IVIF set on the evaluation universe E as follows:

$$A=\{<e_i,\max_{1\le j\le|U|}\mu_R(u_j,e_i),\min_{1\le j\le|U|}\gamma_R(u_j,e_i)>\},$$

where $|U|$ denotes the cardinality of the universe set U.

Third, by Equations (1) and (2), we can compute the generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ of the optimum normal decision object A. Thus, we obtain two most close values $\overline{R}(A)$ and $\underline{R}(A)$ to the decision alternative u_i of the universe set U.

Fourth, Atanassov and Gargov [3,4] introduced the notion of IVIF sets, and gave two operations on two IVIF sets, shown as follows, for all $F, G \in IVIF(U)$,

• Union operation:

$$\begin{aligned} F \cup G = \{< u, & [\mu_F^-(u) \vee \mu_G^-(u), \mu_F^+(u) \vee \mu_G^+(u)], \\ & [\gamma_F^-(u) \wedge \gamma_G^-(u), \gamma_F^+(u) \wedge \gamma_G^+(u)] > | u \in U\}, \end{aligned}$$

• Intersection operation:

$$\begin{aligned} F \cap G = \{< u, & [\mu_F^-(u) \wedge \mu_G^-(u), \mu_F^+(u) \wedge \mu_G^+(u)], \\ & [\gamma_F^-(u) \vee \gamma_G^-(u), \gamma_F^+(u) \vee \gamma_G^+(u)] > | u \in U\}. \end{aligned}$$

In general, the union operation and intersection operation on IVIF sets may result in loss of information in practical decision making problem which affects the accuracy of decision making. Therefore, inspired by the concept of $\oplus$-union operation of intuitionistic fuzzy subset, we also introduce the concept of $\oplus$-union operation of IVIF subset.

Definition 4.1 *Let $F, G \in IVIF(U)$. The $\oplus$-union operation about IVIF sets F and G can be defined as follows:*

$$\begin{aligned} F \oplus G = \{< u, & [\mu_F^-(u) + \mu_G^-(u) - \mu_F^-(u) \cdot \mu_G^-(u), \mu_F^+(u) + \mu_G^+(u) - \mu_F^+(u) \cdot \mu_G^+(u)], \\ & [\gamma_F^-(u) \cdot \gamma_G^-(u), \gamma_F^+(u) \cdot \gamma_G^+(u)] > | u \in U\}. \end{aligned}$$

By using the $\oplus$-union operation rather than the union and intersection operations, we can obtain the choice set as follows

$$\begin{aligned} H = \overline{R}(A) \oplus \underline{R}(A) = \{< u, & [\mu_{\overline{R}(A)}^-(u) + \mu_{\underline{R}(A)}^-(u) - \mu_{\overline{R}(A)}^-(u) \cdot \mu_{\underline{R}(A)}^-(u), \\ & \mu_{\overline{R}(A)}^+(u) + \mu_{\underline{R}(A)}^+(u) - \mu_{\overline{R}(A)}^+(u) \cdot \mu_{\underline{R}(A)}^+(u)], \\ & [\gamma_{\overline{R}(A)}^-(u) \cdot \gamma_{\underline{R}(A)}^-(u), \gamma_{\overline{R}(A)}^+(u) \cdot \gamma_{\underline{R}(A)}^+(u)] > | u \in U\}. \end{aligned}$$

Denote $H = \{< u, \mu_H(u), \gamma_H(u) >\}$.

Finally, define an IVIF value $\lambda = (\mu, \gamma) \in L$, where $\mu = \sup_{1 \le j \le |U|} [\mu_H^-(u_j), \mu_H^+(u_j)]$, $\gamma = \inf_{1 \le j \le |U|} [\gamma_H^-(u_j), \gamma_H^+(u_j)]$. Obviously, IVIF value $\lambda = (\mu, \gamma)$ is the maximum choice value in the choice set H. Hence we take the object u_j in universe U with the maximum choice value as the optimum decision for the given decision making problem. That is to say, if $\mu_H(u_j) \ge_{L^I} \mu$ and $\gamma_H(u_j) \le_{L^I} \gamma$, the optimum decision is u_j.

In general, if there exist two or more objects with the same maximum choice value , then we can take one of them as the optimum decision for the given decision making problem.

To illustrate the new method given above, let us consider the example as follows.

Example 4.2 *Reconsider Example 3.7. Now all the available information on houses under consideration can be formulated as an IVIF soft relation describing attractiveness of house that Mr.X is going to buy. By using the second step of the algorithm for generalized IVIF soft rough sets in decision making presented in this section, we can obtain the optimum normal decision object A as follows*

$$\begin{aligned} A = \{ &< e_1, [0.8, 0.9], [0.0, 0.1] >, < e_2, [0.6, 0.7], [0.1, 0.2] >, \\ &< e_3, [0.6, 0.8], [0.1, 0.2] >, < e_4, [0.4, 0.8], [0.1, 0.2] > \}. \end{aligned}$$

According to Equations (1) and (2), we can conclude that

$$\begin{aligned} \overline{R}(A) = \{ &< u_1, [0.7, 0.8], [0.1, 0.2] >, < u_2, [0.6, 0.7], [0.1, 0.2] >, < u_3, [0.5, 0.8], [0.1, 0.2] >, \\ &< u_4, [0.5, 0.7], [0.1, 0.2] >, < u_5, [0.8, 0.9], [0.0, 0.1] > \} \end{aligned}$$

and

$$\begin{aligned} \underline{R}(A) = \{ &< u_1, [0.4, 0.8], [0.1, 0.2] >, < u_2, [0.4, 0.8], [0.1, 0.2] >, < u_3, [0.4, 0.8], [0.1, 0.2] >, \\ &< u_4, [0.5, 0.7], [0.1, 0.2] >, < u_5, [0.4, 0.8], [0.1, 0.2] > \}. \end{aligned}$$

Now by Definition 4.1, we have

$$\begin{aligned} H = \overline{R}(A) \oplus \underline{R}(A) = \{ &< u_1, [0.82, 0.96], [0.01, 0.04] >, < u_2, [0.76, 0.94], [0.01, 0.04] >, \\ &< u_3, [0.70, 0.96], [0.01, 0.04] >, < u_4, [0.75, 0.91], [0.01, 0.04] >, \\ &< u_5, [0.88, 0.98], [0.00, 0.02] > \}. \end{aligned}$$

Obviously, IVIF value $\lambda = ([0.88, 0.98], [0.00, 0.02])$ *is the maximum choice value in the choice set H. Thus the optimal decision is u_5. Hence, Mr X will buy the house u_5.*

5 Conclusion

Recently, there has been a growing interest in soft set theory. Some extensions of soft sets have been obtained by combining soft set theory with other mathematical models, including fuzzy soft sets, interval-valued fuzzy soft sets, intuitionistic fuzzy soft sets and interval-valued intuitionistic fuzzy soft sets. Among them, the interval-valued intuitionistic fuzzy soft set is the most generalized one. This paper is devoted to the discussion of the combinations of interval-valued intuitionistic fuzzy soft set and rough set. By using an

interval-valued intuitionistic fuzzy soft relation, we present a new soft rough set model, called generalized IVIF soft rough sets. Furthermore, the generalized upper and lower IVIF soft rough approximation operators are represented by crisp soft rough approximation operators. Finally, a practical application is provided to illustrate the validity of the generalized IVIF soft rough set.

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GENERALIZATIONS OF HEINZ MEAN OPERATOR INEQUALITIES INVOLVING POSITIVE LINEAR MAP

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Abstract. In this paper, we study the Heinz mean inequalities of two positive operators involving positive linear map. We obtain a generalized conclusion based on operator Diaz-Metcalf type inequality. The conclusion is presented as follows: Let Φ be a unital positive linear map, if $0 < {m_1}^2 \le A \le {M_1}^2$ and $0 < {m_2}^2 \le B \le {M_2}^2$ for some positive real numbers $m_1 \le M_1$, $m_2 \le M_2$, then for $\alpha \in [0,1]$ and $p \ge 2$, the following inequality holds :

$$\begin{aligned}&(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^p\\ \le& 2^{-(p+4)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).\end{aligned}$$

1. Introduction and preliminaries

We represent the set of all bounded operators on $\mathcal{H}$ by $B(\mathcal{H})$. If an operator A satisfies $\langle Ax, x\rangle \ge 0$ for any $x \in \mathcal{H}$, then A is called a positive operator. For two self-adjoint operators A and B, $A \ge B$ means $A - B \ge 0$. The notation $A > 0$ means A is an invertible positive operator.

A linear map Φ: $B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ is called positive (strictly positive), if $\Phi(A) \ge 0$ ($\Phi(A) > 0$) whenever $A \ge 0$ ($A > 0$), and Φ is said to be unital if $\Phi(I) = I$. Take $A, B > 0$ and $\alpha \in [0, 1]$, the weighted arithmetic operator mean $A\nabla_\alpha B$, geometric mean $A\sharp_\alpha B$ and harmonic mean $A!_\alpha B$ are defined as follows :

$$A\nabla_\alpha B = (1-\alpha)A + \alpha B,\ A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}},\ A!_\alpha B = [(1-\alpha)A^{-1} + \alpha B^{-1}]^{-1}$$

when $\alpha = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively. The Heniz mean is defined by $H_\alpha(A, B) = \frac{A\sharp_\alpha B + A\sharp_{1-\alpha} B}{2}$, where $A, B > 0$ and $\alpha \in [0, 1]$. Recently, M. S. Moslehian, R. Nakamoto and Y. Seo [1, Theorem 2.1, part (ii)] showed that

Theorem 1.1 Let Φ be positive linear map, if $0 < {m_1}^2 \le A \le {M_1}^2$ and $0 < {m_2}^2 \le B \le {M_2}^2$ for some positive real numbers $m_1 \le M_1$ and $m_2 \le M_2$, we can get operator Diaz-Metcalf type inequality:

$$\frac{M_2m_2}{M_1m_1}\Phi(A) + \Phi(B) \le (\frac{M_2}{m_1} + \frac{m_2}{M_1})\Phi(A\sharp B).$$

Thus $A\sharp B \le H_\alpha(A, B)$ implies the following.

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Remark 1.2 Let Φ be positive linear map, if $0 < {m_1}^2 \le A \le {M_1}^2$ and $0 < {m_2}^2 \le B \le {M_2}^2$ for some positive real numbers $m_1 \le M_1$ and $m_2 \le M_2$, then for $\alpha \in [0,1]$, the following inequality holds:

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B) \le (\frac{M_2}{m_1} + \frac{m_2}{M_1})\Phi(H_\alpha(A,B)).$$

In 2015, Mohammad Sal Moslehian and Xiaohui Fu obtained a second powering of the operator Diaz-Metcalf type inequality:

Theorem 1.3 [9] Let Φ be positive linear map, if $0 < {m_1}^2 \le A \le {M_1}^2$ and $0 < {m_2}^2 \le B \le {M_2}^2$ for some positive real numbers $m_1 \le M_1$ and $m_2 \le M_2$, then the following inequality holds:

$$(\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B))^2 \le \left(\frac{(M_1 m_1({M_2}^2 + {m_2}^2) + M_2 m_2({M_1}^2 + {m_1}^2))^2}{8\sqrt{M_1 m_1 M_2 m_2} M_1^2 m_1^2 M_2 m_2}\right)^2 (\Phi(A \sharp B))^2.$$

In the paper we shall give further generalizations of Remark 1.2 in the following section, along with presenting p-th powering of some inequality for Heniz mean based on Remark 1.2 and the following consideration: It is easy to see that the Heniz operator mean interpolates the arithmetic-geometric operator mean inequality: $A!B \le A\sharp B \le H_\alpha(A,B) \le A\nabla B$, and the geometric mean has so-called maximal characterization [2], which says that $\begin{bmatrix} A & A\sharp B \\ A\sharp B & B \end{bmatrix}$ is positive, and moreover, if the operator matrix $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ is positive with X being self-adjoint, then $A\sharp B \ge X$.

2. Results and Proofs

In order to prove the first main theorem of the paper, first we give the following lemmas.

lemma 2.1. [3] Let Φ be a unital strictly positive linear map and $A > 0$, then $\Phi(A)^{-1} \le \Phi(A^{-1})$.

lemma 2.2. [5] Let $A, B \ge 0$, then the following norm inequality holds : $\|AB\| \le \frac{1}{4}\|A+B\|^2$.

lemma 2.3. [4] Let $A, B \ge 0$, then for $1 \le r < +\infty$, $\|A^r + B^r\| \le \|(A+B)^r\|$.

lemma 2.4. [7] (L-H inequality) If $0 \le \alpha \le 1$, $A \ge B \ge 0$, then $A^\alpha \ge B^\alpha$.

Theorem 2.5. Let Φ be a unital positive linear map, if $0 < {m_1}^2 \le A \le {M_1}^2$ and $0 < {m_2}^2 \le B \le {M_2}^2$ for some positive real numbers $m_1 \le M_1$, $m_2 \le M_2$, then for $\alpha \in [0,1]$ and $p \ge 2$, the following inequality holds :

$$\begin{aligned}&(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^p\\&\le 2^{-(p+4)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).\end{aligned}\tag{2.1}$$

Proof. Obviously (2.1) is equivalent to

$$\begin{aligned}&\|(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\&\le 2^{-(\frac{p}{2}+2)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{p}.\end{aligned}$$

Note that

$$({M_1}^2-A)({m_1}^2-A)A^{-1}\le 0,$$

implies

$${M_1}^2{m_1}^2A^{-1}-{M_1}^2-{m_1}^2+A\le 0,$$

therefore

$${M_1}^2{m_1}^2\Phi(A^{-1})+\Phi(A)\le {M_1}^2+{m_1}^2,$$

which equals to

$$M_1m_1M_2m_2\Phi(A^{-1})+\tfrac{M_2m_2}{M_1m_1}\Phi(A)\le\tfrac{M_2m_2}{M_1m_1}({M_1}^2+{m_1}^2).\tag{2.2}$$

Similarly, we have

$${M_2}^2{m_2}^2\Phi(B^{-1})+\Phi(B)\le {M_2}^2+{m_2}^2.\tag{2.3}$$

Since

$$H_\alpha^{-1}(A,B)\le(A!B)^{-1}=\frac{A^{-1}+B^{-1}}{2},$$

therefore

$$\begin{aligned}&H_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2})\\&=\frac{(\frac{1}{M_2m_2M_1m_1})^{1-\alpha}(\frac{1}{{M_2}^2{m_2}^2})^\alpha(A\sharp_\alpha B)+(\frac{1}{M_2m_2M_1m_1})^{\alpha}(\frac{1}{{M_2}^2{m_2}^2})^{1-\alpha}(A\sharp_{1-\alpha}B)}{2}\\&\le\max\{(\frac{1}{M_2m_2M_1m_1})^{1-\alpha}(\frac{1}{M_2m_2})^{2\alpha},(\frac{1}{M_2m_2M_1m_1})^{\alpha}(\frac{1}{M_2m_2})^{2-2\alpha}\}H_\alpha(A,B)\\&=\frac{H_\alpha(A,B)}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{1+\alpha},(M_1m_1)^\alpha(M_2m_2)^{2-\alpha}\}}.\end{aligned}\tag{2.4}$$

If we put

$$\beta=\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{1+\alpha},(M_1m_1)^\alpha(M_2m_2)^{2-\alpha}\},$$

then

$$\begin{aligned}
&\beta\Phi^{-1}(H_\alpha(A,B))\\
\leq&\Phi^{-1}(H_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2}))\\
\leq&\Phi({H^{-1}}_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2}))\\
\leq&\frac{1}{2}\Phi(M_2m_2M_1m_1A^{-1}+{M_2}^2{m_2}^2B^{-1})\\
=&\frac{1}{2}(M_2m_2M_1m_1\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1})).
\end{aligned}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
&\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)))^{\frac{p}{2}}\beta^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\
\leq&\frac{1}{4}\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)))^{\frac{p}{2}}+\beta^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|^2\\
\leq&\frac{1}{4}\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))+\beta\Phi^{-1}(H_\alpha(A,B)))^{\frac{p}{2}}\|^2\\
=&\frac{1}{4}\|\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))+\beta\Phi^{-1}(H_\alpha(A,B))\|^p\\
\leq&\frac{1}{4}\|\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)+M_2m_2M_1m_1\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1}))\|^p\\
\leq&2^{-(p+2)}({M_2}^2+{m_2}^2+\frac{M_2m_2}{M_1m_1}({M_1}^2+{m_1}^2))^p.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\
\leq&2^{-(\frac{p}{2}+2)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^p.
\end{aligned}$$

Corollary 2.6. In Theorem 2.5, if $1\leq p\leq 2$, we get

$$\begin{aligned}
&(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^p\\
\leq&2^{-3p}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).
\end{aligned}$$

Theorem 2.7. Let Φ be a unital positive linear map, if $0<{m_1}^2\leq A\leq {M_1}^2$ and $0<{m_2}^2\leq B\leq {M_2}^2$ for some positive real numbers $m_1\leq M_1$, $m_2\leq M_2$, then for $\alpha\in[0,1]$ and $p\geq 2$, the following inequality holds :

$$(\Phi(A)\nabla_\alpha\Phi(B))^p\leq 2^{-(p+4)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)). \quad (2.5)$$

Proof. Obviously (2.5) is equivalent to

$$\begin{aligned}&\|(\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\ \leq&2^{-(\frac{p}{2}+2)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{\alpha},(M_1m_1)^{\alpha}(M_2m_2)^{1-\alpha}\}}\right]^p.\end{aligned}$$

Note that

$$({M_1}^2-(1-\alpha)A)({m_1}^2-A)A^{-1}\leq 0,$$

implies

$${M_1}^2{m_1}^2A^{-1}-{M_1}^2-(1-\alpha){m_1}^2+(1-\alpha)A\leq 0.$$

Therefore

$${M_1}^2{m_1}^2\Phi(A^{-1})+(1-\alpha)\Phi(A)\leq {M_1}^2+(1-\alpha){m_1}^2. \tag{2.6}$$

Similarly, we have

$${M_2}^2{m_2}^2\Phi(B^{-1})+\alpha\Phi(B)\leq {M_2}^2+\alpha{m_2}^2. \tag{2.7}$$

Since

$$H_\alpha^{-1}(A,B)\leq(A!B)^{-1}=\frac{A^{-1}+B^{-1}}{2},$$

and by analogy to (2.4)

$$\begin{aligned}&H_\alpha(\frac{A}{{M_1}^2{m_1}^2},\frac{B}{{M_2}^2{m_2}^2})\\ =&\frac{H_\alpha(A,B)}{\min\{(M_1m_1)^{2-2\alpha}(M_2m_2)^{2\alpha},(M_1m_1)^{2\alpha}(M_2m_2)^{2-2\alpha}\}}.\end{aligned}$$

By puting

$$h=\min\{(M_1m_1)^{2-2\alpha}(M_2m_2)^{2\alpha},(M_1m_1)^{2\alpha}(M_2m_2)^{2-2\alpha}\},$$

we have

$$\begin{aligned}&h\Phi^{-1}(H_\alpha(A,B))\\ \leq&h\Phi^{-1}(H_\alpha(\frac{A}{M_1^2m_1^2},\frac{B}{{M_2}^2{m_2}^2}))\\ \leq&h\Phi({H^{-1}}_\alpha(\frac{A}{M_1^2m_1^2},\frac{B}{{M_2}^2{m_2}^2}))\\ \leq&\frac{1}{2}\Phi(M_1^2m_1^2A^{-1}+{M_2}^2{m_2}^2B^{-1})\\ =&\frac{1}{2}(M_1^2m_1^2\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1})).\end{aligned}$$

By (2.6) and (2.7), we have

$$\begin{aligned}
&\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}h^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\
\leq&\frac{1}{4}\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}+h^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|^2\\
\leq&\frac{1}{4}\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B)+h\Phi^{-1}(H_\alpha(A,B)))^{\frac{p}{2}}\|^2\\
=&\frac{1}{4}\|\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B)+h\Phi^{-1}(H_\alpha(A,B))\|^p\\
\leq&\frac{1}{4}\|\frac{1}{2}((1-\alpha)\Phi(A)+\alpha\Phi(B)+M_1^2m_1^2\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1}))\|^p\\
\leq&2^{-(p+2)}({M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2)^p.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|(\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\
\leq&2^{-(\frac{p}{2}+2)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^p.
\end{aligned}$$

Theorem 2.8. Let Φ be a unital positive linear map, if $0<{m_1}^2\leq A\leq {M_1}^2$ and $0<{m_2}^2\leq B\leq {M_2}^2$ for some positive real numbers $m_1\leq M_1$, $m_2\leq M_2$, δ is a arbitrary mean less than or equal to arithmetic mean, then for $\alpha\in[0,1]$ and $p\geq 2$, the following inequality holds :

$$(\Phi(A)\delta\Phi(B))^p\leq 2^{-(2p+4)}\left[\frac{{M_1}^2+{M_2}^2+{m_1}^2+{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).$$

Proof. By the similar method of proofing Theorem 2.7.

Corollary 2.9. In Theorem 2.8, we easily get

$${H_\alpha}^p(\Phi(A),\Phi(B))\leq 2^{-(2p+4)}\left[\frac{{M_1}^2+{M_2}^2+{m_1}^2+{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).$$

Theorem 2.10. [8] Let $0<m\leq A,\ B\leq M$, with the scalars $m,M>0$ and σ,τ two arbitrary means beween harmonic and arithmetic means, then for every positive unital linear map Φ, $2\leq p<\infty$,

$$\Phi^p(A\sigma B)\leq(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm})^p(\Phi(A)\tau\Phi(B))^p.$$

By $A!B\leq H_\alpha(A,B)\leq A\nabla B$, we obtain the following inequality.

Remark 2.11. Let $0<m\leq A,B\leq M$, then for every positive unital linear map Φ and $0<\alpha<1$, $K(h)=\frac{(h+1)^2}{4h}$, $h=\frac{M}{m}$, $p\geq 2$, the following inequality holds :

$$\Phi^p(H_\alpha(A,B))\leq 2^{2p-4}K^p(h){H_\alpha}^p(\Phi(A),\Phi(B)). \tag{2.8}$$

lemma 2.12. [6] For any bounded operator X,

$$|X| \leq tI \Longleftrightarrow \|X\| \leq t \Longleftrightarrow \begin{bmatrix} tI & X \\ X^* & tI \end{bmatrix} \geq 0 \ (t \geq 0).$$

Theorem 2.13. Let $0 < m \leq A, B \leq M$, then for every positive unital linear map Φ and $0 < \alpha < 1$, $K(h) = \frac{(h+1)^2}{4h}, h = \frac{M}{m}$, $p \geq 2$, the following inequality holds :

$$\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) \leq 2^{p-1}K^{\frac{p}{2}}(h). \quad (2.9)$$

Proof. By (2.8) we get

$$\|\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\| \leq 2^{p-2}K^{\frac{p}{2}}(h). \quad (2.10)$$

By (2.10) and Lemma 2.12, we obtain

$$\begin{bmatrix} 2^{p-2}K^{\frac{p}{2}}(h)I & \Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) \\ {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) & 2^{p-2}K^{\frac{p}{2}}(h)I \end{bmatrix} \geq 0,$$

and

$$\begin{bmatrix} 2^{p-2}K^{\frac{p}{2}}(h)I & {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) \\ \Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) & 2^{p-2}K^{\frac{p}{2}}(h)I \end{bmatrix} \geq 0.$$

Summing up these two operator matrices above, put

$$2^{p-2}K^{\frac{p}{2}}(h) = t,$$

$$\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) = X.$$

We have

$$\begin{bmatrix} 2tI & X \\ X^* & 2tI \end{bmatrix} \geq 0.$$

Since $\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B))$ is self-adioint, (2.9) follows from the maximal characterization of geometric mean.

Corollary 2.14. Let Φ be a unital positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$, then for $\alpha \in [0,1]$ and $p \geq 2$, the following inequality holds :

$$\begin{aligned} &{H_\alpha}^{\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{-\frac{p}{2}}(H_\alpha(A,B)) + \Phi^{-\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{\frac{p}{2}}(\Phi(A),\Phi(B)) \\ \leq &2^{-(p+1)}\left[\frac{{M_1}^2+{M_2}^2+{m_1}^2+{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{\alpha},(M_1m_1)^{\alpha}(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)). \end{aligned}$$

Proof. By Corollary 2.9 and the similar method of proofing Theorem 2.13, we can easily get.

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Existence and uniqueness results of nonlocal fractional sum-difference boundary value problems for fractional difference equations involving sequential fractional difference operators.

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Abstract

In this article, we study some new existence results for a nonlinear fractional difference equation with fractional sum-difference boundary conditions. Our problem containing sequential fractional difference operators that have different orders. The existence and uniqueness results are based on Banach contraction mapping principle and Schaefer's fixed point theorem. Finally, we present some examples to show the importance of these results.

Keywords: Fractional difference equations; boundary value problems; existence.

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1 Introduction

In this paper we consider a fractional sum-difference boundary value problem of a fractional difference equation of the form

$$\begin{cases} \Delta^{\alpha}u(t) = f(t+\alpha-1, u(t+\alpha-1), \Delta^{\mu}\Delta^{\nu}u(t+\alpha-\mu-\nu+1)), \\ u(\alpha-2) = \Delta^{\theta}u(\alpha-\theta-2) = p\,y(u), \\ u(T+\alpha) = q\,\Delta^{-\beta}u(\eta+\beta), \end{cases} \tag{1.1}$$

where $t \in \mathbb{N}_{0,T} := \{0,1,...,T\}$, $p,q > 0$, $2 < \alpha \leq 3$, $0 < \beta,\theta,\mu,\nu \leq 1$, $1 < \mu+\nu \leq 2$, $\eta \in \mathbb{N}_{\alpha-1,T+\alpha-1}$, $f \in (\mathbb{N}_{\alpha-3,T+\alpha} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function, and $y : C\left(\mathbb{N}_{\alpha-3,T+\alpha}, \mathbb{R}\right) \to \mathbb{R}$ is a given functional.

Mathematicians have used this fractional calculus in recent years to model and solve various related problems. In particular, fractional calculus is a powerful tool for the processes which appears in nature, e.g. biology, ecology and other areas.

Fractional difference equations have been interested many researchers since can use for describing many problems in the real-world phenomena such as physics, chemistry,

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mechanics, control systems, flow in porous media, and electrical networks can be found in [1] and [2] and the references therein. An excellent papers dealing with discrete fractional boundary value problems, which has helped to establish some of the basic theory of this field, one may see the papers [3]-[17], and references cited therein.

For example, Kang *et* al. [3] obtained sufficient conditions for the existence of solutions for the nonlocal boundary value problem as follows,

$$\begin{cases} -\Delta^{\mu}y(t) = \lambda h(t+\mu-1)\, f(y(t+\mu-1)), \quad t \in \mathbb{N}_{0,b} := \{0,1,...,b\}, \\ y(\mu-2) = \Psi(y), \quad y(\mu+b) = \Phi(y), \end{cases} \tag{1.2}$$

where $1 < \mu \leq 2$, $f \in C([0,\infty),[0,\infty))$ and $h \in C(\mathbb{N}_{\mu-1,\mu+b-1},[0,\infty))$ are given functions, and $\Psi, \Phi : \mathbb{R}^{b+3} \to \mathbb{R}$ are given functionals.

Presently, Chasreechai *et* al. [15] examined a Caputo fractional sum-difference equation with nonlocal fractional sum boundary value conditions of the form

$$\begin{cases} \Delta_C^{\alpha}u(t) = f(t+\alpha-1, u(t+\alpha-1), (\Psi^{\beta}u)(t+\alpha-2)), \quad t \in \mathbb{N}_{0,T}, \\ u(\alpha-2) = y(u), \\ u(T+\alpha) = \Delta^{-\gamma}g(T+\alpha+\gamma-3)\, u(T+\alpha+\gamma-3), \end{cases} \tag{1.3}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $2 < \gamma \leq 3$. For $U \subseteq \mathbb{R}$, $g \in C(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R}^{+} \cap U)$, $f \in C(\mathbb{N}_{\alpha-2,T+\alpha} \times U \times U, U)$ are given functions, $y : C(\mathbb{N}_{\alpha-2,T+\alpha}, U) \to U$ is a given functional, and for $\varphi : \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha} \to [0,\infty)$,

$$(\Psi^{\beta}u)(t) := [\Delta^{-\beta}\varphi\, u](t+\beta) = \frac{1}{\Gamma(\beta)} \sum_{s=\alpha-\beta-2}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}}\varphi(t,s+\beta)\, u(s+\beta).$$

The plan of this paper is as follows. In Section 2, we recall some definitions and basic lemmas. Also, we derive a representation of the solution to (1.1) by converting the problem to an equivalent fractional sum equation. In Section 3, the existence and uniqueness results of the boundary value problem (1.1) are established by Banach contraction mapping principle and Schaefer's fixed point theorem. An illustrative example is presented in Section 4.

2 Preliminaries

In this section, we introduce notations, definitions, and lemmas that are used in the main results.

Definition 2.1. *We define the generalized falling function by* $t^{\underline{\alpha}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, *for any* t *and* α *for which the right-hand side is defined. If* $t+1-\alpha$ *is a pole of the Gamma function and* $t+1$ *is not a pole, then* $t^{\underline{\alpha}} = 0$.

Lemma 2.1. *[10] If $t \le r$, then $t^{\underline{\alpha}} \le r^{\underline{\alpha}}$ for any $\alpha > 0$.*

Definition 2.2. *For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order fractional sum of f is defined by*

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

for $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 2.3. *For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order Riemann-Liouville fractional difference of f is defined by*

$$\Delta^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha \le N$.

Lemma 2.2. *[10] Let $0 \le N-1 < \alpha \le N$. Then*

$$\Delta^{-\alpha}\Delta^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \ldots + C_N t^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.

To define the solution of the boundary value problem (1.1) we need the following lemma that deals with a linear variant of the boundary value problem (1.1) and gives a representation of the solution.

Lemma 2.3. *Let $\Lambda \neq 0$, $p, q > 0$, $2 < \alpha \le 3$, $0 < \beta, \theta \le 1$, $\eta \in \mathbb{N}_{\alpha-1,\alpha+T-1}$, functions $h : \mathbb{N}_{\alpha-1,\alpha+T-1} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ be given. Then the problem*

$$\begin{cases} \Delta^{\alpha} u(t) = h(t+\alpha-1), & t \in \mathbb{N}_{0,T}, \\ u(\alpha-2) = \Delta^{\theta} u(\alpha-\theta-2) = p\, y(u), \\ u(T+\alpha) = q\, \Delta^{-\beta} u(\eta+\beta), \end{cases} \tag{2.1}$$

has the unique solution

$$\begin{aligned} u(t) \;=\; & -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)} \left[\frac{q}{\Gamma(\beta)} \sum_{s=\alpha}^{\eta} \sum_{\xi=0}^{s-\alpha} (\eta+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha-1) \right. \\ & \left. -\sum_{s=0}^{T} (T+\alpha-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1) \right] + \frac{p\, y(u)}{\Gamma(\alpha-1)} \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda} \right] \\ & + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1), \end{aligned} \tag{2.2}$$

where

$$\Lambda = \frac{q}{\Gamma(\beta)} \sum_{s=0}^{\eta-\alpha+1} (\eta+\beta-s-\alpha)^{\underline{\beta-1}} (s+\alpha-1)^{\underline{\alpha-1}} - \frac{\Gamma(T+\alpha+1)}{\Gamma(T+2)}, \tag{2.3}$$

$$\Theta = \frac{q}{\Gamma(\beta)} \sum_{s=0}^{\eta-\alpha+2} (\eta+\beta-\alpha-s+1)^{\underline{\beta-1}} (s+\alpha-2)^{\underline{\alpha-2}} - \frac{\Gamma(T+\alpha+1)}{\Gamma(T+3)}. \tag{2.4}$$

Proof. From Lemma 2.2, we find that a general solution for (2.1) can be written as

$$u(t) = C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + C_3 t^{\underline{\alpha-3}} + \Delta^{-\alpha} h(t+\alpha-1), \tag{2.5}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$.

Using the fractional difference of order $0 < \theta \leq 1$ for (2.5), we obtain

$$\begin{aligned}
\Delta^{\theta} u(t) \quad = \quad & \frac{C_1}{\Gamma(-\theta)} \sum_{s=\alpha-1}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-1}} + \frac{C_2}{\Gamma(-\theta)} \sum_{s=\alpha-2}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-2}} \\
& + \frac{C_3}{\Gamma(-\theta)} \sum_{s=\alpha-3}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-3}} \\
& + \frac{1}{\Gamma(-\theta)\Gamma(\alpha)} \sum_{s=\alpha}^{t+\theta} \sum_{\xi=0}^{s-\alpha} (t-\sigma(s))^{\underline{-\theta}} (s-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha-1),
\end{aligned}$$

for $t \in \mathbb{N}_{\alpha-\theta-2,T+\alpha-\theta+1}$.

Applying the condition of (2.1): $u(\alpha-2) = \Delta^{\theta} u(\alpha-\theta-2)$, we have $C_3 = 0$.
So,

$$u(t) = C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \Delta^{-\alpha} h(t+\alpha-1). \tag{2.6}$$

From (2.6) and the second condition of (2.1): $u(\alpha-2) = p\, y(u)$, we have

$$C_2 = \frac{p\, y(u)}{\Gamma(\alpha-1)}. \tag{2.7}$$

Hence,

$$u(t) = C_1 t^{\underline{\alpha-1}} + \frac{py(u)}{\Gamma(\alpha-1)} t^{\underline{\alpha-2}} + \Delta^{-\alpha} h(t+\alpha-1), \tag{2.8}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$.

Using the fractional sum of order $0 < \beta \leq 1$ for (2.8), we obtain

$$\Delta^{-\beta} u(t) \quad = \quad \frac{C_1}{\Gamma(\beta)} \sum_{s=\alpha-1}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} s^{\underline{\alpha-1}} + \frac{py(u)}{\Gamma(\beta)\Gamma(\alpha-1)} \sum_{s=\alpha-2}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} s^{\underline{\alpha-2}}$$

$$+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t-\beta}\sum_{\xi=0}^{s-\alpha}(t-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha-1), \tag{2.9}$$

for $t \in \mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta}$.

The third condition of (2.1) implies

$$\begin{aligned}
& q\Delta^{-\beta}u(\eta+\beta) \\
= \; & \frac{qC_1}{\Gamma(\beta)}\sum_{s=\alpha-1}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}\, s^{\underline{\alpha-1}} + \frac{p\,q\,y(u)}{\Gamma(\beta)\Gamma(\alpha-1)}\sum_{s=\alpha-2}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}\, s^{\underline{\alpha-2}} \\
& +\frac{q}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\, h(\xi+\alpha-1) \\
= \; & C_1(T+\alpha)^{\underline{\alpha-1}} + \frac{p\,y(u)}{\Gamma(\alpha-1)}(T+\alpha)^{\underline{\alpha-2}} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\, h(s+\alpha-1).
\end{aligned}$$

Solving the above equation for the constant C_1, we get

$$\begin{aligned}
C_1 \;=\; & \frac{-p\,q\,y(u)}{\Lambda\Gamma(\beta)\Gamma(\alpha-1)}\sum_{s=\alpha-2}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}\, s^{\underline{\alpha-2}} + \frac{p\,y(u)}{\Lambda\Gamma(\alpha-1)}(T+\alpha)^{\underline{\alpha-2}} \\
& +\frac{1}{\Lambda\Gamma(\alpha)}\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\, h(s+\alpha-1) \qquad (2.10) \\
& -\frac{q}{\Lambda\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\, h(\xi+\alpha-1),
\end{aligned}$$

where Λ is defined as (2.3). Substituting C_1 into (2.8), we obtain (2.2). □

3 Main Results

In this section, we wish to establish the existence results for problem (1.1). To accomplish this, let $\mathcal{C} = C(\mathbb{N}_{\alpha-3,\alpha+T}, \mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \max\{\|u\|, \|\Delta^{\mu}\Delta^{\nu}u\|\},$$

where $\|u\| = \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}} |u(t)|$ and $\|\Delta^{\mu}\Delta^{\nu}u\| = \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}} |\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)|$. Also define an operator $F : \mathcal{C} \to \mathcal{C}$ by

$$Fu(t)$$

$$
\begin{aligned}
&= -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}f(\xi+\alpha-1,u(\xi+\alpha-1),\\
&\quad \Delta^{\mu}\Delta^{\nu}u(\xi+\alpha-\mu-\nu+1))-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}f(s+\alpha-1,u(s+\alpha-1),\\
&\quad \Delta^{\mu}\Delta^{\nu}u(s+\alpha-\mu-\nu+1))\Bigg]+\frac{p\,y(u)}{\Gamma(\alpha-1)}\left[t^{\underline{\alpha-2}}-\frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right] \qquad (3.1)\\
&\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}f(s+\alpha-1,u(s+\alpha-1),\Delta^{\mu}\Delta^{\nu}u(s+\alpha-\mu-\nu+1)),
\end{aligned}
$$

for $t\in\mathbb{N}_{\alpha-3,\alpha+T}$, where $\Lambda\neq 0,\ \Theta$ are defined as (2.3),(2.4), respectively. The problem (1.1) has solutions if and only if the operator F has fixed points.

Our first result is based on Banach contraction mapping principle.

Theorem 3.1. *Assume that*

(H_1) *There exist constants* $\gamma_1,\gamma_2>0$ *such that, for each* $t\in\mathbb{N}_{\alpha-3,\alpha+T}$ *and for all* $u,v\in\mathcal{C}$,

$$
\begin{aligned}
&|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))-f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))|\\
\leq\ &\gamma_1|u(t)-v(t)|+\gamma_2|\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)-\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2)|.
\end{aligned}
$$

(H_2) *There exists a constant* $\omega>0$ *such that, for all* $u,v\in\mathcal{C}$,

$$|y(u)-y(v)|\leq\omega|u-v|.$$

(H_3) $\gamma\Omega+\omega\Phi<\frac{(T+2)(T+1)}{(T+\alpha+2)(T+\alpha+1)}$,

where

$$
\begin{aligned}
\gamma &= \max\{\gamma_1+\gamma_2\} &(3.2)\\
\Omega &= \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(T)}-\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|+\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} &(3.3)\\
\Phi &= \frac{p(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]. &(3.4)
\end{aligned}
$$

Then the boundary value problem (1.1) *has at least one solution on* $\mathbb{N}_{\alpha-3,\alpha+T}$.

Proof. Denote that,

$$\mathcal{H}|u-v|(t)=\big|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))-f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))\big|.$$

For all $u, v \in \mathcal{C}$, by computing directly, we have

$$\begin{aligned}
&\|Fu - Fv\| \\
= \;& \max_{t \in \mathbb{N}_{\alpha-3,\alpha+T}} \Bigg| -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)} \Bigg[\frac{q}{\Gamma(\beta)} \sum_{s=\alpha}^{\eta} \sum_{\xi=0}^{s-\alpha} (\eta+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} \mathcal{H}|u-v|(\xi) \\
& - \sum_{s=0}^{T} (T+\alpha-\sigma(s))^{\underline{\alpha-1}} \mathcal{H}|u-v|(s) \Bigg] + \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda} \right] \frac{p\,|y(u)-y(v)|}{\Gamma(\alpha-1)} \\
& + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} \mathcal{H}|u-v|(s) \Bigg| \\
\leq \;& (\gamma\|u-v\|_{\mathcal{C}}) \left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|} \left| \frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} \right| \right] \\
& + \left[1 + (T+4) \left| \frac{\Theta}{\Lambda} \right| \right] \frac{(\omega\|u-v\|_{\mathcal{C}})\; p\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} \\
= \;& (\gamma\|u-v\|_{\mathcal{C}})\; \Omega + (\omega\|u-v\|_{\mathcal{C}})\; \Phi,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \Delta^{\mu}\Delta^{\nu}Fu - \Delta^{\mu}\Delta^{\nu}Fv \right\| \\
= \;& \max_{t \in \mathbb{N}_{\alpha-3,\alpha+T}} \left| \left(\Delta^{\mu}\Delta^{\nu}Fu\right)(t-\mu-\nu+2) - \left(\Delta^{\mu}\Delta^{\nu}Fv\right)(t-\mu-\nu+2) \right| \\
< \;& \left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|} \sum_{s=\alpha-\nu}^{T+\alpha-\nu+2} \sum_{\xi=\alpha-1}^{s+\nu} (T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}} (s-\sigma(\xi))^{\underline{-\nu-1}} \right) \times \\
& (T+\alpha+2)^{\underline{\alpha-1}} \Bigg[\frac{(\gamma\|u-v\|_{\mathcal{C}})}{|\Lambda|} \left| \frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} \right| \\
& + \frac{p\;(\omega\|u-v\|_{\mathcal{C}})}{\Gamma(\alpha-1)} \left| \frac{\Theta}{\Lambda} \right| \Bigg] + p\;(\omega\|u-v\|_{\mathcal{C}})\; \frac{(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} \times \\
& \left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|} \sum_{s=\alpha-\nu}^{T+\alpha-\nu+2} \sum_{\xi=\alpha-2}^{s+\nu} (T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}} (s-\sigma(\xi))^{\underline{-\nu-1}} \right) \\
& + \left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|} \sum_{s=\alpha-\nu}^{T+\alpha-\nu+2} \sum_{r=\alpha}^{s+\nu} (T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}} (s-\sigma(r))^{\underline{-\nu-1}} \right) \times \\
& \frac{(\gamma\|u-v\|_{\mathcal{C}})}{\Gamma(\alpha)} \sum_{\xi=0}^{T+2} (T+\alpha+2-\sigma(\xi))^{\underline{\alpha-1}} \\
< \;& \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)} \left[\gamma\Omega + \omega\Phi \right] \|u-v\|_{\mathcal{C}}.
\end{aligned}$$

Thus, $\|Fu - Fv\|_{\mathcal{C}} \leq \dfrac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)} \left[\gamma\Omega + \omega\Phi \right] \|u-v\|_{\mathcal{C}}$.

By (H_3), we get that F is a contraction mapping, and then Theorem 3.1 implies that boundary value problem (1.1) has unique solution on $\mathbb{N}_{\alpha-3,\alpha+T}$. This completes the proof. □

The second result is based on Schaefer's fixed point theorem.

Theorem 3.2. *(Arzelá-Ascoli Theorem) [18] A set of function in $C[a,b]$ with the sup norm, is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

Theorem 3.3. *[18] If a set is closed and relatively compact then it is compact.*

Theorem 3.4. ***[Schaefer's fixed point theorem]*** *[19] Let X be a Banach space and $T : X \to X$ be a continuous and compact mapping. If the set*

$$\{x \in X \ : \ x = \lambda T(x), \ \textit{for some } \lambda \in (0,1)\}$$

is bounded, then T has a fixed point.

We shall use Schaefer's fixed point theorem to prove that the operator F defined as (3.1), has a fixed point.

Theorem 3.5. *Suppose that there exist constants $L_1, L_2 > 0$ such that, for each $t \in \mathbb{N}_{\alpha-3,\alpha+T}$ and $u \in \mathcal{C}$,*

$$\begin{aligned} |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))| &\leq L_1 \max\{\|u\|, \|\Delta^{\mu}\Delta^{\nu}u\|\}, \\ |y(u)| &\leq L_2. \end{aligned}$$

Then the problem (1.1) has at least one solution on $\mathbb{N}_{\alpha-3,\alpha+T}$.

Proof. We divide the proof into four steps.

Step I. Verify F map bounded sets into bounded sets in $C(\mathbb{N}_{\alpha-3,\alpha+T})$.
Let $u \in B_L = \{u \in C(\mathbb{N}_{\alpha-3,\alpha+T}) : \|u\|_{\mathcal{C}} \leq L\}$, and choosing a constant

$$L \geq \frac{L_2\Phi(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1) - L_1\Omega\,(T+\alpha+2)(T+\alpha+1)}.$$

Denote that

$$\begin{aligned} \mathcal{H}|u-v|(t) &:= |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)) - f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))| \\ &\leq \|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)) - f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))\| \\ &=: \mathcal{H}\|u-v\|(t). \end{aligned}$$

For each $u \in B_L$, we obtain

$$\begin{aligned}
&\|Fu\| \\
= \;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}} \Bigg| -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{H}|u-v|(\xi) \\
& -\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg] + \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right]\frac{p\,|y(u)|}{\Gamma(\alpha-1)} \\
& +\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg| \\
\le \;& L_1\|u\|_{\mathcal{C}}\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] \\
& +\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]\frac{pL_2\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} \\
\le \;& L_1 L\,\Omega + L_2\,\Phi.
\end{aligned}$$

and

$$\begin{aligned}
&\|\Delta^{\mu}\Delta^{\nu}Fu\| = \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\left|\,(\Delta^{\mu}\Delta^{\nu}Fu)\,(t-\mu-\nu+2)\right| \\
= \;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\Bigg\{\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{\xi=\alpha-1}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\times \\
& \xi^{\underline{\alpha-1}}\Bigg[\frac{(L_1\|u\|_{\mathcal{C}})}{|\Lambda|\Gamma(\alpha)}\Bigg|\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}} \\
& -\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\Bigg| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\Bigg] \\
& +\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{\xi=\alpha-2}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\xi^{\underline{\alpha-2}}\times \\
& \left[\frac{pL_2}{\Gamma(\alpha-1)}\,(T-\alpha+2)^{\underline{\alpha-2}}\right] + \frac{(L_1\|u\|_{\mathcal{C}})}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{r=\alpha}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}\times \\
& (s-\sigma(r))^{\underline{-\nu-1}}\left[\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{r-\alpha}(r-\sigma(\xi))^{\underline{\alpha-1}}\right]\Bigg\} \\
< \;& \left\{\frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\right\}L_1L\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\times\right. \\
& \left.\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] + \left\{\frac{(T+\alpha+2)(T+\alpha+1)}{(T+3)(T+2)}\right\}\times
\end{aligned}$$

$$
\begin{aligned}
& \frac{pL_2}{\Gamma(\alpha-1)}(T+\alpha+2)^{\underline{\alpha-2}}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right] \\
< \; & \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[L_1 L\Omega + L_2\Phi\right].
\end{aligned}
$$

Hence, $\|Fu\|_{\mathcal{C}} \leq L$ where Ω and Φ are defined on 3.3 and 3.4, respectively. Thus F is uniformly bounded.

Step II. Show that F is continuous on B_L.
Let $\epsilon > 0$ there exists $\delta = \max\{\delta_1, \delta_2\} > 0$ such that, for each $t \in \mathbb{N}_{\alpha-3,\alpha+T}$ and for all $u, v \in B_L$ with

$$\max\{|u(t)-v(t)|, |\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2) - \Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2)|\} < \delta_1,$$

we have

$$\mathcal{H}|u-v| < \frac{\epsilon\,(T+2)(T+1)}{2\Omega\,(T+\alpha+2)(T+\alpha+1)},$$

and for all $u, v \in B_L$ with $|u-v| < \delta_2$, we have

$$|y(u)-y(v)| < \frac{\epsilon\,(T+2)(T+1)}{2\Phi\,(T+\alpha+2)(T+\alpha+1)}.$$

Then, we have

$$
\begin{aligned}
& \|Fu(t)-Fv(t)\| \\
= \; & \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\Bigg| -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{H}|u-v|(\xi) \\
& -\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg] + \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right]\frac{p\,|y(u)-y(v)|}{\Gamma(\alpha-1)} \\
& +\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg| \\
\leq \; & \mathcal{H}\|u-v\|\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\cdot\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] \\
& +\|y(u)-y(v)\|\,\frac{p\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right] \\
= \; & \Omega\,\mathcal{H}\|u-v\| + \Phi\,\|y(u)-y(v)\|.
\end{aligned}
$$

Similarly to the proof above and Theorem 3.1, we obtain

$$\Big\|\left(\Delta^{\mu}\Delta^{\nu}Fu\right)(t-\mu-\nu+2) - \left(\Delta^{\mu}\Delta^{\nu}Fv\right)(t-\mu-\nu+2)\Big\|$$

$$< \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Big[\Omega\,\mathcal{H}\|u-v\| + \Phi\,\|y(u)-y(v)\|\Big] \;<\; \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

Hence, $\|Fu - Fv\|_{\mathcal{C}} \leq \epsilon$. This means that F is continuous on B_L.

Step III. Examine $F(B_L)$ is equicontinuous with B_L. For any $\epsilon > 0$, there exists $\delta = \max\{\delta_1,\delta_2,\delta_3\} > 0$ such that, for $t_1, t_2 \in \mathbb{N}_{\alpha-3,\alpha+T}$

$$\left|t_2^{\underline{\alpha}} - t_1^{\underline{\alpha}}\right| \;<\; \frac{\epsilon\,\Gamma(\alpha+1)\,(T+2)(T+1)}{3L_1(T+\alpha+2)(T+\alpha+1)} \qquad \text{whenever} \quad |t_2-t_1| < \delta_1,$$

$$\left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right| \;<\; \frac{\epsilon\,|\Lambda|\,(T+2)(T+1)}{3(T+\alpha+2)(T+\alpha+1)\left[L_1\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha-1)}\right| + \frac{pL_2\,|\Theta|}{\Gamma(\alpha-1)}\right]} \quad \text{whenever} \quad |t_2-t_1| < \delta_2,$$

$$\left|t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right| \;<\; \frac{\epsilon\Gamma(\alpha-1)(T+2)(T+1)}{3pL_2(T+\alpha+2)(T+\alpha+1)} \qquad \text{whenever} \quad |t_2-t_1| < \delta_3.$$

Then, we have

$$\begin{aligned}
&|Fu(t_2) - Fu(t_1)| \\
= &\;\Bigg| -\frac{t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}} \times \\
&f(\xi+\alpha-1, u(\xi+\alpha-1), \Delta^{\mu}\Delta^{n}u(\xi+\alpha-\mu-\nu+1)) - \sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}} \times \\
&f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1))\Bigg] \\
&+\frac{p\,y(u)}{\Gamma(\alpha-1)}\left[\left(t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right) - \left(t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right)\frac{\Theta}{\Lambda}\right] \\
&+\frac{1}{\Gamma(\alpha)}\Bigg[\sum_{s=0}^{t_2-\alpha}(t_2-\sigma(s))^{\underline{\alpha-1}} f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1)) \\
&-\sum_{s=0}^{t_1-\alpha}(t_1-\sigma(s))^{\underline{\alpha-1}} f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1))\Bigg]\Bigg| \\
\leq &\;\left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right|\left[\frac{L_1}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right] \\
&+\frac{L_1}{\Gamma(\alpha)}\left[\sum_{s=0}^{t_2-\alpha}(t_2-\sigma(s))^{\underline{\alpha-1}} + \sum_{s=0}^{t_1-\alpha}(t_1-\sigma(s))^{\underline{\alpha-1}}\right] + \left|t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right|\frac{pL_2}{\Gamma(\alpha-1)} \\
= &\;\left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right|\left[\frac{L_1}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right]
\end{aligned}$$

$$+\frac{L_1}{\Gamma(\alpha+1)}\big|t_2^{\underline{\alpha}}-t_1^{\underline{\alpha}}\big|+\frac{pL_2}{\Gamma(\alpha-1)}\big|t_2^{\underline{\alpha-2}}-t_1^{\underline{\alpha-2}}\big|.$$

So $\|Fu-Fv\|<\epsilon$.

Similarly to the proof above and Theorem 3.1, we obtain

$$\begin{aligned}
&\big\|\Delta^{\mu}\Delta^{\nu}Fu-\Delta^{\mu}\Delta^{\nu}Fu\big\|\\
< \;& \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Bigg\{\big|t_2^{\underline{\alpha-1}}-t_1^{\underline{\alpha-1}}\big|\left[\frac{L_1}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)}-\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right.\\
&\left.+\frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right]+\frac{L_1}{\Gamma(\alpha+1)}\big|t_2^{\underline{\alpha}}-t_1^{\underline{\alpha}}\big|+\frac{pL_2}{\Gamma(\alpha-1)}\big|t_2^{\underline{\alpha-2}}-t_1^{\underline{\alpha-2}}\big|\Bigg\}\\
< \;& \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}$$

Thus, $\|Fu(t_2)-Fu(t_1)\|_{\mathcal{C}}\le\epsilon$. This means that $F(B_L)$ is an equicontinuous set.

As a consequence of Steps I to III together with the Arzelá-Ascoli theorem, its imply that $F: C(\mathbb{N}_{\alpha-3,\alpha+T})\to C(\mathbb{N}_{\alpha-3,\alpha+T})$ is completely continuous.

Step IV. *A priori bounds.* We show that the set

$$E=\{u\in C(\mathbb{N}_{\alpha-3,\alpha+T}): u=\lambda Fu \quad \text{for some} \quad 0<\lambda<1\} \text{ is bounded.}$$

Let $u\in E$. Then $u(t)=\lambda(Fu)(t)$ for some $0<\lambda<1$. Thus, for each $t\in\mathbb{N}_{\alpha-3,\alpha+T}$, we have

$$|\lambda Fu(t)|<|Fu(t)|<L_1L\,\Omega+L_2\,\Phi:=\Im.$$

So, we have $\|\lambda Fu\|<\Im$. Similarly to the proof above and Theorem 3.1, we obtain

$$\big\|\lambda\,\Delta^{\mu}\Delta^{\nu}Fu\big\| \;<\; \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Im \;=:\; \widetilde{\Im}.$$

Hence, $\|\lambda Fu\|_{\mathcal{C}}\le\widetilde{\Im}$. This shows that E is bounded.

By of the Schaefer's fixed point theorem, we conclude that F has a fixed point which is a solution of the problem (1.1). □

4 Some examples

In this section, in order to illustrate our results, we consider some examples.

Example 4.1. *Consider the following boundary value problem*

$$\Delta^{\frac{5}{2}}u(t) = \frac{e^{-\sin^2(t+\frac{3}{2})}}{\left(t+\frac{15}{2}\right)^2}\cdot\frac{\left|u\left(t+\frac{3}{2}\right)\right| + |\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{25}{12}\right)|}{\left|u\left(t+\frac{3}{2}\right)\right|+1}, \quad t\in\mathbb{N}_{0,4}, \tag{4.1}$$

$$u\left(\frac{1}{2}\right) = \Delta^{\frac{1}{4}}u\left(\frac{1}{4}\right) = \frac{1}{2}\sum_{i=0}^{7}C_i u(t_i), \quad t_i = i-\frac{1}{2}, \tag{4.2}$$

$$u\left(\frac{13}{2}\right) = \frac{1}{3}\Delta^{-\frac{1}{3}}u\left(\frac{29}{6}\right). \tag{4.3}$$

where C_i *are given positive constants with* $\sum_{i=0}^{7}C_i < \frac{1}{10\,e^{20}}$.

Here $p=\frac{1}{2},\ q=\frac{1}{3},\ \theta=\frac{1}{4},\ \alpha=\frac{5}{2},\ \beta=\frac{1}{3},\ \mu=\frac{2}{3},\ \nu=\frac{3}{4},\ \eta=\frac{9}{2},\ T=4,$ $f\left(t,u(t),\Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right) = \frac{e^{-\sin^2 t}}{(t+6)^2}\cdot\frac{|u(t)|+|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{7}{12}\right)|}{|u(t)|+1}$ and $y(u)=\sum_{i=0}^{7}C_i u(t_i),\ t_i = i-\frac{1}{2}$.

Let $t\in\mathbb{N}_{-\frac{1}{2},\frac{13}{2}}$ and $u,v\in\mathbb{R}$, then

$$|\Lambda| = 7.781 \neq 0, \quad \Theta = 1.278, \quad \Omega \approx 106.039, \quad \Phi\approx 3.119.$$

Since $\quad |f\left(t,u(t),\Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right) - f\left(t,v(t),\Delta^{\mu}\Delta^{\nu}v\left(t-\mu-\nu+2\right)\right)|$
$\leq \frac{4}{1849}\left|u(t)-v(t)\right| + \frac{4}{1849}\left|\Delta^{\mu}\Delta^{\nu}u\left(t+\frac{7}{12}\right) - \Delta^{\mu}\Delta^{\nu}v\left(t+\frac{7}{12}\right)\right|$
is satisfied with $\gamma=\max\{\gamma_1+\gamma_2\} = \frac{8}{1849}$.

Also, we get $|y(u)-y(v)| = \left|\sum_{i=0}^{7}C_i u(t_i) - \sum_{i=0}^{7}C_i v(t_i)\right| \leq \sum_{i=0}^{7}C_i|u(t_i)-v(t_i)|$, so (H_2) holds with $\omega=\sum_{i=0}^{7}C_i < \frac{1}{10\,e^{20}}$.

We can show that

$$\frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[\gamma\Omega+\omega\Phi\right] \approx 0.975 < 1.$$

Hence, by Theorem 3.1, the problem (4.1)-(4.3) has unique solution. □

Example 4.2. *Consider the following boundary value problem*

$$\Delta^{\frac{5}{2}}u(t) = \frac{t+\frac{3}{2}}{10\pi}\left[2\sin\left|u\left(t+\frac{3}{2}\right)\right| + \cos\left|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{25}{12}\right)\right|\right],\ t\in\mathbb{N}_{0,4}, \tag{4.4}$$

$$u\left(\frac{1}{2}\right) = \Delta^{\frac{1}{4}}u\left(\frac{1}{4}\right) = \frac{1}{4}\sum_{i=0}^{7}C_i\frac{|u(t_i)|}{1+|u(t_i)|}, \quad t_i = i-\frac{1}{2}, \tag{4.5}$$

$$u\left(\frac{13}{2}\right) = \frac{1}{5}\Delta^{-\frac{1}{3}}u\left(\frac{29}{6}\right), \tag{4.6}$$

where C_i *are given positive constants with* $\sum_{i=0}^{7}C_i < \frac{1}{e}$.

Here $p=\frac{1}{4},\ q=\frac{1}{5},\ \alpha=\frac{5}{2},\ \beta=\frac{1}{3},\ \theta=\frac{1}{4},\ \mu=\frac{2}{3},\ \nu=\frac{3}{4},\ \eta=\frac{9}{2},\ T=4,$ $f\left(t,u(t),\Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right)=\frac{t}{10\pi}\left[2\sin|u(t)|+\cos\left|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{7}{12}\right)\right|\right]$ and $y(u)=\sum_{i=0}^{7}C_i\frac{|u(t_i)|}{1+|u(t_i)|},\quad t_i=i-\frac{1}{2}$. Clearly for $t\in\mathbb{N}_{-\frac{1}{2},\frac{13}{2}}$, we have

$$\begin{aligned}|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right))| &\leq \frac{13}{20\,\pi}\max\{2,1\}\approx 0.414 \qquad \left(L_1=\frac{13}{20\,\pi}\right)\\ |y(u)| &\leq \sum_{i=0}^{7}C_i\frac{|u(t_i)|}{1+|u(t_i)|}<\frac{1}{e}=L_2.\end{aligned}$$

Hence, by Theorem 3.5, the problem (4.4)-(4.6) has at least one solution. □

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Hesitant fuzzy mighty filters of BE-algebras

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Abstract. The notion of hesitant fuzzy mighty filter of a BE-algebra is introduced and related properties are investigated. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy mighty filter. We construct a new quotient structure of a transitive BE-algebra using a hesitant fuzzy filter and study some properties of it.

1. Introduction

In 2007, Kim and Kim [5] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Song et al. [8] considered the fuzzification of ideals in BE-algebras. They introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. They gave characterizations of a fuzzy ideal in BE-algebras.

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [9] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [3, 7, 11, 12, 13, 14, 15]). In [4], Y. B. Jun and S. S. Ahn introduced the notion of a hesitant fuzzy filter and investigated some properties of it. The authors [2] defined a hesitant fuzzy implicative filter in a BE-algebra and discussed some properties of it.

In this paper, we introduce the notion of hesitant fuzzy mighty filter of a BE-algebra, and investigate some properties of it. We consider characterizations of a hesitant fuzzy mighty filter of a BE-algebra. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy mighty filter. We construct a new quotient structure of a transitive BE-algebra using a hesitant fuzzy filter and study some properties of it.

2. Preliminaries

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By a *BE-algebra* ([5]) we mean a system $(X;*,1)$ of type $(2,0)$ which the following axioms hold:

(2.1) $(\forall x \in X)\,(x * x = 1)$,
(2.2) $(\forall x \in X)\,(x * 1 = 1)$,
(2.3) $(\forall x \in X)\,(1 * x = x)$,
(2.4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)$ (exchange).

We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X;*,1)$ is said to be *transitive* if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X;*,1)$ is said to be *self distributive* if it satisfies: for any $x, y, z \in X$, $x*(y*z) = (x*y)*(x*z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [5]).

Every self distributive BE-algebra $(X;*,1)$ satisfies the following properties:

(2.5) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,
(2.6) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,
(2.7) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$,

Definition 2.1. Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X ([5]) if

(F1) $1 \in F$;
(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

F is a *mighty filter* ([6]) of X if it satisfies (F1) and

(F3) $(\forall x, y, z \in X)(z * (y * x), z \in F \Rightarrow ((x * y) * y) * x \in F)$.

Theorem 2.2. ([6]) *A filter F of a BE-algebra X is mighty if and only if*

(2.8) $(\forall x, y \in X)(y * x \in F \Rightarrow ((x * y) * y) * x \in F)$.

Definition 2.3. ([9]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0,1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) | e \in E\}$$

where $h_E : E \to \mathscr{P}([0,1])$.

Definition 2.4. Given a non-empty subset A of a BE-algebra X, a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) | x \in X\}$$

on satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A$$

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is called a *hesitant fuzzy set related to* A (briefly, A-*hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

For a hesitant set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X and a subset γ of $[0,1]$, the hesitant fuzzy γ-inclusive set of H_X, denoted by $H_X(\gamma)$, is defined to be the set

$$H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}.$$

For any hesitant fuzzy set $H_X = \{(x, h_X(x)|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$, we call H_X a *hesitant fuzzy subset* of G_X, denoted by $H_X \widetilde{\subseteq} G_X$, if $h_X(x) \subseteq g_X(x)$ for all $x \in X$.

3. Hesitant fuzzy mighty filters

Definition 3.1. Given a non-empty subset (subalgebra as much as possible) A of a BE-algebra X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy subalgebra of* X *related to* A (briefly, A-*hesitant fuzzy subalgebra* of X) ([4]) if it satisfies the following condition: $h_A(x) \cap h_A(y) \subseteq h_A(x * y)$ for any $x, y \in A$. An A-hesitant fuzzy subalgebra of X with $A = X$ is called a *hesitant fuzzy subalgebra* of X. An A-hesitant fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a *hesitant fuzzy filter of* X *related to* A (briefly, A-*hesitant fuzzy filter* of X) ([4]) if it satisfies the following condition:

(3.1) $(\forall x \in A)(h_A(x) \subseteq h_A(1))$,

(3.2) $(\forall x, y \in A)(h_A(x * y) \cap h_A(x) \subseteq h_A(y))$.

An A-hesitant fuzzy filter of X with $A = X$ is called a *hesitant fuzzy filter* of X.

Proposition 3.2. ([4]) *Let* $H_A := \{(x, h_A(x))|x \in X\}$ *be an* A*-hesitant fuzzy filter of a* BE*-algebra* X *where* A *is a subalgebra of* X*. Then the following assertions are valid.*

(i) $(\forall x, y \in A)(x \le y \Rightarrow h_A(x) \subseteq h_A(y))$,

(ii) $(\forall x, y, z \in A)(z \le x * y \Rightarrow h_A(y) \supseteq h_A(x) \cap h_A(z))$,

(iii) $(\forall x, y, z \in A)(h_A(x * (y * z)) \cap h_A(y) \subseteq h_A(x * z))$,

(iv) $(\forall a, x \in A)(h_A(a) \subseteq h_A((a * x) * x)$.

Proposition 3.3. *Every hesitant fuzzy filter of a* BE*-algebra* X *is a hesitant fuzzy subalgebra of* X*.*

Proof. Let $H_X = \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy filter of X. For any $x, y \in X$, we have $h_X(x) \cap h_X(y) \subseteq h_X(1) \cap h_X(y) = h_X(y * (x * y)) \cap h_X(y) \subseteq h_X(x * y)$. Hence H_X is a hesitant fuzzy subalgebra of X. □

The converse of Proposition 3.3 may not be true in general (see Example 3.4).

Example 3.4. Let $X = \{0, 1, a, b, c\}$ be a BE-algebra ([4]) with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \big\{(1, [0, 1]), (a, (0, \tfrac{1}{8})), (b, (\tfrac{1}{4}, \tfrac{3}{4}])), (c, (0, \tfrac{1}{2}))\big\}$$

Then H_X is a hesitant fuzzy subalgebra of X, but not a hesitant fuzzy filter of X since $h_X(b * a) \cap h_X(b) = h_X(1) \cap h_X(b) = [0, 1] \cap (\frac{1}{4}, \frac{3}{4}] \not\subseteq h_X(a) = (0, \frac{1}{8})$.

Definition 3.5. Given a non-empty subset (subalgebra as much as possible) A of a BE-algebra X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy mighty filter of X related to A* (briefly, *A-hesitant fuzzy mighty filter* of X) if it satisfies (3.1) and

(3.3) $(\forall x, y, z \in A)(h_A(z * (y * x)) \cap h_A(z) \subseteq h_A(((x * y) * y) * x)$.

An A-hesitant fuzzy mighty filter of X with $A = X$ is called a *hesitant fuzzy mighty filter* of X.

Example 3.6. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra ([6]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	d	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0, 1]), (a, [\tfrac{3}{4}, 1]), (b, [\tfrac{1}{2}, 1]), (c, [\tfrac{1}{2}, 1]), (d, \{\tfrac{3}{4}, 1\}), (0, \{\tfrac{1}{2}, 1\})\}$$

It is easy to check that H_X is a hesitant fuzzy fuzzy mighty filter of X.

Proposition 3.7. *Every hesitant fuzzy mighty filter of a BE-algebra X is a hesitant fuzzy filter of X.*

Proof. Let $H_X = \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy mighty filter of X. Putting $y := 1$ in (3.3), we have $h_X(z * (1 * x)) \cap h_X(z) = h_X(z * x) \cap h_X(z) \subseteq h_X(((x * 1) * 1) * x) = h_X(x)$. Hence H_X is a hesitant fuzzy filter of X. □

The converse of Proposition 3.7 may not be true in general (see Example 3.8).

Example 3.8. Let $X = \{1, a, b, c, d\}$ be a BE-algebra ([5]) with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0, 1]), (a, [\tfrac{1}{2}, 1]), (b, [\tfrac{1}{4}, 1]), (c, [\tfrac{1}{5}, 1]), (d, \{\tfrac{3}{4}, 1\})\}.$$

It is easy to check that H_X is a hesitant fuzzy fuzzy filter of X, but not a hesitant fuzzy mighty filter of X since $h_X(1 * (c * a)) \cap h_X(1) = h_X(1) = [0, 1] \nsubseteq h_X(((a * c) * c) * a) = h_X(a) = [\frac{1}{2}, 1]$.

Theorem 3.9. *Any hesitant fuzzy filter $H_X = \{(x, h_X(x))|x \in X\}$ of a BE-algebra X is mighty if and only if it satisfies*

(3.4) $(\forall x, y \in X)(h_X(y * x) \subseteq h_X(((x * y) * y) * x))$.

Proof. Assume that a hesitant fuzzy filter H_X is mighty. Setting $z := 1$ in (3.3), we have $h_X(1 * (y * x)) \cap h_X(1) = h_X(y * x) \subseteq h_X(((x * y) * y) * x)$. Hence (3.4) holds.

Conversely, suppose that the hesitant fuzzy filter $H_X = \{(x, h_X(x))|x \in X\}$ satisfies the condition (3.4). Using (3.2) and (3.4), we have $h_X(z * (y * x)) \cap h_X(z) \subseteq h_X(y * x) \subseteq h_X(((x * y) * y) * x)$, for any $x, y \in X$. Hence H_X is a hesitant fuzzy mighty filter of X. □

Proposition 3.10. *Let $H_X = \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy mighty filter of a BE-algebra X. Then $X_{H_X} := \{x \in X|h_X(x) = h_X(1)\}$ is a mighty filter of X.*

Proof. Clearly, $1 \in X_{H_X}$. Let $z*(y*x), z \in X_{H_X}$. Then $h_X(z*(y*x)) = h_X(1)$ and $h_X(z) = h_X(1)$. It follows from (3.3) that $h_X(z * (y * x)) \cap h_X(z) = h_X(1) \subseteq h_X(((x * y) * y) * x)$. By (3.1), we get $h_X(((x * y) * y) * x) = h_X(1)$. Hence $((x * y) * y) * x \in X_{H_X}$. Therefore X_{H_X} is a mighty filter of X. □

Theorem 3.11. *Let $H_X = \{(x, h_X(x))|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$ be hesitant fuzzy filters of a transitive BE-algebra such that $H_X \widetilde{\subseteq} G_X$ and $h_X(1) = g_X(1)$. If H_X is mighty, then so is G_X.*

Proof. Let $x, y \in X$. Note that $y * ((y * x) * x) = (y * x) * (y * x) = 1$. Since H_X is a hesitant fuzzy mighty filter of a BE-algebra X, by (3.4) and $H_X \widetilde{\subseteq} G_X$ we have $h_X(1) = h_X(y * ((y * x) * x)) \subseteq h_X(((((y*x)*x)*y)*y)*((y*x)*x)) \subseteq g_X(((((y*x)*x)*y)*y)*((y*x)*x))$. Since $h_X(1) = g_X(1)$, we get $g_X((y * x) * ((((y * x) * x) * y) * y) * x)) = g_X(((((y * x) * x) * y) * y) * ((y * x) * x)) = g_X(1)$.

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It follows from (3.1) and (3.2) that

$$\begin{aligned} g_X(y*x) =& g(1) \cap g_X(y*x) \\ =& g_X((y*x)*(((((y*x)*x)*y)*y)*x)) \cap g_X(y*x) \\ \subseteq& g_X(((((y*x)*x)*y)*y)*x). \end{aligned} \tag{3.5}$$

Since X is transitive, we get

$$\begin{aligned} [((((y*x)*x)*y)*y)*x]*&[((x*y)*y)*x] \\ &\geq ((x*y)*y)*((((y*x)*x)*y)*y) \\ &\geq (((y*x)*x)*y)*(x*y) \\ &\geq x*((y*x)*x) \\ &= (y*x)*(x*x) \\ &= (y*x)*1 = 1. \end{aligned}$$

It follows from Proposition 3.2 that $g_X(((((y*x)*x)*y)*y)*x) \cap g_X(1) = g_X(((((y*x)*x)*y)*y)*x) \subseteq g_X(((x*y)*y)*x)$. Using (3.5), we have $g_X(y*x) \subseteq g_X(((((y*x)*x)*y)*y)*x) \subseteq g_X(((x*y)*y)*x)$. Therefore $g_X(y*x) \subseteq g_X(((x*y)*y)*x)$. By Theorem 3.9, G_X is a hesitant fuzzy mighty filter of X. □

Corollary 3.12. *Every hesitant fuzzy filter H_X of a transitive BE-algebra X is mighty if and only if the hesitant fuzzy filter $H_{\{1\}}$ is mighty.*

Proof. Straightforward. □

Let $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy filter of a transitive BE-algebra X. Define a binary relation " $\sim_{h_X}$ " on X by putting $x \sim_{h_X} y$ if and only if $h_X(x*y) = h_X(y*x) = h_X(1)$ for any $x, y \in X$.

Lemma 3.13. *The relation " $\sim_{h_X}$ " is an equivalence relation on a transitive BE-algebra X.*

Proof. For any $x \in X$, $x*x = 1$ by (2.1). So $h_X(x*x) = h_X(1)$, hence $x \sim_{h_X} x$, which $\sim_{h_X}$ is reflexive. Suppose that $x \sim_{h_X} y$ for any $x, y \in X$. Then $h_X(x*y) = h_X(y*x) = h_X(1)$. Hence $\sim_{h_X}$ is symmetric. Assume that $x \sim_{h_X} y$ and $y \sim_{h_X} z$ for any $x, y, z \in X$. Then $h_X(x*y) = h_X(y*x) = h_X(1)$ and $h_X(y*z) = h_X(z*y) = h_X(1)$. By transitivity, $(x*y)*[(y*z)*(x*z)] = 1$ and $(z*y)*[(y*x)*(z*x)] = 1$. By Proposition 3.2, we have $h_X(x*y) \cap h_X(y*z) = h_X(1) \subseteq h_X(x*z)$ and $h_X(z*y) \cap h_X(y*x) = h_X(1) \subseteq h_X(z*x)$. Hence $h_X(z*x) = h_X(z*x) = h_X(1)$, i.e., $x \sim_{h_X} z$. Thus $\sim_{h_X}$ is an equivalence relation on X. □

Lemma 3.14. *The relation " $\sim_{h_X}$ " is a congruence relation on a transitive BE-algebra X.*

Proof. If $x \sim_{h_X} y$ and $u \sim_{h_X} v$ for any $x, y, u, v \in X$, then $h_X(x*y) = h_X(y*x) = h_X(1)$ and $h_X(u*v) = h_X(v*u) = h_X(1)$. By transitivity, $(u*v)*[(x*u)*(x*v)] = 1$ and

$(v*u)*[(x*v)*(x*u)] = 1$, it follows from Proposition 3.2 that $h_X(1) = h_X(u*v) \subseteq h_X((x*u)*((x*v))$ and $h_X(1) = h_X(v*u) \subseteq h_X((x*v)*(x*u))$. Hence $h_X((x*u)*(x*v)) = h_X(1)$ and $h_X((x*v)*(x*u)) = h_X(1)$. Therefore $x*u \sim_{h_X} x*v$. By a similar way, we can prove that $x*v \sim_{h_X} y*v$. Therefore $\sim_{h_X}$ is a congruence relation on X. $\square$

X is decomposed by the congruence relation $\sim_{h_X}$. The class containing x is denoted by $[x]_{h_X}$. Denote $X/h_X := \{[x]_{h_X} | x \in X\}$. We define a binary relation $*'$ on X/h_X by $[x]_{h_X} *' [y]_{h_X} := [x*y]_{h_X}$. This definition is well defined since $\sim_{h_X}$ is a congruence relation on X.

Lemma 3.15. $[1]_{h_X} = X_{H_X}$.

Proof. $[1]_{h_X} = \{x \in X | 1 \sim_{h_X} x\} = \{x \in X | h_X(1*x) = h_X(x*1) = h_X(1)\} = \{x \in X | h_X(x) = h_X(1)\} = X_{H_X}$. $\square$

Theorem 3.16. *Let X be a transitive BE-algebra X. Then $(X/h_X; *', [1]_{h_X})$ is a transitive BE-algebra.*

Proof. Straightforward. $\square$

Theorem 3.17. *A hesitant fuzzy filter of a transitive BE-algebra X is mighty if and only if every filter of the quotient algebra X/h_X is mighty.*

Proof. Assume that a hesitant fuzzy filter H_X is mighty and let $x, y \in X$ be such that $[y]_{h_X} *' [x]_{h_X} \in [1]_{h_X}$. Then $h_X(y*x) = h_X(1)$. It follows from (2.3) and (3.3) that $h_X(1*(y*x)) \cap h_X(1) = h_X(y*x) = h_X(1) \subseteq h_X(((x*y)*y)*x)$. Hence $h_X(((x*y)*y)*x) = h_X(1)$. So $((([x]_{h_X} *' [y]_{h_X}) *' [y]_{h_X})) *' [x]_{h_X} = [((x*y)*y)*x]_{h_X} \in [1]_{h_X}$ which proves that $\{[1]_{h_X}\}$ is a mighty filter of X/h_X. By Corollary 3.13, every filter of X/h_X is mighty.

Conversely, suppose that every filter of the quotient algebra X/h_X is mighty and let $x, y \in X$ be such that $y*x \in [1]_{h_X}$. Then $h_X(y*x) = h_X(1)$ and so $[y]_{h_X} *' [x]_{h_X} \in [1]_{h_X}$. Since $\{[1]_{h_X}\}$ is a mighty filter of X/h_X, it follows from Theorem 2.2 that $[((x*y)*y)*x]_{h_X} = (([x]_{h_X} *' [y]_{h_X}) *' [y]_{h_X}) *' [x]_{h_X} \in [1]_{h_X}$. Hence $h_X((((x*y)*y)*)*x) = h_X(1)$. Therefore $h_X(y*x) = h_X(((x*y)*y))*x)$. Thus H_X is a hesitant fuzzy filter of Theorem 3.9. $\square$

Theorem 3.18. *A hesitant fuzzy set $H_X := \{(x, h_X(x) | x \in X\}$ of a BE-algebra X is a hesitant fuzzy mighty filter of X if and only if the set $H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}$ is a mighty filter of X for all $\gamma \in \mathscr{P}([0,1])$ whenever it is nonempty.*

Proof. Suppose that H_X is a hesitant fuzzy mighty filter of X. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $z*(y*x) \in H_X(\gamma)$ and $z \in H_X(\gamma)$. Then $h_X(z*(y*x)) \supseteq \gamma$ and $h_X(z) \supseteq \gamma$. It follows from (3.1) and (3.3) that $h_X(1) \supseteq h_X(((x*y)*y)*x) \supseteq h_X(z*(y*x)) \cap h_X(z) \supseteq \gamma$. Hence $1 \in H_X(\gamma)$ and $((x*y)*y)*x \in H_X(\gamma)$, and therefore $H_X(\gamma)$ is a mighty filter of X.

Conversely, assume that $H_X(\gamma)$ is a mighty filter of X for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$. For any $x \in X$, let $h_X(x) = \gamma$. Then $x \in H_X(\gamma)$. Since $H_X(\gamma)$ is a mighty filter of X, we have

$1 \in h_X(\gamma)$ and so $h_X(x) = \gamma \subseteq h_X(1)$. For any $x, y, z \in X$, let $h_X(z*(y*x)) = \gamma_{z*(y*x)}$ and $h_X(z) = \gamma_z$. Let $\gamma := \gamma_{z*(y*x)} \cap \gamma_z$. Then $z*(y*x) \in H_X(\gamma)$ and $z \in H_X(\gamma)$ which imply that $((x*y)*y)*x \in H_X(\gamma)$. Hence $h_X(((x*y)*y)*x) \supseteq \gamma = \gamma_{z*(y*x)} \cap \gamma_z = h_X(z*(y*x)) \cap h_X(z)$. Thus H_X is a hesitant fuzzy mighty filter of X. □

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A Class of New General Iteration Approximation of Common Fixed Points for Total Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract. In this paper, we introduce and study a class of new general iteration processes for two finite families of total asymptotically nonexpansive mappings in hyperbolic spaces, which includes asymptotically nonexpansive mapping, (generalized) nonexpansive mapping of all normed linear spaces, Hadamard manifolds and CAT(0) spaces as special cases. Some important related properties to the new general iterative processes are also given and analyzed, and Δ-convergence and strong convergence of the iteration in hyperbolic spaces are proved. Furthermore, some meaningful illustrations for clarifying our results and two open questions are proposed. The results presented in this paper extend and improve the corresponding results announced in the current literature.

Key Words and Phrases: common fixed point, new general iterative approximation, Δ-convergence and strong convergence, total asymptotically nonexpansive mapping, hyperbolic space.

AMS Subject Classification: 47H09, 47H10, 54E70.

1 Introduction and preliminaries

Let $(\mathcal{H}, d)$ be a metric space, $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$ be two finite families of nonlinear mappings on nonempty set $K \subset \mathcal{H}$. Suppose that $\{\alpha_{in}\}$ and $\{\beta_{in}\}$ are two real sequences in $[a, b]$ for some $a, b \in (0, 1)$ and $\theta_{in} := \frac{\beta_{in}}{1-\alpha_{in}}$. For $r \geq 2$ and $n \geq 1$, in this paper, we consider the following general iterative sequence $\{x_n\}$:

$$\begin{aligned}
x_{n+1} &= W(T_1^n y_{n+r-2}, W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), \alpha_{1n}),\\
y_{n+r-2} &= W(T_2^n y_{n+r-3}, W(y_{n+r-3}, S_2^n y_{n+r-3}, \theta_{2n}), \alpha_{2n}),\\
y_{n+r-3} &= W(T_3^n y_{n+r-4}, W(y_{n+r-4}, S_3^n y_{n+r-4}, \theta_{3n}), \alpha_{3n}),\\
&\ \ \vdots\\
y_{n+1} &= W(T_{r-1}^n y_n, W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), \alpha_{(r-1)n}),\\
y_n &= W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}).
\end{aligned} \tag{1.1}$$

Remark 1.1 For appropriate and suitable choices of the nonlinear mappings $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$, the positive integer r and the underlying spaces, the iteration (1.1) includes a number of known iterative processes, which were studied previously by many authors. For more details, see [1–20] and the references therein, and the following examples:

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Example 1.1 If $\beta_{in} = 0$ for $i = 1, 2, 3, \cdots, r$ and all $n \geq 1$, and $\{\alpha_{in}\}$ is a real sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, then the sequence $\{x_n\}$ in (1.1) reduces to

$$\begin{aligned}
x_{n+1} &= \alpha_{1n} y_{n+r-2} + (1-\alpha_{1n}) T_1^n y_{n+r-2},\\
y_{n+r-2} &= \alpha_{2n} y_{n+r-3} + (1-\alpha_{2n}) T_2^n y_{n+r-3},\\
y_{n+r-3} &= \alpha_{3n} y_{n+r-4} + (1-\alpha_{3n}) T_3^n y_{n+r-4},\\
&\vdots\\
y_{n+1} &= \alpha_{(r-1)n} y_n + (1-\alpha_{(r-1)n}) T_{r-1}^n y_n,\\
y_n &= \alpha_{rn} x_n + (1-\alpha_{rn}) T_r^n x_n,
\end{aligned} \tag{1.2}$$

which was considered by Yildirim and Ozdemir [1] when $\{T_i\}_{i=1}^r$ is a family of asymptotically quasi-nonexpansive self-mappings on $K \subset \mathcal{H}$ and $\mathcal{H}$ is a Banach space. Further, the iteration process (1.2) was introduced and studied by Basarir and Sahin [2] for a generalized nonexpansive mapping of the CAT(0) spaces.

Example 1.2 For $r = 3$ and $\alpha_{in} = 0$, then (1.1) changes into the iterative process introduced by Noor [3], which was dealt for variational inequalities of the Hilbert spaces. Moreover, a unified treatment regarding the iterative process for nonexpansive mapping in hyperbolic spaces was considered by Akbulut and Gündüz [4]. For many more, see, for example, the research works of Sahin and Basarir [5], Suantai [6] and many others in the literature.

Example 1.3 Let $r = 2$, and $\alpha_{1n} = 1$, and $\alpha_{2n} = 0$, and $T_2 = S_2$, then (1.1) becomes to the following iteration:

$$x_{n+1} = T_1^n y_n, y_n = W(x_n, T_2^n x_n, \theta_{2n}). \tag{1.3}$$

The iteration (1.3) is called a modified hybrid Picard-Mann iteration process, which was introduced and studied by Thakur et al. [7] in CAT(0) space. This process (1.3) is independent of Picard and Mann iterative process and the convergence process is faster than Picard and Mann iteration process. For more on (hybrid) Picard-Mann iteration process and a comparison between different process of modified hybrid Picard-Mann iteration process, see, for example, [7, 8] and the references therein.

Example 1.4 Let $r = 2$, and $\alpha_{1n} = 0$, and $\beta_{1n} = 1$, $\alpha_{2n} = 1$, then (1.1) is equivalent to

$$x_{n+1} = W(x_n, S^n x_n, \theta_n),$$

which is well-known modified Mann iteration process, and was studied by Schu [9] in Banach spaces.

In 2013, Fukhar-ud-din and Khan [21] pointed out "structural properties of the space under consideration are very important in establishing the fixed point property of the space, for example, strict convexity, uniform convexity and uniform smoothness etc". In fact, in recent decades, motivated and governed by questions in most of science problems about hyperbolic groups, the study on hyperbolic spaces has been developed unremittingly in geometric group theory and metric fixed point theory in normed linear spaces or Banach spaces. Especially, the concept of hyperbolic spaces introduced by Kohlenbach [22] and defined below, is more restrictive and more general than that of being considered in [23] and in [24], respectively (see also [25]). Furthermore, all normed linear spaces, convex subsets wherein Hadamard manifolds and CAT(0) spaces are the special cases of the class of hyperbolic spaces due to Kohlenbach [22].

Definition 1.1 A hyperbolic spaces is a metric space $(\mathcal{H}, d)$ together with a mapping $W : \mathcal{H}^2 \times [0,1] \to \mathcal{H}$ satisfying

(i) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha) d(u, y)$,

(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,

(iii) $W(x, y, \alpha) = W(y, x, (1-\alpha))$,

(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1-\alpha) d(z, w)$ for all $u, x, y, z, w \in \mathcal{H}$ and $\alpha, \beta \in [0,1]$.

Remark 1.1 (1) The class of hyperbolic spaces is general in nature and its important example is the open unit ball B in a complex domain C with respect to the Poincare metric (also called

"Poincare distance")

$$d_B(x,y) := \arg\tanh\left|\frac{x-y}{1-x\overline{y}}\right| = \arg\tanh(1-\sigma(x,y))^{\frac{1}{2}},$$

where $\sigma(x,y) := \frac{(1-|x|^2)(1-|y|^2)}{|1-x\overline{y}|^2}$ for all $x, y \in B$. Further, the above example can be extended from C to general complex Hilbert spaces $(H, \langle\cdot\rangle)$ (see [21, 22]).

(2) A metric space $(\mathcal{H}, d)$ satisfying only (i) in Definition 1.1 is a convex metric space introduced by Takahashi [26]. A nonempty subset K of a hyperbolic space $\mathcal{H}$ is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$. For more on hyperbolic spaces and a comparison between different notions of hyperbolic space, see, for example, [27] and the references therein.

(3) A hyperbolic space is uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, and all $u, x, y \in \mathcal{H}$, there exists $\delta \in (0, 1]$ such that

$$d(W(x,y,\frac{1}{2}),u) \leq (1-\delta)r,$$

provided $\max\{d(x,u), d(y,u)\} \leq r$ and $d(x,y) \geq r\epsilon$ (see [28, 29]). A map $\eta : (0,+\infty)\times(0,2] \to (0,1]$, which provides such $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$, is known as a modulus of uniform convexity of $\mathcal{H}$. We call η monotone if it decreases with r (for fixed ϵ), i.e., for all $\epsilon > 0$, $r_2 \geq r_1 > 0(\eta(r_2,\epsilon) \leq \eta(r_1,\epsilon))$. CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ (see [28, 30]). Thus, the class of uniformly convex hyperbolic spaces includes both uniformly convex normed spaces and CAT(0) spaces as special cases.

In the sequel, let $(\mathcal{H}, d)$ be a metric space, and let K be a nonempty subset of $\mathcal{H}$. We shall denote the fixed point set of a self-mapping on K of T by $F(T) = \{x \in K : Tx = x\}$.

Definition 1.2 A mapping $T : K \to K$ is said to be

(i) semi-compact if every bounded sequence $\{x_n\} \subset K$, satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence;

(ii) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in K$;

(iii) quasi-nonexpansive if $d(Tx, p) \leq d(x, p)$ for all $x \in K$ and $p \in F(T) \neq \emptyset$;

(iv) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0,+\infty)$ and $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, T^n y) \leq (1+k_n)d(x,y), \quad \forall x,y \in K, n \geq 1;$$

(v) asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [0,+\infty)$ and $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, p) \leq (1+k_n)d(x,p), \quad \forall x \in K, p \in F(T), n \geq 1;$$

(vi) $(\{\mu_n\}, \{\xi_n\}, \rho)$-total asymptotically nonexpansive, if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\rho : [0,+\infty) \to [0,+\infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x,y) + \mu_n \rho\big(d(x,y)\big) + \xi_n, \quad \forall x,y \in K, n \geq 1;$$

(vii)$(\{\mu_n\}, \{\xi_n\}, \rho)$-total asymptotically quasi-nonexpansive, if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\rho : [0,+\infty) \to [0,+\infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, p) \leq d(x,p) + \mu_n \rho\big(d(x,p)\big) + \xi_n, \quad \forall x \in K, p \in F, n \geq 1;$$

(viii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x,y), \quad \forall x,y \in K, n \geq 1.$$

Remark 1.2 From Definition 1.2, it follows that a (quasi-)nonexpansive mapping is an asymptotically (quasi-)nonexpansive mapping with $k_n \equiv 0$ for $n \geq 1$, and each asymptotically (quasi-)nonexpansive mapping is a $(\{\mu_n\}, \{\xi_n\}, \rho)$-total asymptotically (quasi-)nonexpansive mapping with $\xi_n = 0$, and $\rho(t) = t \geq 0$. However, in general, the converse of these statement is not true.

As all we know, the study of such types of problems on the iterative approximation of (common) fixed points for generalizations of nonexpansive mappings in hyperbolic spaces, is motivated by an increasing interest in the problems of finding a common fixed point of some nonlinear mappings, which is the only main tool for analysis of generalized nonexpansive mappings and provides us a general and unified framework for studying the existence of fixed points of various nonlinear mappings arising in many branches of nonlinear analysis, topology and applied mathematics, etc.

Inspired and motivated and by the above recent works, in this paper, we shall study some important related properties to the new general iterative process (1.1) for two finite families of total asymptotically nonexpansive mappings as well as two finite families of total asymptotically quasi-nonexpansive mappings in hyperbolic spaces. Results concerning Δ-convergence as well as strong convergence of this iteration are proved. The results presented in the paper extend and improve some recent results given in [1, 2, 4–7, 9, 21].

In order to define the concept of Δ-convergence in the general setup of hyperbolic spaces, in the next moment, we first give some basic concepts.

In 1976, Lim [31] introduced the notion of asymptotic center and, consequently, coined the concept of Δ-convergence in a general setting of a metric space. Kirk and Panyanak [32] proposed an analogous version of convergence in geodesic spaces, namely Δ-convergence, which was originally introduced by Lim [31]. Further, Kirk and Panyanak [32] showed that Δ-convergence coincides with the usual weak convergence in Banach spaces and both concepts share many useful properties.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $\mathcal{H}$. For $x \in \mathcal{H}$, we define a continuous functional $r(\cdot, \{x_n\}) : \mathcal{H} \to [0, +\infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $\hat{r}(\{x_n\})$ of $\{x_n\}$ is given by

$$\hat{r}(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in \mathcal{H}\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to $K \subset \mathcal{H}$ is defined as follows:

$$A_K(\{x_n\}) = \{x \in \mathcal{H} : r(x, \{x_n\}) \le r(y, \{x_n\}), \forall y \in K\},$$

which is the set of minimizers for $r(\cdot, \{x_n\})$. Further, it is simply denoted by $A(\{x_n\})$ when the asymptotic center is taken with respect to $\mathcal{H}$, and a sequence $\{x_n\}$ in $\mathcal{H}$ is said to Δ-converge to $x \in \mathcal{H}$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

It is well known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets". The following lemma ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 ([30]) Let $(\mathcal{H}, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in $\mathcal{H}$ has a unique asymptotic center with respect to any nonempty closed convex subset K of $\mathcal{H}$.

In the sequel, we need the following lemmas.

Lemma 1.2 ([10]) Let $(\mathcal{H}, d, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in \mathcal{H}$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{H}$ such that for some $c \ge 0$,

$$\limsup_{n\to\infty} d(x_n, x) \le c, \quad \limsup_{n\to\infty} d(y_n, x) \le c, \quad \lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c,$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.3 ([10]) Let K be a nonempty closed convex subset of uniformly convex hyperbolic space, and let $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \zeta$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \zeta$, then $\lim_{m\to\infty} y_m = y$.

Lemma 1.4 ([33]) Let $\{a_n\}$, $\{b_n\}$ and $\{\omega_n\}$ be nonnegative real sequences satisfying

$$a_{n+1} \leq (1+\omega_n)a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \omega_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exist. If there exists a subseqence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

2 Some important related properties

Throughout in this paper, we assume that $I = \{1, 2, \cdots, r\}$, $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$ are two finite families of total asymptotically nonexpansive mappings on a nonempty subset K of the hyperbolic space $\mathcal{H}$ defined by Definition 1.2, for each $i \in I$ and all $n \geq 1$, $\{\alpha_{in}\}$, $\{\beta_{in}\}$ and $\{\theta_{in}\}$ are the same as in (1.1). We start with the following important related property of the general iterative process (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space.

Theorem 2.1 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. For $i \in I$, let $T_i : K \to K$ be a $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \mu_n^i = 0$ and $\lim_{n\to\infty} \xi_n^i = 0$, and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ satisfying $\rho^i(0) = 0$, and let $S_i : K \to K$, be a $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \hat{\mu}_n^i = 0$ and $\lim_{n\to\infty} \hat{\xi}_n^i = 0$, and a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ satisfying $\hat{\rho}^i(0) = 0$. Assume that $F = \bigcap_{i=1}^r (F(T_i) \cap F(S_i)) \neq \emptyset$, and for each $i \in I$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\mu}_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\xi}_n^i < +\infty$;

(ii) there exists a constant $M^* > 0$ such that

$$\rho^i(r) \leq M^* r, \quad \hat{\rho}^i(r) \leq M^* r, \quad \forall r > 0.$$

Then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.

Proof. Set $\mu_n = \max_{i\in I}\{\mu_n^i, \hat{\mu}_n^i\}$, and $\xi_n = \max_{i\in I}\{\xi_n^i, \hat{\xi}_n^i\}$, $\rho = \max_{i\in I}\{\rho^i, \hat{\rho}^i\}$. By condition (i), we know that $\sum_{n=1}^{\infty} \mu_n < +\infty$, $\sum_{n=1}^{\infty} \xi_n < +\infty$. For any $p \in F$ and all $n \geq 1$, it follows from (1.1) that

$$\begin{aligned}
d(y_n, p) \quad & \leq \alpha_{rn} d(T_r^n x_n, p) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), p) \\
& \leq \alpha_{rn} d(T_r^n x_n, p) + \beta_{rn} d(x_n, p) + (1-\alpha_{rn}-\beta_{rn}) d(S_r^n x_n, p) \\
& \leq \alpha_{rn}[d(x_n, p) + \mu_n^r \rho^r\big(d(x_n, p)\big) + \xi_n^r] + \beta_{rn} d(x_n, p) \\
& \quad + (1-\alpha_{rn}-\beta_{rn})[d(x_n, p) + \hat{\mu}_n^r \hat{\rho}^r\big(d(x_n, p)\big) + \hat{\xi}_n^r] \\
& \leq \alpha_{rn}[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] + \beta_{rn} d(x_n, p) \\
& \quad + (1-\alpha_{rn}-\beta_{rn})[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] \\
& \leq \alpha_{rn}[(1+\mu_n M^*) d(x_n, p) + \xi_n] + \beta_{rn} d(x_n, p) \\
& \quad + (1-\alpha_{rn}-\beta_{rn})[(1+\mu_n M^*) d(x_n, p) + \xi_n] \\
& \leq (1+\mu_n M^*) d(x_n, p) + \xi_n
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
d(y_{n+1}, p) \quad & \leq \alpha_{(r-1)n} d(T_{r-1}^n y_n, p) + (1-\alpha_{(r-1)n}) d(W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), p) \\
& \leq \alpha_{(r-1)n} d(T_{r-1}^n y_n, p) + \beta_{(r-1)n} d(y_n, p) \\
& \quad + (1-\alpha_{(r-1)n}-\beta_{(r-1)n}) d(S_{r-1}^n y_n, p) \\
& \leq \alpha_{(r-1)n}[d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] + \beta_{(r-1)n} d(y_n, p) \\
& \quad + (1-\alpha_{(r-1)n}-\beta_{(r-1)n})[d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] \\
& \leq \alpha_{(r-1)n}[(1+\mu_n M^*) d(y_n, p) + \xi_n] + \beta_{(r-1)n} d(y_n, p) \\
& \quad + (1-\alpha_{(r-1)n}-\beta_{(r-1)n})[(1+\mu_n M^*) d(y_n, p) + \xi_n] \\
& \leq (1+\mu_n M^*) d(y_n, p) + \xi_n.
\end{aligned} \tag{2.2}$$

Similarly, we have

$$d(y_{n+r-2},p) \le (1+\mu_n M^*)d(y_{n+r-3},p)+\xi_n,$$
$$d(x_{n+1},p) \le (1+\mu_n M^*)d(y_{n+r-2},p)+\xi_n.$$

Thus,

$$\begin{aligned} d(x_{n+1},p) \quad &\le (1+\mu_n M^*)^r d(x_n,p)+\sum_{j=1}^{r-1}(1+\mu_n M^*)^j\xi_n \\ &\le d(x_n,p)\Big[1+\tbinom{r}{1}\mu_n M^*+\tbinom{r}{2}(\mu_n M^*)^2+\tbinom{r}{3}(\mu_n M^*)^3 \\ &\quad +\cdots+\tbinom{r}{r}(\mu_n M^*)^r\Big]+\sum_{j=1}^{r-1}(1+\mu_n M^*)^j\xi_n \\ &\le (1+a_n^r\mu_n)d(x_n,p)+\sum_{j=1}^{r-1}(1+\mu_n M^*)^j\xi_n \\ &\le (1+M_1\mu_n)d(x_n,p)+M_2\xi_n, \end{aligned}$$

where $a_n^r = \binom{r}{1}M^* + \binom{r}{2}(M^*)^2\mu_n + \binom{r}{3}(M^*)^3(\mu_n)^2 + \cdots + \binom{r}{r}(M^*)^r(\mu_n)^{r-1}$, and by virtue of condition(i), there exist positive constants M_1 and M_2 such that $a_n^r \le M_1$, $\sum_{j=1}^{r-1}(1+\mu_n M^*)^j \le M_2$ for each $n \ge 1$. Applying Lemma 1.4 to the above inequality, we have $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F$. □

In 1993, Bruck et al. [34] introduced a notion of asymptotically nonexpansive mapping in the intermediate sense. More accurately, a mapping $T : K \to K$ is said to be asymptotically nonexpansive mapping in the intermediate sense, provided that T is uniformly continuous and $\limsup_{n\to\infty}\sup_{x,y\in K}\{d(T^n x, T^n y)-d(x,y)\} \le 0$. Put $\xi_n = \max\{0, \sup_{x,y\in K}\{d(T^n x,T^n y)-d(x,y)\}\}$ and $\sum_{n=1}^{\infty}\xi_n^i < +\infty$, then $d(T^n x, T^n y) \le d(x,y)+\xi_n$ for any $n \ge 1$ and $x,y \in K$. In more detail, see, for example, [20] and the references therein.

The following result can be obtained from Theorem 2.1 immediately.

Corollary 2.1 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. For $i \in I$, let $T_i : K \to K$ be a $\{\xi_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense and let $S_i : K \to K$ be a $\{\hat{\xi}_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense. If $\sum_{n=1}^{\infty}\xi_n^i < +\infty$, $\sum_{n=1}^{\infty}\hat{\xi}_n^i < +\infty$ for $i \in I$ and $F = \bigcap_{i=1}^r(F(T_i)\cap F(S_i)) \ne \emptyset$, then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F$.

Proof. Let $\xi_n = \max_{i\in I}\{\xi_n^i, \hat{\xi}_n^i\}$, then $\sum_{n=1}^{\infty}\xi_n < +\infty$. The rest of the proof is trivial. □

Corollary 2.2 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. Let $T_i : K \to K$ be a $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}k_n^i < +\infty$ and $S_i : K \to K$ be a $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}\hat{k}_n^i < +\infty$ for $i \in I$. Assume that $F = \bigcap_{i=1}^r(F(T_i)\cap F(S_i)) \ne \emptyset$. Then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F$.

Proof. Taking $k_n = \max_{i\in I}\{k_n^i, \hat{k}_n^i\}$, then $\sum_{n=1}^{\infty}k_n < +\infty$. Let $\rho^i(t) = \hat{\rho}^i(t) = t$, $\xi_n^i = \hat{\xi}_n^i = 0$, $\mu_n^i = k_n^i$ in Theorem 2.1 for $i \in I$. Then all the conditions in Theorem 2.1 are satisfied and so the result holds. □

Theorem 2.2 Let K be a nonempty closed and convex subset of a uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\},\{\xi_n^i\},\rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty}\mu_n^i = 0$ and $\lim_{n\to\infty}\xi_n^i = 0$, and a strictly increasing continuous function $\rho^i : [0,+\infty) \to [0,+\infty)$ satisfying $\rho^i(0) = 0$, and let $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $(\{\hat{\mu}_n^i\},\{\hat{\xi}_n^i\},\hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty}\hat{\mu}_n^i = 0$ and $\lim_{n\to\infty}\hat{\xi}_n^i = 0$, and a strictly increasing continuous function $\hat{\rho}^i : [0,+\infty) \to [0,+\infty)$ satisfying $\hat{\rho}^i(0) = 0$. Suppose that $F = \bigcap_{i=1}^r(F(T_i)\cap F(S_i)) \ne \emptyset$ and the conditions (i) and (ii) in Theorem 2.1 hold. Then, for $i \in I$ and the sequence $\{x_n\}$ generated by (1.1), we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0.$$

Proof. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} d(x_n, p) = c > 0$. Otherwise the proof is trivial.

Take lim sup on both sides of inequalities (2.1) and (2.2). Since $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, we have $\limsup_{n\to\infty} d(y_n, p) \le c$ and $\limsup_{n\to\infty} d(y_{n+1}, p) \le c$. Similarly, we get $\limsup_{n\to\infty} d(y_{n+r-2}, p) \le c$, and so in total

$$\limsup_{n\to\infty} d(y_{n+k-1}, p) \le c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.3}$$

Carry lim inf on both side of (2.4). Since

$$d(x_{n+1}, p) \le (1 + \mu_n M^*)^{r-1} d(y_n, p) + \sum_{j=1}^{r-2} (1 + \mu_n M^*)^j \xi_n \tag{2.4}$$

we have

$$\liminf_{n\to\infty} d(y_n, p) \ge c,$$
$$d(x_{n+1}, p) \le (1 + \mu_n M^*)^{r-k} d(y_{n+k-1}, p) + \sum_{j=1}^{r-k-1} (1 + \mu_n M^*)^j \xi_n, \quad \forall k = 2, 3, \cdots, r-1.$$

Also taking lim inf on both side of the above estimate, then we get

$$\liminf_{n\to\infty} d(y_{n+k-1}, p) \ge c, \quad \forall k = 2, 3, \cdots, r-1.$$

Thus, in total,

$$\liminf_{n\to\infty} d(y_{n+k-1}, p) \ge c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.5}$$

Combining (2.3) and (2.5), we have

$$\lim_{n\to\infty} d(y_{n+k-1}, p) = c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.6}$$

For $k = 1$ in (2.6), we get

$$\lim_{n\to\infty} d(W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}), p) = c. \tag{2.7}$$

Moreover,

$$\begin{aligned} d(W(x_n, S_r^n x_n, \theta_{rn}), p) \quad & \le \theta_{rn} d(x_n, p) + (1 - \theta_{rn}) d(S_r^n x_n, p) \\ & \le \theta_{rn} d(x_n, p) + (1 - \theta_{rn})[(1 + \mu_n M^*) d(x_n, p) + \xi_n] \\ & \le (1 + \mu_n M^*) d(x_n, p) + \xi_n \end{aligned}$$

implies that

$$\limsup_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) \le c. \tag{2.8}$$

Obviously,

$$\limsup_{n\to\infty} d(T_r^n x_n, p) \le c. \tag{2.9}$$

It follows from (2.7)-(2.9) and Lemma 1.2 that

$$\lim_{n\to\infty} d(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn})) = 0. \tag{2.10}$$

Again, for $k = 2, 3, \cdots, r-1$, (2.6) can be expressed as

$$\lim_{n\to\infty} d(W(T^n_{r-(k-1)}y_{n+k-2}, W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n}), p) = c. \tag{2.11}$$

By (2.3) and the inequality

$$\begin{aligned}
&d(W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), p)\\
&\le \theta_{(r-k+1)n}d(y_{n+k-2}, p) + (1-\theta_{(r-k+1)n})d(S^n_{r-(k-1)}y_{n+k-2}, p)\\
&\le \theta_{(r-k+1)n}d(y_{n+k-2}, p) + (1-\theta_{(r-k+1)n})[(1+\mu_n M^*)d(y_{n+k-2}, p) + \xi_n]\\
&\le (1+\mu_n M^*)d(y_{n+k-2}, p) + \xi_n,
\end{aligned}$$

now we know that

$$\limsup_{n\to\infty} d(W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), p) \le c. \tag{2.12}$$

Further,

$$\limsup_{n\to\infty} d(T^n_{r-(k-1)}y_{n+k-2}, p) \le c, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.13}$$

From (2.11)-(2.13) and Lemma 1.2, it follows that

$$\lim_{n\to\infty} d(T^n_{r-(k-1)}y_{n+k-2}, W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n})) = 0 \tag{2.14}$$

for $k = 2, 3, \cdots, r-1$ and for $k = r$, we have

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), \alpha_{1n}), p) = c. \tag{2.15}$$

Applying (2.3), the following estimate

$$\begin{aligned}
&d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), p)\\
&\le \theta_{1n}d(y_{n+r-2}, p) + (1-\theta_{1n})d(S^n_1 y_{n+r-2}, p)\\
&\le \theta_{1n}d(y_{n+r-2}, p) + (1-\theta_{1n})[(1+\mu_n M^*)d(y_{n+r-2}, p) + \xi_n]\\
&\le (1+\mu_n M^*)d(y_{n+r-2}, p) + \xi_n
\end{aligned}$$

implies that

$$\limsup_{n\to\infty} d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), p) \le c. \tag{2.16}$$

Also,

$$\limsup_{n\to\infty} d(T^n_1 y_{n+r-2}, p) \le c. \tag{2.17}$$

Hence, (2.15)-(2.17) and Lemma 1.2 imply that

$$\lim_{n\to\infty} d(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n})) = 0. \tag{2.18}$$

Observe that

$$\begin{aligned}
d(x_{n+1}, T^n_1 y_{n+r-2}) &= d(W(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), \alpha_{1n}), T^n_1 y_{n+r-2})\\
&\le (1-\alpha_{1n})d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), T^n_1 y_{n+r-2})\\
&\quad +\alpha_{1n}d(T^n_1 y_{n+r-2}, T^n_1 y_{n+r-2}).
\end{aligned}$$

Based on (2.18), this implies

$$\lim_{n\to\infty} d(x_{n+1}, T^n_1 y_{n+r-2}) = 0. \tag{2.19}$$

Similarly, since $a \le \alpha_{in}, \beta_{in} \le b$ for all $i \in I$, we have

$$\begin{aligned}
d(x_{n+1}, p) &\le \alpha_{1n} d(T_1^n y_{n+r-2}, p) + (1-\alpha_{1n}) d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\
&\le \alpha_{1n} d(x_{n+1}, p) + \alpha_{1n} d(x_{n+1}, T_1^n y_{n+r-2}) \\
&\quad +(1-\alpha_{1n}) d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\
&\le \frac{\alpha_{1n}}{1-\alpha_{1n}} d(x_{n+1}, T_1^n y_{n+r-2}) + d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\
&\le \frac{b}{1-b} d(x_{n+1}, T_1^n y_{n+r-2}) + d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p). \qquad (2.20)
\end{aligned}$$

Taking lim inf on both side of the estimate (2.20) and using (2.19), we have

$$\liminf_{n\to\infty} d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \ge c. \qquad (2.21)$$

Combining (2.16) and (2.21), we get

$$\lim_{n\to\infty} d(W(y_{n+r-2}, , S_1^n y_{n+r-2}, \theta_{1n}), p) = c. \qquad (2.22)$$

By Lemma 1.2 and (2.22), we have

$$\lim_{n\to\infty} d(y_{n+r-2}, S_1^n y_{n+r-2}) = 0.$$

In a similar way, for $k = 2, 3, \cdots, r-1$, we compute

$$\begin{aligned}
& d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) \\
& = d(W(T_{r-(k-1)}^n y_{n+k-2}, W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n}), \\
& T_{r-(k-1)}^n y_{n+k-2}) \\
& \le (1-\alpha_{(r-k+1)n}) d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), T_{r-(k-1)}^n y_{n+k-2}) \\
& \quad +\alpha_{(r-k+1)n} d(T_{r-(k-1)}^n y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}).
\end{aligned}$$

Utilizing (2.14), we have

$$\lim_{n\to\infty} d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \qquad (2.23)$$

For $k = 1$, we calculate

$$\begin{aligned}
d(y_n, T_r^n x_n) &= d(W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}), T_r^n x_n) \\
&\le \alpha_{rn} d(T_r^n x_n, T_r^n x_n) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), T_r^n x_n).
\end{aligned}$$

Now, using (2.10), we have

$$\lim_{n\to\infty} d(y_n, T_r^n x_n) = 0. \qquad (2.24)$$

Reasoning as above, we get that

$$d(y_n, p) \le \frac{b}{1-b} d(T_r^n x_n, y_n) + d(W(x_n, S_r^n x_n, \theta_{rn}), p). \qquad (2.25)$$

Setting lim inf on both sides of the estimate (2.25) and utilizing (2.6) and (2.24), we know

$$\liminf_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) \ge c. \qquad (2.26)$$

Inequalities (2.8) and (2.26) collectively imply that

$$\lim_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) = c. \qquad (2.27)$$

Consequently, Lemma 1.2 and (2.27) imply that

$$\lim_{n\to\infty} d(x_n, S_r^n x_n) = 0. \tag{2.28}$$

Note that

$$\begin{aligned} d(x_n, T_r^n x_n) &\le d(x_n, y_n) + d(y_n, T_r^n x_n) \\ &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), x_n) + d(y_n, T_r^n x_n) \\ &\le (1-\theta_{rn}) d(x_n, S_r^n x_n) + \frac{1}{1-\alpha_{rn}} d(y_n, T_r^n x_n) \\ &\le \frac{1-2a}{1-b} d(x_n, S_r^n x_n) + \frac{1}{1-b} d(y_n, T_r^n x_n). \end{aligned}$$

From (2.24) and (2.28), we have

$$\lim_{n\to\infty} d(x_n, T_r^n x_n) = 0. \tag{2.29}$$

Moreover

$$\begin{aligned} d(x_n, y_n) &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}) d(x_n, W(x_n, S_r^n x_n, \theta_{rn})) \\ &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}-\beta_{rn}) d(x_n, S_r^n x_n) \\ &\le b d(x_n, T_r^n x_n) + (1-2a) d(x_n, S_r^n x_n). \end{aligned}$$

By (2.28) and (2.29), we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.30}$$

Again, reasoning as above, we have

$$\begin{aligned} d(y_{n+k-1}, p) &\le d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), p) \\ &\quad + \frac{b}{1-b} d(T_{r-(k-1)}^n y_{n+k-2}, y_{n+k-1}). \end{aligned}$$

Now, Utilizing (2.6) and (2.23), we get

$$\liminf_{n\to\infty} d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), p) \ge c. \tag{2.31}$$

Thus, (2.12) and (2.31) imply in total

$$\lim_{n\to\infty} d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), p) = c,$$

and by Lemma 1.2, we conclude that

$$\lim_{n\to\infty} d(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.32}$$

Also,

$$\begin{aligned} & d(y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}) \\ & \le d(y_{n+k-2}, y_{n+k-1}) + d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}), \\ & S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n})) + d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) \\ & \le d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) + \alpha_{(r-k+1)n} d(y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}) \\ & \quad + (1-\alpha_{(r-k+1)n}) d(y_{n+k-2}, W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n})) \\ & \le d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) + \alpha_{(r-k+1)n} d(y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}) \\ & \quad + (1-\alpha_{(r-k+1)n} - \beta_{(r-k+1)n}) d(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}) \\ & \le \frac{1}{1-b} d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) + \frac{1-2a}{1-b} d(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}). \end{aligned}$$

Now, it follows from (2.23) and (2.32) that

$$\lim_{n\to\infty} d(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.33}$$

For $k = 2, 3, \cdots, r-1$, we have

$$d(y_{n+k-2}, y_{n+k-1}) \le d(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}) + d(T^n_{r-(k-1)}y_{n+k-2}, y_{n+k-1}).$$

Hence, (2.23) and (2.33) imply that

$$\lim_{n\to\infty} d(y_{n+k-2}, y_{n+k-1}) = 0. \tag{2.34}$$

Additionally,

$$d(x_n, y_{n+k-1}) \le d(x_n, y_n) + d(y_n, y_{n+1}) + \cdots + d(y_{n+k-2}, y_{n+k-1}).$$

By (2.30) and (2.34), we have

$$\lim_{n\to\infty} d(x_n, y_{n+k-1}) = 0, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.35}$$

Let $L = \max_{i\in I}\{L_i, \hat{L}_i\}$, where L_i and $\hat{L}_i$ are Lipschitz constants for T_i and S_i for $i \in I$, respectively. Since each T_i is uniformly L-Lipschitzian for $i \in I$, we have

$$\begin{aligned} d(x_n, T^n_i x_n) &\le d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T^n_i x_n) \\ &\le d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T^n_i y_{n+r-i-1}) + d(T^n_i y_{n+r-i-1}, T^n_i x_n) \\ &\le (1+L)d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T^n_i y_{n+r-i-1}) \end{aligned}$$

for $1 \le i \le r-1$.

It follows from (2.33) and (2.35) that

$$\lim_{n\to\infty} d(x_n, T^n_i x_n) = 0, \quad \forall 1 \le i \le r-1. \tag{2.36}$$

Moreover,

$$\begin{aligned} d(x_n, T_i x_n) &\le d(x_n, T^n_i x_n) + d(T^n_i x_n, T^n_i y_{n+r-i-1}) + d(T^n_i y_{n+r-i-1}, T_i x_n) \\ &\le d(x_n, T^n_i x_n) + Ld(x_n, y_{n+r-i-1}) + Ld(T^{n-1}_i y_{n+r-i-1}, x_n) \\ &\le d(x_n, T^n_i x_n) + 2Ld(x_n, y_{n+r-i-1}) + Ld(T^{n-1}_i y_{n+r-i-1}, y_{n+r-i-1}). \end{aligned}$$

Thus, (2.33), (2.35) and (2.36) (or (2.29)) imply that $d(x_n, T_i x_n) \to 0$ as $n \to \infty$ and so

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \forall 1 \le i \le r.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad \forall 1 \le i \le r.$$

This completes the proof. □

The following results can be obtained from Theorem 2.2 immediately. The proof is similar to Corollaries 2.1 and 2.2, respectively, and so they are omitted.

Corollary 2.3 Assume that K and F are the same as in Theorem 2.2. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{\xi^i_n\}$-asymptotically nonexpansive mapping in the intermediate sense and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{\xi}^i_n\}$-asymptotically nonexpansive mapping in the intermediate sense. If $\sum_{n=1}^{\infty} \xi^i_n < +\infty$ and $\sum_{n=1}^{\infty} \hat{\xi}^i_n < +\infty$ for $i \in I$, then, for the sequence $\{x_n\}$ in (1.1),

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad \forall i \in I.$$

Corollary 2.4 Suppose that K and F are the same as in Theorem 2.2. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} k_n^i < +\infty$, and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \hat{k}_n^i < +\infty$. Then,

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i \in I,$$

where $\{x_n\}$ is the sequence defined by (1.1).

Remark 2.1 (1) It is worth mentioning that Theorems 2.1-2.2 can easily be extended to a more general class of total asymptotically quasi-nonexpansive mappings for the iteration process (1.1). And the proofs of Theorems 2.1-2.2 are greatly differ from those of Lemmas 2.1 and 2.2 in [21]. Further, Corollaries 2.1 and 2.3 (Corollaries 2.2 and 2.4, respectively) are so.

(2) Moreover, conclusion of the Theorem 2.2 (Corollaries 2.3 and 2.4, respectively) can be extended to a more general class of weakly total-asymptotically quasi-nonexpansive mappings (weakly asymptotically quasi-nonexpansive mappings asymptoticallyin in the intermediate sense and weakly quasi-nonexpansive mappings). For concepts of the weakly properly, see, for example, Fukhar-ud-din and Khan [21].

3 Approximation of common fixed points

In this section, we approximate common fixed points of two finite families of total asymptotically nonexpansive mappings in a hyperbolic space. More briefly, we establish Δ-convergence and strong convergence of the iteration process (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space.

Theorem 3.1 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$, $i \in I = \{1, 2, 3, \cdots, r\}$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \mu_n^i = 0$ and $\lim_{n\to\infty} \xi_n^i = 0$, and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ satisfying $\rho^i(0) = 0$, and let $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \hat{\mu}_n^i = 0$ and $\lim_{n\to\infty} \hat{\xi}_n^i = 0$, and with a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ satisfying $\hat{\rho}^i(0) = 0$. Assume that $F = \bigcap_{i=1}^{r}(F(T_i) \cap F(S_i)) \neq \emptyset$ and for $i \in I$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\mu}_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\xi}_n^i < +\infty$.

(ii)There exists a constant $M^* > 0$ such that $\rho^i(r) \leq M^* r$ and $\hat{\rho}^i(r) \leq M^* r$ for all $r > 0$.

Then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Proof. Since the sequence $\{x_n\}$ is bounded (by Theorem 2.1), therefore Lemma 1.1 asserts that $\{x_n\}$ has a unique asymptotic center in K. That is, $A(\{x_n\}) = \{x\}$. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. Then, by Theorem 2.2, we have

$$\lim_{n\to\infty} d(v_n, T_i v_n) = \lim_{n\to\infty} d(v_n, S_i v_n) = 0, \quad \forall i \in I. \tag{3.1}$$

We claim that v is the common fixed point of $\{T_i\}_{i=1}^{r}$ and $\{S_i\}_{i=1}^{r}$.

For each $i \in I$, define a sequence $\{z_m\}$ in K by $z_m = T_i^m v$. Then, we calculate

$$\begin{aligned} d(z_m, v_n) &\leq d(T_i^m v, T_i^m v_n) + d(T_i^m v_n, T_i^{m-1} v_n) + \cdots + d(T_i v_n, v_n) \\ &\leq [d(v, v_n) + \mu_m^i \rho^i(d(v, v_n)) + \xi_m^i] + \sum_{j=0}^{m-1} d(T_i^{j+1} v_n, T_i^j v_n). \end{aligned}$$

Since each T_i is uniformly L_i-Lipschitzian with the Lipschitz constant L_i for $i \in I$, the above estimate yields

$$d(z_m, v_n) \leq [(1 + \mu_m M^*) d(v, v_n) + \xi_m] + mLd(T_i v_n, v_n), \tag{3.2}$$

where $L = \max_{i\in I}\{L_i, \hat{L}_i\}$.

Taking lim sup on both sides of (3.2) and using (3.1), we have

$$r(z_m, \{v_n\}) = \limsup_{n\to\infty} d(z_m, v_n) \le \limsup_{n\to\infty} d(v, v_n) = r(v, \{v_n\}),$$

which implies that $|r(z_m, \{v_n\}) - r(v, \{v_n\})| \to 0$ as $m \to \infty$. It follows Lemma 1.3 that $\lim_{m\to\infty} T_i^m v = v$. by the uniform continuity of T_i, we know that

$$T_i(v) = T(\lim_{m\to\infty} T_i^m v) = \lim m \to \infty T_i^{m+1} v = v.$$

From the arbitrariness of $i \in I$, we conclude that v is the common fixed point of $\{T_i\}_{i=1}^r$. Similarly, we can show that v is the common fixed point of $\{S_i\}_{i=1}^r$. Hence, $v \in F$.

Next, we claim that the common fixed point v is the unique asymptotic center for each subsequence $\{v_n\}$ of $\{x_n\}$.

Contrarily, $v \ne x$. It follows Theorem 2.1 that $\lim_{n\to\infty} d(x_n, v)$ exists, and by the uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) \quad &< \limsup_{n\to\infty} d(v_n, x) \le \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

a contradiction. Therefore $v = x$. Since $\{v_n\}$ is an arbitrary subsequence of $\{x_n\}$, $A(\{v_n\}) = \{x\}$ for all subsequence $\{v_n\}$ of $\{x_n\}$, this proves that $\{x_n\}$ Δ-converges to a common fixed point x of $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$. □

From Theorem 3.1, we have the following result.

Corollary 3.1 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{\xi_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{\xi}_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense. If for all $i \in I$, $\sum_{n=1}^\infty \xi_n^i < +\infty$ and $\sum_{n=1}^\infty \hat{\xi}_n^i < +\infty$, and $F = \bigcap_{i=1}^r (F(T_i) \cap F(S_i)) \ne \emptyset$, then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Corollary 3.2 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^\infty k_n^i < +\infty$, and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^\infty \hat{k}_n^i < +\infty$. Assume that $F = \bigcap_{i=1}^r (F(T_i) \cap F(S_i)) \ne \emptyset$. Then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Proof. Based on Corollaries 2.2 and 2.4, and the proof of Theorem 3.1 in [21], the result holds. □

In order to prove strong convergence of the iteration (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space, we first give the following conditions:

(**H**) There exists a nondecreasing self-mapping on $[0, +\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, +\infty)$ such that $d(x, Tx) \ge f(d(x, F(T)))$ for all $x \in K$, where $T : K \to K$ is a nonlinear mapping with $F(T) \ne \emptyset$ and $d(x, F(T)) = \inf\{d(x, y) : y \in F(T)\}$.

The condition (**H**) was introduced by Senter and Dotson [35]. Further, based on works of [21, 36, 37], for two finite families of total asymptotically nonexpansive mappings $\{T_i, i \in I\}_{i=1}^r$ and $\{S_i, i \in I\}_{i=1}^r$ on $K \subset \mathcal{H}$ with $F = \bigcap_{i=1}^n (F(T_i) \cap F(S_i)) \ne \emptyset$, condition (**H**) becomes as follows:

(**A**) $d(x, Tx) \ge f(d(x, F))$ or $d(x, Sx) \ge f(d(x, F))$ holds for $x \in K$ and for at least one $T \in \{T_i\}_{i=1}^r$ or $S \in \{S_i\}_{i=1}^r$, where $d(x, F) = \inf\{d(x, y) : y \in F\}$.

(**B**) $d(x, T_i x) + d(x, S_i x) \ge f(d(x, F))$ for $x \in K$ and $i \in I$.

($\mathbf{C_1}$) $\frac{1}{2r}\left(\sum_{i=1}^r d(x, T_i x) + \sum_{i=1}^r d(x, S_i x)\right) \ge f(d(x, F))$ for $x \in K$.

$(\mathbf{C_2})$ $\frac{1}{2}\big(\max_{1\le i\le r} d(x,T_ix)+\max_{1\le i\le r} d(x,S_ix)\big)\ge f(d(x,F))$ for $x\in K$.

$(\mathbf{C_3})$ $\max\big\{\max_{1\le i\le r} d(x,T_ix),\max_{1\le i\le r} d(x,S_ix)\big\}\ge f(d(x,F))$ for $x\in K$.

Note that the conditions $(\mathbf{A})$, $(\mathbf{B})$ and $(\mathbf{C_1})$-$(\mathbf{C_3})$ are equivalent to the condition $(\mathbf{H})$, if $T_i=S_i$ for $i\in I$. We shall use condition $(\mathbf{C_1})$ or $(\mathbf{C_2})$ or $(\mathbf{C_3})$ to study strong convergence of the iteration (1.1).

Now we give the following lemma for proving the strong convergence.

Lemma 3.1 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and $\{x_n\}$ be as in Theorem 3.1. Then $\{x_n\}$ converges strongly to some $p\in F$ if and only if

$$\liminf_{n\to\infty} d(x_n,F)=0.$$

Proof. If $\{x_n\}$ converges strongly to $p\in F$, then $\lim_{n\to\infty} d(x_n,p)=0$. Since $0\le d(x_n,F)\le d(x_n,p)$, we have $\liminf_{n\to\infty} d(x_n,F)=0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F)=0$. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n,F)$ exists. Now $\liminf_{n\to\infty} d(x_n,F)=0$ reveals that $\lim_{n\to\infty} d(x_n,F)=0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. By last inequalities in the proof of Theorem 2.1

$$d(x_{n+1},p)\le(1+M_1\mu_n)d(x_n,p)+M_2\xi_n,$$

taking infimum on $p\in F$ on both sides in the above inequality, we have

$$d(x_{n+1},F)\le(1+M_1\mu_n)d(x_n,F)+M_2\xi_n.$$

On account of $\sum_{n=1}^\infty\mu_n<\infty$, $\sum_{n=1}^\infty\xi_n<\infty$, set $e^{M_1\sum_{n=1}^\infty\mu_n}=M$. Let $\forall\varepsilon>0$. Since $\lim_{n\to\infty} d(x_n,F)=0$, for any given $\varepsilon>0$, there exists a positive integer n_0 such that

$$d(x_{n_0},F)<\frac{\varepsilon}{4(M+1)}\quad\text{and}\quad\sum_{n=n_0}^\infty\xi_n<\frac{\varepsilon}{2MM_2}\tag{3.3}$$

The first inequality in (3.3) implies that there exists $p_0\in F$ such that $d(x_{n_0},p_0)<\frac{\varepsilon}{2(M+1)}$. Hence, for any $n\ge n_0$ and $m\ge1$, we have

$$\begin{aligned}
d(x_{n_0+m},x_{n_0}) &\le d(x_{n_0+m},p_0)+d(x_{n_0},p_0)\\
&\le [e^{M_1\sum_{k=n_0}^{n_0+m-1}\mu_k}+1]d(x_{n_0},p_0)+M_2[\xi_{n_0+m-1}\\
&\quad+\xi_{n_0+m-2}e^{M_1\mu_{n_0+m-1}}+\xi_{n_0+m-3}e^{M_1\sum_{k=n_0+m-2}^{n_0+m-1}\mu_k}\\
&\quad+\cdots+\xi_{n_0}e^{M_1\sum_{k=n_0+1}^{n_0+m-1}\mu_k}]\\
&\le (M+1)d(x_{n_0},p_0)+MM_2\sum_{n=n_0}^\infty\xi_n\\
&< (M+1)\frac{\varepsilon}{2(M+1)}+MM_2\frac{\varepsilon}{2MM_2}=\varepsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}$. Sine K is a closed subset of a complete hyperbolic space $\mathcal{H}$, it is complete. We can assume that $\lim_{n\to\infty} x_n=q$, and $q\in K$. It is easy to see that $F(T)$ is a close subset in K, so is $F(T)$. Since $\lim_{n\to\infty} d(x_n,F)=0$, we obtain $q\in F(T)$. This completes the proof. □

We now establish strong convergence of the iteration process (1.1) based on Theorem 2.2.

Theorem 3.2 Suppose that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Theorem 3.1, and $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p\in F$.

Proof. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n,F)$ exists. Moreover, Theorem 2.2 implies that $\lim_{n\to\infty} d(x_n,T_ix_n)=\lim_{n\to\infty} d(x_n,S_ix_n)=0$ for each $i\in I$. Thus, the condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$) guarantees that $\lim_{n\to\infty} f(d(x_n,F))=0$. Since f is nondecreasing with $f(0)=0$,

it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Then, Lemma 3.1 implies that $\{x_n\}$ converges strongly to a common fixed point $p \in F$. □

From Theorem 3.2, we have the following results.

Corollary 3.3 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Corollary 3.1. Suppose that $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p \in F$.

Corollary 3.4 Assume that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F are the same as in Corollary 3.2, and $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p \in F$.

Theorem 3.3 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Theorem 3.1. Suppose that either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Proof. Let $T_l \in \{T_i\}_{i=1}^r$ is semi-compact. By Theorem 2.2, we know that $\lim_{n\to\infty} d(T_i x_n, x_n) = 0$ for all $i \in I$. By Theorem 2.1, $\{x_n\}$ is bounded and T_l is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. By continuity of T_i and Theorem 2.2, we obtain

$$d(q, T_i q) = \lim_{j\to\infty} d(x_{n_j}, T_i x_{n_j}) = 0, \quad i \in I.$$

This implies that q is the common fixed point of $\{T_i\}_{i=1}^r$. Similarly, we can show that q is the common fixed point of $\{S_i\}_{i=1}^r$. Hence, $q \in F$. Again, by Theorem 2.1, $\lim_{n\to\infty} d(x_n, q)$ exists. Therefore, q is the strong limit of the sequence $\{x_n\}$. As a result, $\{x_n\}$ converges strongly to a point q. □

From Theorem 3.3, we have the following results.

Corollary 3.5 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Corollary 3.1. Suppose that either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Corollary 3.6 Suppose that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and $\{x_n\}$ be the same as in Corollary 3.2, and either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Remark 3.1 (1) If the uniformly convex hyperbolic spaces with modulus of uniform convexity reduce to CAT(0) spaces, and iterative process (1.1) reduce to iterative process (1.3), Theorem 3.1, Lemma 3.1, Theorem 3.2 reduce to Theorems 3.1-3.3 proved by Thakur et al. [7], respectively.

(2) If $r = 3$ and $\alpha_{in} = 0$ and $S_1 = S_2 = \cdots = S_r = T$, Theorem 3.1, Lemma 3.1, Theorem 3.2 and Theorem 3.3 become to Theorems 1-4 in [5], respectively.

(3) If the uniformly convex hyperbolic spaces with modulus of uniform convexity reduce to CAT(0) spaces, and $r = 3$ and $\alpha_{in} = 0$ and $S_1^n = S_2^n = \cdots = S_r^n = T$, where T is a nonexpansive mappings on $K \subset \mathcal{H}$, Theorem 3.1, Lemma 3.1, Theorem 3.2 are equivalent to Theorems 1-3 of [6], respectively.

4 Concluding remarks

In this paper, we introduced and studied the following new general iteration for two finite families of total asymptotically nonexpansive mappings in hyperbolic spaces $\mathcal{H}$:

$$\begin{aligned}
x_{n+1} &= W(T_1^n y_{n+r-2}, W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), \alpha_{1n}),\\
y_{n+r-2} &= W(T_2^n y_{n+r-3}, W(y_{n+r-3}, S_2^n y_{n+r-3}, \theta_{2n}), \alpha_{2n}),\\
y_{n+r-3} &= W(T_3^n y_{n+r-4}, W(y_{n+r-4}, S_3^n y_{n+r-4}, \theta_{3n}), \alpha_{3n}),\\
&\ \ \vdots\\
y_{n+1} &= W(T_{r-1}^n y_n, W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), \alpha_{(r-1)n}),\\
y_n &= W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}),
\end{aligned} \tag{4.1}$$

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where $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$ be two finite families of total asymptotically nonexpansive mappings on nonempty closed and convex subset $K \subset \mathcal{H}$, $\{\alpha_{in}\}$ and $\{\beta_{in}\}$ are two double real sequences in $[0, 1]$, and for each $i \in I = \{1, 2, \cdots, r\}$, $r \geq 2$ and $n \geq 1$, $\theta_{in} := \frac{\beta_{in}}{1-\alpha_{in}}$.

In order to prove Δ-convergence and strong convergence of the iteration (4.1) in hyperbolic spaces, we gave and analyzed some important related properties to the new general iterative processes (4.1), and proposed some meaningful illustrations for clarifying the results presented in this paper, which show that our results extend and improve the corresponding results of iterative approximation for asymptotically (quasi-)nonexpansive mapping, (generalized) (quasi-)nonexpansive mapping of all normed linear spaces, Hadamard manifolds and CAT(0) spaces as special cases. Our results extended and improved the corresponding results of [1, 2, 4–7, 9, 21].

It is well known that iterative processes as ubiquitous in the area of abstract nonlinear analysis and still remain as a main tool for approximation of fixed points of generalizations of nonexpansive maps. Furthermore, the analysis of general iterative processes, in a more general setup, is a problem of interest in theoretical numerical analysis. Therefore, on two finite families of total asymptotically nonexpansive mappings in the setting of the general iteration (4.1), the following two **open questions** will be worth further studying:.

(1) If some errors are added in the iteration (4.1), such as the iterative approximating scheme (3.1) in [11], can the Δ-convergence and strong convergence presented in this paper be proved?

(2) When T_i and S_i $(i \in I)$ in (4.1) become total asymptotically quasi-nonexpansive mappings, whether do the results of Theorems 3.1-3.3 hold?

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On Simpson's type inequalities utilizing fractional integrals

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Abstract

In the present article, we establish an integral identity for Riemann-Liouville fractional integrals. Some Simpson type integral inequalities utilizing this integral identity are obtained. It is worth mentioning that the presented results have close connection with those in [M. Z Sarikaya, E. Set, M. E Ozdemir, On new inequalities of Simpson's type for s-convex functions, Computers and Mathematics with Applications, 60 (2010), 2191–2199)].
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keywords: Simpson's Inequality, Convex Functions, Power-mean Inequality, Riemann-Liouville Fractional Integral.

1. Introduction

The following definition for convex functions is well known in the mathematical literature:

A function $f : \Phi \neq I \subseteq R \to R.$ is said to be convex on I, if inequality

$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$, for all $x, y \in I, t \in [0,1]$

Many inequalities have been established for convex functions but the most famous is the Simpson's inequality, due to its rich geometrical significance and applications, which is stated as [9]:

Theorem 1 *Let $f : [a,b] \to R$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty} = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$, then we have the following inequality:*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4. \tag{1}$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see [[9]-[11]].

In [10], Dragomir et. al proved the following recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth.

Theorem 2 *Let $f : [a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f' \in L[a,b]$. Then we have the following inequality:*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{3}\left\|f'\right\|_1, \tag{2}$$

where $\left\|f'\right\|_1 = \int_a^b |f'(x)\,dx|$.

The bound of (2) for L-Lipschitzian mapping was given in [8] by $\frac{5}{36}(b-a)$.

In [8], Sarikaya et. al presented inequalities for differentiable convex functions which are linked with Simpson's inequality, and the main inequality in [8], pointed out, is as follows.

Theorem 3 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 (interior of I) such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|$ is $s-$convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad \leq (b-a)\frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+1}(s+1)(s+2)}\left(\,|f'(a)| + |f'(b)|\,\right).\end{aligned} \tag{3}$$

Proposition 1 *Under the assumptions of Theorem 3 with $s = 1$, we have the following inequality,*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{5(b-a)}{72}\left(\,|f'(a)| + |f'(b)|\,\right). \tag{4}$$

Proposition 2 *Under the assumptions of Theorem 3 with $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, we have the following inequality,*

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$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{5\,(b-a)}{72}(\ |f'(a)|+|f'(b)|\ ). \tag{5}$$

Theorem 4 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a<b$. If $|f'|^q$ is $s-$convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\times$$
$$\left\{\left(\left[\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(b)|^q+\left[\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(a)|^q\right)^{1/q}\right.$$
$$\left.+\left(\left[\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(b)|^q+\left[\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(a)|^q\right)^{1/q}\right\}.$$

Proposition 3 *Under the assumptions of Theorem 4 with $s=1$, we have the following inequality,*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\times$$
$$\left\{\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}+\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}\right\}.$$

Proposition 4 *Under the assumptions of Theorem 4 with $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$, we have the following inequality,*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{72}(5)^{1-1/q}\times$$
$$\left\{\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}+\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}\right\}$$

Definition 1 *Let $f \in L^1[a,b]$. The left$-$sided and right$-$sided Riemann$-$Liouville fractional integrals of order $\alpha>0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad a<x$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x<b$$

respectively, where $\Gamma(.)$ is Gamma function and its definition is $\Gamma(\alpha)=\int_0^\infty e^{-u}u^{\alpha-1}du$. It is to be noted that $J_{a+}^0f(x)=J_{b-}^0f(x)=f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Properties relating to this operator can be found in [5] and for useful details on Simpson's type inequalities connected with fractional integral inequalities, the interested readers are directed to [1]

The main aim of this paper is to establish new Simpson's type inequalities for Riemann−Liouville fractional integral using the convexity as well as concavity, for the class of functions whose derivatives in absolute value at certain powers are convex functions. we will derive a general integral identity for convex functions.

2. Main Results

In order to prove our main results we need the following integral identity:

Lemma 1 *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \le 1$ on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following identity for Riemann−Liouville fractional integrals holds:*

$$\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] = \frac{b-a}{2^{\alpha+1}}\left[I_1 + I_2 + (2^{\alpha}-1)(I_3+I_4)\right],$$

where

$I_1 = \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt,$

$I_2 = \int_0^1 \left(\frac{1}{2}(1-t)^{\alpha} - \frac{1}{6}\right) f'(ta + (1-t)\frac{a+b}{2})dt,$

$I_3 = \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) f'(tb + (1-t)\frac{a+b}{2})dt,$

$I_4 = \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)} - \frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} + \frac{1}{3}\right) f'(ta + (1-t)\frac{a+b}{2})dt.$

Proof. Integrating by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt \\
&= \left.\frac{2\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt}{b-a}\right|_0^1 \\
&\qquad - \frac{2\alpha}{b-a}\int_0^1 (1-t)^{\alpha+1} f(tb + (1-t)\frac{a+b}{2})dt \\
&= \frac{2}{b-a}\left[\frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2\alpha}{b-a}\int_0^1 (1-t)^{\alpha+1} f(tb + (1-t)\frac{a+b}{2})dt \\
&= \frac{2}{b-a}\left[\frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^{\alpha}\alpha}{(b-a)^{\alpha}}J_3.
\end{aligned}$$

$$I_3 = \int_0^1 (\frac{1}{2\left(2^\alpha-1\right)}\left(1+t\right)^\alpha - \frac{1}{2\left(2^\alpha-1\right)} - \frac{1}{3}) f'(tb+(1-t)\frac{a+b}{2})dt$$
$$= \left.\frac{2\left[\frac{1}{2(2^\alpha-1)}\left(1+t\right)^\alpha - \frac{1}{2(2^\alpha-1)} - \frac{1}{3}\right] f(tb+(1-t)\frac{a+b}{2})dt)}{b-a}\right|_0^1$$
$$- \frac{2\alpha}{(b-a)\left(2^\alpha-1\right)}\int_0^1 \left(1+t\right)^{\alpha+1} f(tb+(1-t)\frac{a+b}{2})dt)$$
$$\left(2^\alpha-1\right) I_3 = \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right] + \frac{2\left(\alpha+1\right)}{b-a}\int_0^1 \left(1+t\right)^{\alpha+1} f(tb+(1-t)\frac{a+b}{2})dt)$$
$$= \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^\alpha \alpha}{\left(b-a\right)^{\alpha+1}} J_2.$$

Analogously:
$I_2 = \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^\alpha\alpha}{(b-a)^\alpha} J_1.$
$\left(2^\alpha-1\right) I_4 = \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^\alpha\alpha}{(b-a)^{\alpha+1}} J_4.$
Adding above equalities, we get

$$\frac{2}{b-a}\left[\frac{1}{6}f\left(a\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f\left(b\right)\right] - \frac{\alpha}{2\left(b-a\right)^\alpha}\left[J_1+J_2+J_3+J_4\right]$$
$$= I_1+I_2+\left(2^\alpha-1\right)\left(I_3+I_4\right).$$

Now making suitable substitutions, we have
$J_1 = \int_0^1 \left(1-t\right)^{\alpha+1} f'(ta+(1-t)\frac{a+b}{2})dt = \frac{2^\alpha}{(b-a)^\alpha}\int_a^{a+b/2}\left(u-a\right)^{\alpha-1} f(u)du$
$J_2 = \int_0^1 \left(1+t\right)^{\alpha+1} f'(tb+(1-t)\frac{a+b}{2})dt = \frac{2^\alpha}{(b-a)^\alpha}\int_{a+b/2}^{b}\left(u-a\right)^{\alpha-1} f(u)du$
$J_1+J_2 = \frac{2^\alpha}{(b-a)^\alpha}\int_a^b (u-a)^{\alpha-1} f(u)du = \frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha} J_{b-}^\alpha f(a),$
likewise
$J_3 = \int_0^1 \left(1-t\right)^{\alpha+1} f'(tb+(1-t)\frac{a+b}{2})dt = \frac{2^\alpha}{(b-a)^\alpha}\int_{a+b/2}^{b}\left(b-u\right)^{\alpha-1} f(u)du$
$J_4 = \int_0^1 \left(1+t\right)^{\alpha+1} f'(ta+(1-t)\frac{a+b}{2})dt = \frac{2^\alpha}{(b-a)^\alpha}\int_a^{a+b/2}\left(b-u\right)^{\alpha-1} f(u)du$
$J_3+J_4 = \frac{2^\alpha}{(b-a)^\alpha}\int_a^b (b-u)^{\alpha-1} f(u)du = \frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha} J_{a+}^\alpha f(b),$
which completes our proof. ⊓⊔

Theorem 5 *Let f and f' be defined as in Theorem 4 and if $|f'|$ is convex on $[a,b]$, then the following identity for Riemann–Liouville fractional integrals holds:*

$$\left[\frac{1}{6}f\left(a\right)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f\left(b\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]$$
$$\leq \frac{(b-a)}{2^\alpha}(\psi_1+\psi_2)(\ |f'(a)|+|f'(b)|\ ). \quad (6)$$

where $\psi_1 = K_1+K_2, \quad \psi_2 = K_3+K_4$

Proof. By using the properties of modulus on Lemma 1, we have

$$\left|\left[\frac{1}{6}f\left(a\right)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f\left(b\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right| \leq \frac{b-a}{2^{\alpha+1}}\times$$
$$\left[\frac{2c-\alpha+2}{6\left(\alpha+1\right)}+\left\{\left(\frac{2^\alpha-1}{3}+\frac{1}{2}\right)(2d-3)-\frac{1}{\alpha+1}\left(\frac{5d}{3}-\frac{2^{\alpha+1}+1}{2}\right)\right\}\right](\ |f'(a)|+|f'(b)|,$$

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where $c = \left(\frac{1}{3}\right)^{\frac{1}{\alpha}}$ and $d^{\alpha} = \frac{2(2^{\alpha}-1)}{3} + 1$.

Using convexity of $|f'|$, we have

$$\begin{aligned}
|I_1| &\leq \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) |f'(tb + (1-t)\frac{a+b}{2})| dt \\
&= \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) |f'(\frac{1+t}{2}b + \frac{1-t}{2}a)| dt \\
&\leq \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) \left\{\left(\frac{1+t}{2}\right) |f'(b)| + \left(\frac{1-t}{2}\right) |f'(a)|\right\} dt \\
&= \frac{K_1}{2} |f'(b)| + \frac{K_2}{2} |f'(a)|.
\end{aligned}$$

Analogously:

$$|I_2| \leq \frac{K_1}{2} |f'(a)| + \frac{K_2}{2} |f'(b)|.$$

Using the convexity on $|f'|$ and the fact that for $\alpha \in (0,1]$ and $\forall\ t \in [0,1]$,

$$\begin{aligned}
|I_3| &\leq \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) |f'(ta + (1-t)\frac{a+b}{2})| dt \\
&= \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) |f'(\frac{1+t}{2}a + \frac{1-t}{2}b)| dt \\
&\leq \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) \left\{\left(\frac{1+t}{2}\right) |f'(a)| + \left(\frac{1-t}{2}\right) |f'(b)|\right\} dt \\
&= \frac{K_3}{2} |f'(a)| + \frac{K_4}{2} |f'(b)|.
\end{aligned}$$

Similarly

$$|I_4| \leq \frac{K_3}{2} |f'(b)| + \frac{K_4}{2} |f'(a)|.$$

To get desired result, adding above inequalities and it is very easy to check

$$\begin{aligned}
K_1 &= \int_0^{1-c} \left(\frac{1}{2}(1-t)^{\alpha} - \frac{1}{6}\right) dt = -\frac{1}{6}(1-c) - \frac{1}{2(\alpha+1)}c^{\alpha+1} + \frac{1}{2(\alpha+1)}, \\
K_2 &= \int_{1-c}^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) dt = \frac{1}{6} - \frac{1}{6}(1-c) - \frac{1}{2(\alpha+1)}c^{\alpha+1},
\end{aligned}$$

$$\begin{aligned}
K_3 &= \int_0^{d-1} \left(\frac{1}{2(2^{\alpha}-1)} - \frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} + \frac{1}{3}\right) dt \\
&= \left[\frac{1}{3} + \frac{1}{2(2^{\alpha}-1)}\right](d-1) - \frac{d^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)} + \frac{1}{2(2^{\alpha}-1)(\alpha+1)}, \\
K_4 &= \int_{d-1}^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) dt \\
&= \frac{2^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)} - \left[\frac{1}{3} + \frac{1}{2(2^{\alpha}-1)}\right] - \frac{d^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)} + \left[\frac{1}{3} + \frac{1}{2(2^{\alpha}-1)}\right](d-1).
\end{aligned}$$

This completes the proof. ◻

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Remark 1 *If we take* $\alpha = 1$ *in Theorem 5 then inequality* (6) *reduces to inequality* (4).

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 6 *Let* f *and* f' *be defined as in Theorem 4 and if* $|f'|^q$ *is a convex on* $[a, b]$, *with* $q \geq 1$, *then the following inequality holds*:

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \leq \frac{(b-a)}{2^\alpha}$$
$$\left[{\psi_1}^{1-1/q}\left\{\left(\frac{K_5|f'(a)|^q + K_6|f'(b)|^q}{2}\right)^{1/q} + \left(\frac{K_5|f'(a)|^q + K_6|f'(b)|^q}{2}\right)^{1/q}\right\} + \right.$$
$$\left.{\psi_2}^{1-1/q}\left\{\left(\frac{K_7|f'(a)|^q + K_8|f'(b)|^q}{2}\right)^{1/q} + \left(\frac{K_7|f'(a)|^q + K_8|f'(b)|^q}{2}\right)^{1/q}\right\}\right]. \quad (7)$$

Proof. Using the well-known power-mean integral inequality for $q > 1$, we have

$$|I_1| \leq \left(\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right| dt\right)^{1-1/q}\left(\int_0^1\left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right|\left|f'\left(ta + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}$$

Using the convexity of $|f'|^q$, we have

$$|I_1| \leq {\psi_1}^{1-1/q}\left(K_5\frac{|f'(a)|^q}{2} + K_6\frac{|f'(b)|^q}{2}\right)^{1/q}.$$

Analogously:

$$|I_2| \leq {\psi_1}^{1-1/q}\left(K_5\frac{|f'(b)|^q}{2} + K_6\frac{|f'(a)|^q}{2}\right)^{1/q}.$$

$$|I_2| \leq {\psi_2}^{1-1/q}\left(\int_0^1((1+t)^{\alpha+1} - 2^\alpha(1+t) + \alpha 2^\alpha(1-t))\left|f'\left(tb + (1-t)\tfrac{a+b}{2}\right)\right|^q dt\right)^{1/q}.$$

By the convexity of $|f'|^q$, we have

$$|I_3| \leq {\psi_2}^{1-1/q}\left(K_7\frac{|f'(a)|^q}{2} + K_8\frac{|f'(b)|^q}{2}\right)^{1/q}.$$

Analogously:

$$|I_4| \leq {\psi_2}^{1-1/q}\left(K_7\frac{|f'(b)|^q}{2} + K_8\frac{|f'(a)|^q}{2}\right)^{1/q}.$$

It is very easy to check that

$$K_5 = \int_0^1\left|\left(\tfrac{1}{6} - \tfrac{1}{2}(1-t)^\alpha\right)\right|(1+t)\,dt = \tfrac{3(\alpha+1)+4\alpha(\alpha+2)c-\alpha(\alpha+1)c^2}{12(\alpha+1)(\alpha+2)} - \tfrac{1}{8},$$
$$K_6 = \int_0^1\left|\left(\tfrac{1}{6} - \tfrac{1}{2}(1-t)^\alpha\right)\right|(1-t)\,dt = \tfrac{2\alpha c^2-\alpha+4}{24(\alpha+2)},$$
$$K_7 = \int_0^1\left|\tfrac{1}{2(2^\alpha-1)} - \tfrac{1}{2(2^\alpha-1)}(1+t)^\alpha + \tfrac{1}{3}\right|(1+t)\,dt,$$
$$= \tfrac{1}{2(2^\alpha-1)}\left[\left(d^2 - \tfrac{5}{2}\right)\left(\tfrac{2^\alpha-1}{3} + \tfrac{1}{2}\right) - \tfrac{1}{(\alpha+2)}\left(\tfrac{5}{3}d^2 - \tfrac{2^{\alpha+1}+1}{2}\right)\tfrac{1}{3} + \tfrac{1}{2(2^\alpha-1)}\right]$$
$$K_8 = \int_0^1\left|\tfrac{1}{2(2^\alpha-1)} - \tfrac{1}{2(2^\alpha-1)}(1+t)^\alpha + \tfrac{1}{3}\right|(1-t))dt$$
$$= \tfrac{1}{2(2^\alpha-1)}\left[\left(\tfrac{1}{2} - (2-d)^2\right)\left(\tfrac{2^\alpha-1}{3} + \tfrac{1}{2}\right) + \tfrac{1}{(\alpha+1)}\left(\tfrac{1}{2} - \tfrac{5d}{3}(2-d)\right) + \tfrac{1}{(\alpha+1)(\alpha+2)}\left(\tfrac{2^{\alpha+2}+1}{2} - \tfrac{5}{3}d^2\right)\right].$$

This completes the proof. ⊔⊓

Remark 2 *If we take $\alpha = 1$ in Theorem 6, then inequality* (7) *reduces to inequality as obtained in Proposition 3.*

In the following theorem, we obtain estimate of Simpson's inequality (1) for concave functions.

Theorem 7 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \le \frac{(b-a)}{2^{\alpha+1}} \times$$
$$\left[\psi_1\left\{\left|f'\left(\frac{K_5 b + K_6 a}{\psi_1}\right)\right| + \left|f'\left(\frac{K_5 a + K_6 b}{\psi_1}\right)\right|\right\}\right.$$
$$\left. + \psi_2\left(2^\alpha - 1\right)\left|f'\left(\frac{K_7 b + K_8 a}{\psi_2}\right)\right| + \left|f'\left(\frac{K_7 a + K_8 b}{\psi_2}\right)\right|\right]. \quad (8)$$

Proof. Using the concavity of $|f'|^q$ and the power-mean inequality, we obtain

$$\begin{aligned} |f'|^q &> t|f'|^q + (1-t)|f'|^q \\ &\ge t|f'|^q + (1-t)|f'|^q. \end{aligned}$$

Hence
$$|f'(tx + (1-t)y)| \ge t|f'(x)| + (1-t)|f'(y)|.$$
So $|f'|$ is also concave. By the Jensen integral inequality, we have

$$\begin{aligned} |I_1| &\le \left(\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right| dt\right) \left|f''\left(\frac{\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right| \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt}{\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right| dt}\right)\right| \\ &= \psi_1 \left|f'\left(\frac{K_5 b + K_6 a}{\psi_1}\right)\right|. \end{aligned}$$

Analogously:

$$\begin{aligned} |I_2| &\le \psi_1 \left|f'\left(\frac{K_5 a + K_6 b}{\psi_1}\right)\right|, \\ |I_3| &\le \psi_2 \left|f'\left(\frac{K_7 b + K_8 a}{\psi_2}\right)\right|, \\ |I_4| &\le \psi_2 \left|f'\left(\frac{K_7 a + K_8 b}{\psi_2}\right)\right|. \end{aligned}$$

This completes the proof. ∎

Corollary 1 *If we take $\alpha = 1$ in Theorem 7, then inequality* (8) *becomes as:*

$$\begin{aligned} &\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\qquad \le \frac{5(b-a)}{72}\left[\left|f'\left(\frac{29a + 61b}{90}\right)\right| + \left|f'\left(\frac{61a + 29b}{90}\right)\right|\right]. \quad (9) \end{aligned}$$

Remark 3 *Inequality* (9) *is an generalization of obtained inequality as in [9, Theorem 8]*

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The permanence and global attractivity in a nonautonomous Gilpin-Ayala competition system with several delayed negative feedbacks

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Abstract: In this paper, a nonautonomous delayed Gilpin-Ayala competition system without instantaneous negative feedbacks (i.e., pure-delay-type system) is investigated. By the techniques of comparison arguments and constructing Lyapunov functionals something different to usual case, several results to guarantee the permanence of the system are derived by means of Ahmad and Lazer's definitions of lower and upper averages of a function. Moreover, the sufficient conditions for the global attractivity of the positive solution are also obtained, in which it is not necessarily to require the exponent of nonlinear intraspecific interference to exceed that of nonlinear interspecific interactions. These results are more general and practical, and possess a wide range of applications. Obviously, they are basically an extension of many existing conclusions for nonlinear competitive systems.

Keywords: Permanence; Global attractivity; Nonlinear competition; Lyapunov functionals; Pure-delays

1 Introduction

The permanence and global stability of ecological systems are always the most important and ubiquitous problems in mathematical biology. As pointed out by Li and Kuang [1], more realistic and interesting models of single or multiple species growth should take into account both the seasonality of the changing environment and the effects of time delays. Moreover, in view of the fact that in real-life species interactions, instantaneous responses are rare or weak relatively to delayed responses, more realistic models should consist of delay differential systems instead of the ones with instantaneous feedbacks. Recently, some model with discrete delay and distributed delay was studied [2–5]. In the meantime, some scholars [6,7] argue that continuously distributed delays as ecologically and biologically are more realistic than discrete delays to species interactions, which is proved true by Caperon [8]. Therefore, a reasonable alternative way is to study the pure-delay-type systems with both discrete delays and continuously distributed delays.

One the other hand, it is well know that for Lotka-Volterra model with delays, the stability is ordinarily delineated in two ways: the one that contain delay independent terms which dominate other intra-specific and inter-specific interaction effects with and without delays, called a "no-pure-delay-type", and the other with only delay feedbacks, is named as "pure-delay-type". For no-pure-delay-type system, one can use the no-delay terms to control the delay terms. Various results have been obtained recently under so-called diagonally dominant conditions and the conditions are often independent of delays (see [9–13]). However, for the pure-delay-type

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systems, the analysis of the permanence and the global asymptotic stability of the system is very difficult, let along the nonlinear type system.

Motivated by the works on Gilpin-Ayala competition systems with delays (see [12, 14–16]), in particular, strongly stimulated by the works [1, 17–19], which all contain several time delay, we consider the following Gilpin-Ayala competitive system with several discrete arguments and continuous time delays

$$\begin{aligned}\dot{x}_i(t) = x_i(t)\Big[r_i(t) - \sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)x_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\\ - \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)x_j^{\beta_{ijl}}(t+s)ds\Big].\end{aligned} \tag{1.1}$$

The aim of this paper is, by developing the analytic technique the analytic technique of the literatures [10, 11, 14–16, 21, 22], to obtain conditions which guarantee the permanence of the system (1.1); after that, by constructing a suitable Lyapunov functional, sufficient conditions about the global attractivity of the positive solution of system (1.1) are gained.

For convenience, we will use following notations in the rest of this paper, let $\tau_{ijk} = \sup\{\tau_{ijk}(t) \mid t \in R\}$ and $\tau = \max\{\tau_{ijk},\ \sigma_{ijl}\}$, then we have $0 < \tau_{ijk},\ \sigma_{ijl} \le \tau$. Denote by $\Psi_{ijk}(t) = t - \tau_{ijk}(t)$, and the functions $\Psi_{ijk}^{-1}(t)$ is the inverse functions of $\Psi_{ijk}(t)$, respectively. In this paper, for system (1.1) we always assume that

(H_1) $\alpha_{ijk} > 0,\ \beta_{ijl} > 0$.

(H_2) $r_i(t),\ a_{ijk}(t),\ \tau_{ijk}(t)$, are positively continuous and bounded functions on $[c, +\infty)$.

(H_3) Functions $b_{ijl}(t, s)$ are defined on $[c, +\infty) \times [-\tau, 0]$ such that they are integrable with respect to s, and $\int_{-\sigma_{ijl}}^{0} b_{ijl}(t, s)ds$ are positive, continuous and bounded above with respect to t on $[c, +\infty)$.

(H_4) $\tau_{ijk}(t)$ are nonnegative, continuous and bounded, $\Psi_{ijk}(t) = t - \tau_{ijk}(t)$ are all invertible. Furthermore, it is differentiable and satisfy $1 - \tau'_{ijk}(t) > 0$ $(t \ge c)$.

Stimulated by the application of system (1.1) to population dynamics, we assume that solutions of system (1.1) satisfy the following initial condition

$$x_i(\theta) = \phi_i(\theta) \ge 0,\ \theta \in [-\tau, 0],\ \ \phi_i(0) > 0,\ \sup_{\theta\in[-\tau,0]} \phi_i(\theta) < +\infty. \tag{1.2}$$

2 Basic results

Let $g(t)$ be a continuous function define on $[c, +\infty)$. Denote

$$g^u = \sup\{g(t) \mid c \le t < +\infty\},\ \ g^l = \sup\{g(t) \mid c \le t < +\infty\}.$$

According to Ahmad and Lazer [10], we define the lower and upper averages of a function $g(t)$. If $c \le t_1 < t_2$, set

$$A[g,\ t_1,\ t_2] = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} g(s)ds.$$

The lower and upper averages of $g(t)$ denoted by $m[g]$ and $M[f]$ are follows

$$m[g] = \lim_{s\to+\infty} \inf\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\},$$

and

$$M[g] = \lim_{s\to+\infty} \sup\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\}.$$

Since the set $\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\}$ decreases as s increases, the limits exist; and since $g^l \le A[g,\ t_1,\ t_2] \le g^u$, it follows that $g^l \le m[g] \le A[g,\ t_1,\ t_2] \le M[g] \le g^u$.

Definition 2.1. The system of differential equation

$$\dot{x}(t) = F(t,\ x(t)),\quad x \in R^n$$

is said to be permanent if there exits a compact set D in $R_+^n = \{(x_1,\ x_2,\ ...,\ x_n) \in R^n \mid x_i > 0\ (i = 1, 2, ..., n)\}$, such that all solutions starting in the interior of R_+^n ultimately enter D.

Now we consider following single species Logistic type equation

$$\dot{x}(t) = x(t)\Big[r(t) - \sum_{k=1}^{n} a_k(t)x^{\alpha_k}(t)\Big]. \tag{2.1}$$

Where $r(t)$ and $a_k(t)$ $(k = 1, 2, ..., n)$ are all continuous functions on $[0, +\infty)$, $r(t)$ may be negative, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in 1,\ 2,\ ...,\ n$ such that $m[a_k] > 0$, and α_k $(k = 1, 2, ..., n)$ are positive constants.

From the Lemma of [11], we have

Lemma 2.1. Suppose that $m[r] > 0$, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then any solution $x(t)$ of (2.1) with initial value $x(t_0) > 0$ is bounded above and below on $[t_0, +\infty)$ and globally attractive. Specially, if $r(t),\ a_k(t)$ $(k = 1, 2, ..., n)$ are continuous T-periodic functions, then (2.1) has a unique positive, global attractive T-periodic solution $x^*(t)$.

As a matter of fact, according to Lemma 2.2 of [11], if $r(t)$ may be negative but $M[r] > 0$, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then we have Lemma 2.2 below corresponding to Lemma 2.1:

Lemma 2.2. Assume that $M[r] > 0$ and $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then any solution $x(t)$ of (2.1) with initial value $x(t_0) > 0$ is bounded above and below by strictly positive real numbers on $[t_0, +\infty)$ and globally attractive. Specially, if $r(t),\ a_k(t)$ $(k = 1, 2, ..., n)$ are all continuous T-periodic functions, then system (2.1) has a unique positive, globally asymptotically stable T-periodic solution $x^*(t)$.

By developing the analytic technique of [11, 16], it is not difficult to verify the following results

Lemma 2.3. If $(H_2) - (H_4)$ are hold, then we have

$$M\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t - \tau_{ijk}(t))\Big] = M\Big[\frac{a_{ijk}\big(\Psi_{ijk}^{-1}(t)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big].$$

$$m\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t - \tau_{ijk}(t))\Big] = m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big].$$

where $X_i(t)$ is the unique solution of the Logistic system corresponding to Eqs. (1.1) with initial condition $X_i(t_0) > 0$.

Proof. From $(H_2) - (H_4)$ and Lemma 2.1, 2.2, we infer that $\tau_{ijk}(t)$, $\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}$ and $X_j^{\alpha_{ijk}}(t)$ are all bounded, we claim that

$$\int_{t_1-\tau_{ijk}(t_1)}^{t_1} \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(s)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(s)\big)}X_j^{\alpha_{ijk}}(s)ds,\ \int_{t_2}^{t_2-\tau_{ijk}(t_2)} \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(s)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(s)\big)}X_j^{\alpha_{ijk}}(s)ds$$

are all bounded above and below. Then from the definition of lower and upper averages of a function, we obtain that for $t_2 > t_1 \geq t_0$

$$M\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\Big] = \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))ds \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1-\tau_{ijk}(t_1)}^{t_2-\tau_{ijk}(t_2)} \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)ds \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\Big(\int_{t_1-\tau_{ijk}(t_1)}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{t_2-\tau_{ijk}(t_2)}\Big)\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1 - \tau'_{ijk}\big(\Psi_{ijk}^{-1}(t)\big)}dt \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}dt \mid t_2-t_1\ge s\Big\} = M\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big].$$

Similarly, we can testify that the equality for the case of $m\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]$ is also true.

Lemma 2.4. If $(H_2)-(H_4)$ hold, then

$$M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big] = M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big],$$
$$m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big] = m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big].$$

where $X_i(t)$ is the unique solution of the Logistic system corresponding to Eqs. (1.1) with initial condition $X_i(t_0)>0$.

Proof. From $(H_2)-(H_4)$ and Lemma 2.1, 2.2, it follows that $b_{ijl}(t,.)$ and $\int_{-\sigma_{ijl}}^0 b_{ijl}(t-s,s)ds,\ X_j^{\beta_{ijl}}(t)$ are all bounded functions, we conclude that

$$\int_{-\sigma_{ijl}}^{0}\int_{t_1+s}^{t_1} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(s)ds,\ \int_{-\sigma_{ijl}}^{0}\int_{t_2}^{t_2+s} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(s)ds$$

are all bounded. Therefore, according to the definition of lower and upper averages of a function, we find that for $t_2>t_1\ge t_0$

$$\begin{aligned}
&M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big]\\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big(\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ij}}(t+s)ds\Big)dt \mid t_2-t_1\ge s\Big\}\\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{-\sigma_{ijl}}^{0}\int_{t_1+s}^{t_2+s} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(t)dt \mid t_2-t_1\ge s\Big\}ds\\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{-\sigma_{ijl}}^{0}\Big(\int_{t_1+s}^{t_1}+\int_{t_1}^{t_2}+\int_{t_2}^{t_2+s}\Big)b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(t)dt \mid t_2-t_1\ge s\Big\}ds\\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big(\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big)dt \mid t_2-t_1\ge s\Big\}\\
&= M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big].
\end{aligned}$$

In a similar way, we can show that the equality for the case of $m\big[\int_{-\sigma_{ijl}}^0 b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]$ is also hold.

3 Permanence

In this section, we are mainly concerned with the permanence of the system (1.1)-(1.2). Firstly, for the sake of the permanence with regarding to the system (1.1), we introduce the following notations

$$a^*_{ijk}(t) = a_{ijk}(t)\exp\Big\{\alpha_{ijk}\int_t^{t-\tau_{ijk}(t)} r_i(s)ds\Big\},$$
$$b^*_{ijl}(t) = \int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\exp\Big\{\beta_{ijl}\int_t^{t+s} r_i(u)du\Big\}ds.$$

Then, let us consider the following logistic type equation corresponding to Eqs. (1.1)

$$\dot{x}_i(t) = x_i(t)\Big[r_i(t)-\sum_{k=1}^{k_{ii}} a^*_{iik}(t)x_i^{\alpha_{iik}}(t)-\sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b^*_{iil}(t,s)dsx_i^{\beta_{iil}}(t)\Big]. \tag{3.1}$$

Theorem 3.1. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)\ M\Big[r_i(t)-\sum_{j=1,\ j\ne i}^{n}\Big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))+\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big)\Big]>0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Firstly, we show that any positive solution of system (1.1) is ultimately bounded above by some positive constant. Let $x(t) = \big(x_1(t),\ ...,\ x_n(t)\big)$ be any positive solution of system (1.1), then it follows from (1.1) that for all $t \geq 0$

$$\dot{x}_i(t) \leq r_i(t)x_i(t). \tag{3.2}$$

Thus for any $t \geq 0,\ s \leq 0$ and $t+s \geq 0$, by integrating (2.11) over interval $[t+s, t]$ we derive

$$x_i(t+s) \geq x_i(t)\exp\Big\{\int_t^{t+s} r_i(s)ds\Big\} \quad \text{for, } t \geq \tau. \tag{3.3}$$

Integrate with (3.3), we obtain directly from the system (1.3) that

$$\begin{aligned}\dot{x}_i(t) &= x_i(t)\Big[r_i(t) - \sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)x_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) - \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)x_j^{\beta_{ijl}}(t+s)ds\Big]\\ &\leq x_i(t)\Big[r_i(t) - \sum_{k=1}^{k_{ii}} a_{iik}(t)x_i^{\alpha_{iik}}(t-\tau_{iik}(t)) - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)x_i^{\beta_{iil}}(t+s)ds\Big]\\ &\leq x_i(t)\Big[r_i(t) - \sum_{k=1}^{k_{ii}} a^*_{iik}(t)x_i^{\alpha_{iik}}(t) - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b^*_{iil}(t,s)dsx_i^{\beta_{iil}}(t)\Big].\end{aligned} \tag{3.4}$$

By using the comparison theorem, we find

$$x_i(t) \leq X_i(t), \text{ for all } t \geq t_0. \tag{3.5}$$

Where $X_i(t)$ is the positive solution of system (3.1) with initial condition $X_i(0)$ which satisfies $x_i(0) \leq X_i(0)$. From Lemma 2.1, Lemma 2.2 and (3.5), it is not difficult to obtain that

$$\limsup_{t\to+\infty} x_i(t) \leq X_i(t), \text{ for all } t \geq t_0.$$

Hence, for a sufficiently small $\varepsilon > 0$, there exists a $T_{i1}(\varepsilon) > 0$ such that for $t \geq T_{i1}(\varepsilon)$

$$x_i(t) \leq X_i(t) \leq X_i(t) + \varepsilon. \tag{3.6}$$

Now choose $M_0 = \sup\{X_i(t) + \varepsilon \mid t \geq 0,\ i = 1,\ 2,\ ...,\ n\}$, then M_0 does not depend on any solution of system (3.1), also $x_i(t) \leq M_0$, for all $t \geq T_1$, where $T_1 = \max_{1\leq i\leq n}\{T_{i1}\}$.

Secondly, we shall show that any positive solution of system (1.1) is ultimately bounded below by some positive constant. To this end, we proceed with following two steps.

Step 1: We show that there exists $\epsilon_0 > 0$ such that $\limsup_{t\to+\infty} x_i(t) \geq \epsilon_0$, for all $i = 1,\ 2,\ ...,\ n$. For the convenience of the following discuss, for any constant $\varepsilon > 0$, we denote by

$$\begin{aligned}R_i(t,\varepsilon) = r_i(t) &- \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)\Big(X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \varepsilon\Big)\\ &- \sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\Big(X_j^{\beta_{ijl}}(t+s) + \varepsilon\Big)ds\Big]\end{aligned}$$

On the one hand, according to (H_5) in Theorem 3.1, one finds that for any given small number $\varepsilon > 0$, there is $M[R_i(t,\varepsilon)] > 0$ $(i = 1, 2, ..., n)$. Therefore, we can choose a sufficiently small number $\epsilon_0 > 0,\ \delta > 0$ such that

$$M\Big[R_i(t,\varepsilon) - \sum_{k=1}^{k_{ii}} a_{iik}(t)\epsilon_0^{\alpha_{iik}} - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big] \geq \delta,$$

for all $i = 1,\ 2,\ ...,\ n$, i.e.,

$$\lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[R_i(t,\varepsilon) - \sum_{k=1}^{k_{ii}} a_{iik}(t)\epsilon_0^{\alpha_{iik}} - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big]dt \mid t_2 - t_1 \geq s\Big\} \geq \delta.$$

Which implies that

$$\lim_{s\to+\infty}\sup\Big\{\int_{t_1}^{t_2}\Big[R_i(t,\varepsilon)-\sum_{k=1}^{k_{ii}}a_{iik}(t)\epsilon_0^{\alpha_{iik}} -\sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0}b_{iil}(t,s)\epsilon_0^{\beta_{iil}}ds\Big]dt \mid t_2-t_1\geq s\Big\}=+\infty.$$

Therefore, there must exist $\lambda>0$ and a positive number $\gamma_0>0$ such that

$$\int_t^{t+\lambda}\Big[R_i(t,\varepsilon)-\sum_{k=1}^{k_{ii}}a_{iik}(t)\epsilon_0^{\alpha_{iik}}-\sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0}b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big]dt\geq\gamma_0,\ \text{ for all }\ t\geq T_2. \tag{3.7}$$

Now we claim that the following inequality holds

$$\limsup_{t\to+\infty}x_i(t)\geq\epsilon_0,\ \text{ for all }\ i=1,\ 2,\ ...,\ n. \tag{3.8}$$

By way of contradiction, suppose that $\limsup_{t\to+\infty}x_i(t)<\epsilon_0$ for a certain $p\in\{1,\ 2,\ ...,\ n\}$, then there exists $T_2>T_1$ such that $x_p(t)<\delta$, for all $t\geq T_2$. This, together with the (3.6), gives out that for all $t\geq T_2$

$$\begin{aligned}\dot{x}_p(t)&=x_p(t)\Big[r_p(t)-\sum_{j=1}^{n}\Big(\sum_{k=1}^{k_{pj}}a_{pjk}(t)x_j^{\alpha_{pjk}}(t-\tau_{pjk}(t))+\sum_{l=1}^{l_{pj}}\int_{-\sigma_{pjl}}^{0}b_{pjl}(t,s)x_j^{\beta_{pjl}}(t+s)ds\Big)\Big]\\
&\geq x_p(t)\Big[r_p(t)-\sum_{j=1,\ j\neq p}^{n}\sum_{k=1}^{k_{pj}}a_{pjk}(t)\Big(X_j^{\alpha_{pjk}}(t-\tau_{pjk}(t))+\varepsilon\Big)\\
&\quad-\sum_{j=1,\ j\neq p}^{n}\sum_{l=1}^{l_{pj}}\int_{-\sigma_{pjl}}^{0}b_{pjl}(t,s)\Big(X_j^{\beta_{pjl}}(t+s)+\varepsilon\Big)ds\Big]\\
&\quad-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big]\\
&\geq x_p(t)\Big[R_p(t,\varepsilon)-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big].\end{aligned} \tag{3.9}$$

An integration of (3.9) over time interval $[T_2,t]$ leads to

$$x_p(t)\geq x_p(T_2)\exp\Big\{\int_{T_2}^{t}\Big[R_p(t,\varepsilon)-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big]\Big\}. \tag{3.10}$$

Obviously, which, together with (3.7) result into the conclusion that $x_p(t)\to+\infty$ as $t\to+\infty$, which contradicts to the boundedness of $x_i(t)$, for all $t\geq T_{i1}$ in (3.6). Hence, the inequality (3.8) is true.

Step 2: We shall prove that there exists a constant $m_0>0$, m_0 is independent of any solution of system (1.1), i.e., there is a positive constant $m_0>0$ such that for any solution $x(t)=(x_1(t),\ ...,\ x_n(t))$, one has

$$\liminf_{t\to+\infty}x_i(t)\geq m_0,\ \text{ for all }\ i=1,\ 2,\ ...,\ n. \tag{3.11}$$

Assume that it is not true, then there exist a certain integer $q\in\{1,\ 2,\ ...,\ n\}$ and a sequence of initial functions $\{\phi_q^{(k)}(t)\}_{k=1}^{+\infty}$ for system (1.1) such that $x_q^{(k)}(t)=x_q(t,\phi_q^{(k)}),\ k=1,\ 2,\ ...$ satisfy

$$\liminf_{t\to+\infty}x_q^{(k)}(t)\leq\frac{\epsilon_0}{(k+1)^2},\ \text{ for all }\ k=1,\ 2,\ ... \tag{3.12}$$

For each $k=1,\ 2,\ ...$, from (3.8) we claim that $\limsup_{t\to+\infty}x_q^{(k)}(t)\geq\frac{1}{(k+1)}\epsilon_0$. Hence, by (3.12) one can infer that there exists two time sequences $\{s_n^{(k)}\}$ and $\{t_n^{(k)}\}$ such that for each $k=1,\ 2,\ ...$

$$0<s_1^{(k)}<t_1^{(k)}<s_2^{(k)}<t_2^{(k)}<\ ...\ <s_n^{(k)}<t_n^{(k)}<\ ...\ ,\ \text{ for all }\ n=1,\ 2,\ ...,$$

$$s_n^{(k)} \to +\infty, \quad t_n^{(k)} \to +\infty, \text{ as } n \to +\infty, \; x_q^{(k)}(t_n^{(k)}) = \frac{\epsilon_0}{(k+1)^2}, \; x_q^{(k)}(s_n^{(k)}) = \frac{\epsilon_0}{(k+1)}. \tag{3.13}$$

$$\frac{\epsilon_0}{(k+1)^2} < x_q^{(k)}(t) < \frac{\epsilon_0}{(k+1)}, \text{ for all } t \in (s_n^{(k)}, \; t_n^{(k)}). \tag{3.14}$$

It follows from (3.6) that for a given small number ϵ_0, there exists $T_2^{(k)} > T_1$ such that $x_i^{(k)}(t) \le X_i(t) + \epsilon_0, \;\; t \ge T_2^{(k)}$.

Obviously, by (3.13) there exists a large enough integer $N_1^{(k)} > 0$ such that $s_n^{(k)} > T_2^{(k)} + \tau$ for all $n \ge N_1^{(k)}$ for each $k = 1, \; 2, \; \ldots$. Hence, for any $t \in [s_n^{(k)}, \; t_n^{(k)}]$ and $n \ge N_1^{(k)}$, we have

$$\begin{aligned}
\dot{x}_q^{(k)}(t) &= x_q^{(k)}(t)\Big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(x_j^{(k)}(t-\tau_{qj\nu}(t))\Big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(x_j^{(k)}(t+s)\Big)^{\beta_{qjl}} ds\Big] \\
&\ge x_q^{(k)}(t)\Big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(X_j^{(k)}(t-\tau_{qj\nu}(t)) + \varepsilon\Big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(X_j^{(k)}(t+s) + \varepsilon\Big)^{\beta_{qjl}} ds\Big] \ge -\gamma x_q^{(k)}(t).
\end{aligned} \tag{3.15}$$

Where

$$\gamma = \sup_{t\in R}\Big\{\sum_{j=1}^{n}\Big[\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\Big)^{\alpha_{qj\nu}} + \sum_{l=1}^{l_{qj}}\int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(X_j^{(k)}(t+s)+\varepsilon\Big)^{\beta_{qjl}} ds\Big]\Big\}.$$

Therefore, for any $n \ge N_1^{(k)}$ and $k = 1, \; 2, \; \ldots$, an integration of (3.15) over $[s_n^{(k)}, \; t_n^{(k)}]$ makes one lead to

$$\begin{aligned}
\frac{\epsilon_0}{(k+1)^2} &= x_q^{(k)}(t_n^{(k)}) \ge x_q^{(k)}(s_n^{(k)})\exp\big\{-\gamma(t_n^{(k)} - s_n^{(k)})\big\} \\
&= \frac{\epsilon_0}{(k+1)}\exp\big\{-\gamma(t_n^{(k)} - s_n^{(k)})\big\}.
\end{aligned}$$

Which means

$$t_n^{(k)} - s_n^{(k)} \ge \frac{\ln(k+1)}{\gamma}, \text{ for all } n \ge N_1^{(k)}, \; k = 1, \; 2, \; \ldots \tag{3.16}$$

It follows from (3.16) that there exists a sufficient large integer K_0 such that

$$t_n^{(k)} - s_n^{(k)} \ge \lambda, \text{ for all } k \ge K_0, \;\; n \ge N_1^{(k)}. \tag{3.17}$$

Hence, for any $k \ge K_0, \;\; n \ge N_1^{(k)}$ and $t \in [s_n^{(k)}, \; t_n^{(k)}]$, it follows from (3.13) and (3.14) that

$$\begin{aligned}
\dot{x}_q^{(k)}(t) &= x_q^{(k)}(t)\big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\big(x_j^{(k)}(t-\tau_{qj\nu}(t))\big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}}\int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(x_j^{(k)}(t+s)\big)^{\beta_{qjl}} ds\big] \\
&\ge x_q^{(k)}(t)\big[r_q(t) - \sum_{\nu=1}^{\nu_{qq}} a_{qq\nu}(t)\big(\frac{\epsilon_0}{k+1}\big)^{\alpha_{qq\nu}} - \sum_{l=1}^{l_{qq}}\int_{-\sigma_{qql}}^{0} b_{qql}(t,s)ds\big(\frac{\epsilon_0}{k+1}\big)^{\beta_{qql}} \\
&\qquad - \sum_{j=1, \; j\ne q}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\big(X_j^{(k)}(t-\tau_{qj\nu}(t)) + \varepsilon\big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1, \; j\ne p}^{n}\sum_{l=1}^{l_{qj}}\int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(X_j^{(k)}(t+s) + \varepsilon\big)^{\beta_{qjl}} ds\big] \\
&\ge x_q^{(k)}(t)\big[r_q(t) - \sum_{\nu=1}^{\nu_{qq}} a_{qq\nu}(t)\epsilon_0^{\alpha_{qq\nu}} - \sum_{l=1}^{l_{qq}}\int_{-\sigma_{qql}}^{0} b_{qql}(t,s)ds\,\epsilon_0^{\beta_{qql}}
\end{aligned}$$

$$- \sum_{j=1,\ j\neq q}^{n} \sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\big)^{\alpha_{qj\nu}}$$
$$- \sum_{j=1,\ j\neq p}^{n} \sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(X_j^{(k)}(t+s)+\varepsilon\big)^{\beta_{qjl}} ds\big]. \tag{3.18}$$

According to (3.7), (3.13) and (3.14), an integration of (3.18) over time interval $[t_n^{(k)}-\lambda,\ t_n^{(k)}]$ makes it reach

$$\frac{\epsilon_0}{(k+1)^2} = x_q^{(k)}(t_n^{(k)}) \geq x_q^{(k)}(t_n^{(k)}-\lambda)\exp\Big\{\int_{t_n^{(k)}-\lambda}^{t_n^{(k)}} \big[B_q(t,\epsilon_0) - \sum_{j=1,\ j\neq q}^{n} \big(\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)$$
$$\times\big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\big)^{\alpha_{qj\nu}} + \sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(X_j^{(k)}(t+s)+\varepsilon\big)^{\beta_{qjl}} ds\big)\big]dt\Big\}$$
$$> \frac{\epsilon_0}{(k+1)^2}\exp\epsilon_0 > \frac{\epsilon_0}{(k+1)^2}. \tag{3.19}$$

Where

$$B_q(t,\epsilon_0) = r_q(t) - \sum_{\nu=1}^{\nu_{qq}} a_{qq\nu}(t)\epsilon_0^{\alpha_{qq\nu}} - \sum_{l=1}^{l_{qq}} \int_{-\sigma_{qql}}^{0} b_{qql}(t,s)ds\epsilon_0^{\beta_{qql}}.$$

Which is contradiction. This shows that there exists a constant $m_0 > 0$ ($m_0 > 0$ is independent of any initial function) such that the inequality (2.15) is correct. That is to say, any positive solution $x(t)$ of the initial value problem (1.1)-(1.2) is ultimately bounded below by a positive constant $m_0 > 0$. From Definition 2.1, the proof of Theorem 3.1 is complete.

Theorem 3.2. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)'\ M\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n} \sum_{k=1}^{k_{ij}} m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)} X_j^{\alpha_{ijk}}(t)\Big]$$
$$- \sum_{j=1,\ j\neq i}^{n} \sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then the system (1.1)-(1.2) is permanent.

Proof. In order to prove the correct of Theorem 3.2, We only need to show that $(H_5)'$ implies the assumption (H_5). Actually, if take into account the fact that

$$M[X_{i0}(t)+c] = M[X_{i0}(t)]+c, \qquad m[f_i(t)] \leq A[f_i(t),\ t_1,\ t_2].$$

Then we may obtain that

$$M\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n} \Big(\sum_{k=1}^{k_{ij}} m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big] + \sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big]\Big)$$
$$-M\big[r_i(t) - \sum_{j=1,\ j\neq i}^{n} \big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \sum_{l=1}^{l_{ij}} \int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]$$
$$= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[r_i(t) - \sum_{j=1,\ j\neq i}^{n} \big(\sum_{k=1}^{k_{ij}} m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big]$$
$$+\sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big]\big)\Big]dt \mid t_2-t_1 \geq s\Big\} - \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t)$$
$$- \sum_{j=1,\ j\neq i}^{n} \big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \sum_{l=1}^{l_{ij}} \int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt \mid t_2-t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[r_i(t) - \sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} m\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]$$
$$+\sum_{l=1}^{l_{ij}} m\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]\big)\Big]dt \mid t_2-t_1\geq s\Big\} - \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t)$$
$$-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))+\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt \mid t_2-t_1\geq s\Big\}$$
$$\geq \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} r_i(t)dt - \sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]dt$$
$$+\sum_{l=1}^{l_{ij}}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]\big)dt \mid t_2-t_1\geq s\Big\} - \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t)$$
$$-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))+\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt \mid t_2-t_1\geq s\Big\} = 0.$$

Therefore, we claim from Theorem 3.1 that Theorem 3.2 is correct. The proof is complete.

Theorem 3.3. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)''\ M\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} M\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big]$$
$$- \sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Noticing the following facts that

$$M[X_{i0}(t)+c] = M[X_{i0}(t)]+c,\quad m[f_i(t)]\leq M[f_i(t)]\ \text{ and }\ \sum_{i=1}^{n} m[f_i(t)]\leq\sum_{i=1}^{n} M[f_i(t)].$$

We find that the condition $(H_5)''$ means the hypothesis $(H_5)'$, and so it does the assumption (H_5). Hence, one can confirm that the result of Theorem 3.3 is also true.

Theorem 3.4. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)'''\ m\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} M\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big]$$
$$- \sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Taking into account the facts that

$$M[X_{i0}(t)+c] = M[X_{i0}(t)]+c,\quad m[f_i(t)]\leq M[f_i(t)].$$

We declare that the assumption $(H_5)''$ can be deduced from the hypothesis $(H_5)'''$, so it is evident that Theorem 3.3 implies the Theorem 3.4.

Theorem 3.5. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)''''\ m\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big]$$

$$-\sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. As a matter of fact, $m[f_i(t)] \leq M[f_i(t)]$ and assumption $(H_5)''''$ means that the hypothesis (H_5) is true, so it follows from Theorem 3.1 that the conclusion of Theorem 3.5 is right.

Remark. 3.1 It is easy to verify that $M[g] = m[g] = \frac{1}{T}\int_0^T g(t)dt$ for a T-periodic function $g(t)$. So if system (1.1) is a periodic system, i.e., $r_i(t)$, $a_{ijk}(t)$, $b_{ijl}(t,\cdot)$ are the continuous T-periodic functions, then $X_i(t)$ in above mentioned Theorems can be replaced by the unique positive T-periodic solution $X_i^*(t)$ of (3.1), and the assumptions of Theorem 3.1-Theorem 3.5 are equivalent to each other.

Remark. 3.2 Theorems 3.1-3.5 generalize the main results of Zhao et al. [11], Chen et al. [14,15] and Xia et al. [16]. We mention here that for general nonautonomous Lotka-Volterra system (1.1), Teng et al. [21, 22] also obtained some similar results as that of Zhao [11]. It is in this sense, our results can also be seen as the generalization of Theorems of [21, 22].

4 Global attractivity

A very basic and important problem accompanying with the ecological dynamics systems is the global stability of the positive solution for the system. In this section, we will devote ourselves to give some new criteria to guarantee global attractivity of the positive solution.

Definition 4.1. The bounded solution $X^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))$ of system (1.1) with $X^*(t_0) > 0$ is said to be globally attractive, if for any other solution $X(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ with $X(0) > 0$, there is

$$\lim_{t\to+\infty} \mid x_i(t) - x_i^*(t) \mid = 0, \quad i = 1, 2, ..., n.$$

Before we state the main result of this section, we first introduce some notations which will be used in the following discussion. Let $\Phi_{ijk}^{-1}(t)$ be the inverse function of $\Phi_{ijk}(t) = t - \tau_{ijk}(t)$, and

$$A_{ijk}^{(1)}(t) = \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)},\ A_{ijk}^{(2)}(t) = \frac{a_{ijk}\Big(\Phi_{ijk}^{-1}\big(\Phi_{ijk}^{-1}(t)\big)\Big)}{\Big(1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}\big(\Phi_{ijk}^{-1}(t)\big)\big)\Big)\Big(1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)\Big)},$$

$$B_{ijl}^{(1)}(t) = \int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)ds, \quad B_{ijl}^{(2)}(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} b_{ijl}(\theta-s,s)d\theta ds,$$

$$\big(B_{ijl}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} A_{i\tilde{j}\tilde{k}}^{(1)}(\theta-s)b_{ijl}(t-s,s)d\theta ds,$$

$$\big(B_{ijl}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} B_{i\tilde{j}\tilde{l}}^{(1)}(\theta-s)b_{ijl}(t-s,s)d\theta ds.$$

Let $u_i(t) = \ln x_i(t)$, then Eqs. (1.1) can be reformulated as

$$\begin{aligned}\dot{u}_i(t) = r_i(t) &- \sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)\exp\Big\{\alpha_{ijk}u_j\big(t-\tau_{ijk}(t)\big)\Big\} \\ &- \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\exp\big\{\beta_{ijl}u_j(t+s)\big\}ds. \qquad (4.1)\end{aligned}$$

Now we are in the position of stating the sufficient conditions which guarantee the global attractivity of system (1.1).

Theorem 4.1. In addition to $(H_1) - (H_5)$, we assume further that

(H_6) There exist positive constants $\lambda_i > 0$ $(i = 1, 2, ..., n)$, $\zeta > 0$ such that

$$\liminf_{t\to+\infty}\{\Lambda_i(t)\}>\zeta,\ \ \liminf_{t\to+\infty}\{\Delta_i(t)\}>\zeta.$$

Where $\Lambda_i(t)=2\sum_{k=1}^{k_{ii}}\lambda_iA^{(1)}_{iik}(t)-\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha^2_{jik}M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A^{(1)}_{jik}(t)+\sum_{k=1}^{k_{ij}}\frac{\lambda_i}{\alpha_{iik}m_{i0}^{\alpha_{iik}}}A^{(1)}_{ijk}(t)\Big]$

$$-\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha^2_{jik}M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A^{(1)}_{jik}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A^{(1)}_{i\tilde{j}\tilde{k}}(s)ds+\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}A^{(2)}_{i\tilde{j}\tilde{k}}(s)ds\Big)$$

$$+\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha^2_{jik}M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A^{(1)}_{jik}(t)\Big(\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B^{(1)}_{i\tilde{j}\tilde{l}}(\Phi^{-1}_{jik}(\theta))d\theta+\sum_{l=1}^{l_{i\tilde{j}}}B^{(2)}_{i\tilde{j}l}(t)\Big)\Big],$$

$$\Delta_i(t)=2\sum_{l=1}^{l_{ii}}\lambda_iB^{(1)}_{iil}(t)-\sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta^2_{jil}M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B^{(1)}_{jil}(t)+\sum_{l=1}^{l_{ij}}\frac{\lambda_i}{\beta_{iil}m_{i0}^{\beta_{iil}}}B^{(1)}_{ijl}(t)\Big]$$

$$-\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta^2_{jil}M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B^{(1)}_{jil}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A^{(1)}_{i\tilde{j}k}(s)ds+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B^{(1)}_{jil}(t)B^{(2)}_{i\tilde{j}\tilde{l}}(t)\Big)$$

$$+\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta^2_{jil}M_{i0}^{2\beta_{jil}}}{\beta_{iil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\big(B^{(2)}_{jil}\cdot A^{(1)}_{i\tilde{j}\tilde{k}}\big)(t)+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\big(B^{(2)}_{jil}\cdot B^{(1)}_{i\tilde{j}\tilde{l}}\big)(t)\Big)\Big].$$

Then the solution $X^*(t)=(x_1^*(t),x_2^*(t),...,x_n^*(t))$ of $(1.1)-(1.2)$ is globally attractive.

Proof. Let $X^*(t)=\big(x_1^*(t),...,x_n^*(t)\big)$ with $x_i^*(t_0)>0$ be a positive solution of (1.1), and $X(t)=\big(x_1(t),...,x_n(t)\big)$ with $x_i(t_0)>0$ be an any given solution of system (1.1). In order to show the global attractivity of the bounded solution $X^*(t)$ of system (1.1), we shall show that the solution $U^*(t)=(u_1^*(t),\ ...,\ u_n^*(t)$ of system (4.1) is globally attractive. Let $U(t)=(u_1(t)\ ,...,\ u_n(t))$ be any other positive solution of system (4.1). According to Theorem 3.1, there exist positive constants $m_{i0},\ M_{i0}\ (i=1,2,...,n)$ and enough large $T>0$ such that for all $t\geq T$, there are

$$m_{i0}\leq u_i(t),\quad u_i^*(t)\leq M_{i0}\ (i=1,2,...,n). \tag{4.2}$$

Obviously, So to prove the global attractivty of the system (1.1), it is suffices to verify that system (4.1) is globally attractive. Firstly, construct a Lyapunov functional as follows

$$V_1(t)=\sum_{i=1}^{n}\lambda_i\Big[\big(u_i(t)-u_i^*(t)\big)-\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A^{(1)}_{ijk}(t)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds$$

$$-\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(\theta)\}-\exp\{\beta_{ijl}u_j^*(\theta)\}\Big)d\theta ds\Big]^2.$$

By calculating the right upper derivative of $V_1(t)$, we find

$$\dot{V}_1(t)=-2\sum_{i=1}^{n}\lambda_i\Big[\big(u_i(t)-u_i^*(t)\big)-\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A^{(1)}_{ijk}(s)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds$$

$$-\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(\theta)\}-\exp\{\beta_{ijl}u_j^*(\theta)\}\Big)d\theta ds\Big]$$

$$\times\Big[\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}A^{(1)}_{ijk}(t)\Big(\exp\{\alpha_{ijk}u_j(t)\}-\exp\{\alpha_{ijk}u_j^*(t)\}\Big)$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}B^{(1)}_{ijl}(t)\Big(\exp\{\beta_{ijl}u_j(t)\}-\exp\{\beta_{ijl}u_j^*(t)\}\Big)\Big]$$

$$\leq-2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_iA^{(1)}_{iik}(t)\Big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\Big)\big(u_i(t)-u_i^*(t)\big)$$

$$
\begin{aligned}
&-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big(\exp\{\alpha_{jik}u_i(t)\}-\exp\{\alpha_{jik}u_i^*(t)\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big(\exp\{\beta_{jil}u_i(t)\}-\exp\{\beta_{jil}u_i^*(t)\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j(t)\}-\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\}\Big)\Big]\\
&\qquad\times\Big[\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j(t)\}-\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\}\Big)\Big]\\
&\qquad\times\Big[\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(s)\}-\exp\{\beta_{ijl}u_j^*(s)\}\Big)d\theta ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\{\beta_{i\tilde{j}\tilde{l}}u_j(t)\}-\exp\{\beta_{i\tilde{j}\tilde{l}}u_j^*(t)\}\Big)\Big]\\
&\qquad\times\Big[\sum_{j=1,j\neq\tilde{j}}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\{\beta_{i\tilde{j}\tilde{l}}u_j(t)\}-\exp\{\beta_{i\tilde{j}\tilde{l}}u_j^*(t)\}\Big)\Big]\\
&\qquad\times\Big[\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(s)\}-\exp\{\beta_{ijl}u_j^*(s)\}\Big)d\theta ds\Big]. \qquad (4.3)
\end{aligned}
$$

That is

$$
\begin{aligned}
\dot{V}_1(t)\le&-2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big(\exp\{\alpha_{jik}u_i(t)\}-\exp\{\alpha_{jik}u_i^*(t)\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big(\exp\{\beta_{jil}u_i(t)\}-\exp\{\beta_{jil}u_i^*(t)\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\}-\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\}\Big)\\
&\qquad\times\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds\\
&+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\}-\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\}\Big)\Big]\\
&\qquad\times\Big[\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(s)\}-\exp\{\beta_{ijl}u_j^*(s)\}\Big)d\theta ds
\end{aligned}
$$

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$$+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^*(t)\big\}\Big)$$
$$\times\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)ds$$
$$+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^*(t)\big\}\Big)$$
$$\times\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)d\theta ds. \tag{4.4}$$

By further using the inequality $a^2+b^2\geq 2ab$, it follows from (4.4) that

$$\dot{V}_1(t)\leq -2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\big\{\alpha_{iik}u_i(t)\big\}-\exp\big\{\alpha_{iik}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)$$
$$-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\big\{\beta_{iil}u_i(t)\big\}-\exp\big\{\beta_{iil}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)$$
$$+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big[\Big(\exp\big\{\alpha_{jik}u_i(t)\big\}-\exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2+\big(u_j(t)-u_j^*(t)\big)^2\Big]$$
$$+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big[\Big(\exp\big\{\beta_{jil}u_i(t)\big\}-\exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2+\big(u_j(t)-u_j^*(t)\big)^2\Big]$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big[\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}^*(t)\big\}\Big)^2$$
$$+\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)^2ds\Big]$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big[B_{ijl}^{(2)}(t)\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}^*(t)\big\}\Big)^2$$
$$+\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)^2d\theta ds\Big]$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big[\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^*(t)\big\}\Big)^2$$
$$+\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)^2ds\Big]$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big[B_{ijl}^{(2)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^*(t)\big\}\Big)^2$$
$$+\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)^2d\theta ds\Big]$$

Now let us define the Lyapunov functional $V_2(t)$ as follows

$$V_2(t)=\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)\big(\Phi_{ijk}^{-1}\big)\int_{s}^{t}A_{ijk}^{(1)}(r)$$
$$\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^*(r)\big\}\Big)^2drds$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(\theta-s)\int_{\theta}^{t}b_{ijl}(r-s,s)\\
&\qquad\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2drd\theta ds\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{ijk}^{-1}(\theta)\big)\int_{\theta}^{t}A_{ijk}^{(1)}(r)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^*(r)\big\}\Big)^2drd\theta\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}(\theta-s)\int_{\theta}^{t}b_{ijl}(r-s,s)\\
&\qquad\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2drd\theta ds.
\end{aligned}
$$

Calculating the derivative of $V_2(t)$ along the positive solution of system (1.1), it follows:

$$
\begin{aligned}
\dot{V}_2(t)=&\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)dsA_{ijk}^{(1)}(t)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(t)\big\}-\exp\big\{\alpha_{ijk}u_j^*(t)\big\}\Big)^2\\
&-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_iA_{i\tilde{j}\tilde{k}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)^2ds\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\big(B_{ijl}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{ijl}u_j(t)\big\}-\exp\big\{\beta_{ijl}u_j^*(t)\big\}\Big)^2\\
&-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_iA_{i\tilde{j}\tilde{k}}^{(1)}(t)\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(r-s,s)\\
&\qquad\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2drds\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{ijk}^{-1}(\theta)\big)d\theta A_{ijk}^{(1)}(t)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(t)\big\}-\exp\big\{\alpha_{ijk}u_j^*(t)\big\}\Big)^2\\
&-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_iB_{i\tilde{j}\tilde{l}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(r)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^*(r)\big\}\Big)^2dr\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\big(B_{ijl}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{ijl}u_j(t)\big\}-\exp\big\{\beta_{ijl}u_j^*(t)\big\}\Big)^2\\
&-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_iB_{i\tilde{j}\tilde{l}}^{(1)}(t)\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(r-s,s)\\
&\qquad\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2drds. \qquad (4.5)
\end{aligned}
$$

Finally, we consider the following Lyapunov functional $V(t)$

$$V(t) = V_1(t) + V_2(t). \tag{4.6}$$

Calculating the upper right derivative of $V(t)$ along the solution of system (1.2), and integrating with the above-mentioned analysis, one claims that

$$\begin{aligned}
D^+V(t) \le & -2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\big\{\alpha_{iik}u_i(t)\big\} - \exp\big\{\alpha_{iik}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
& -2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\big\{\beta_{iil}u_i(t)\big\} - \exp\big\{\beta_{iil}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
& +\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ij}}\lambda_i A_{ijk}^{(1)}(t)\big(u_i(t)-u_i^*(t)\big)^2 + \sum_{l=1}^{l_{ij}}\lambda_i B_{ijl}^{(1)}(t)\big(u_i(t)-u_i^*(t)\big)^2\Big]\\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{k_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_{\tilde{j}} A_{\tilde{j}i\tilde{k}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i(t)\big\} - \exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_{\tilde{j}} A_{\tilde{j}i\tilde{k}}^{(1)}(t)B_{ijl}^{(2)}(t)\Big(\exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i(t)\big\} - \exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_{\tilde{j}} B_{\tilde{j}i\tilde{l}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i(t)\big\} - \exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_{\tilde{j}} B_{\tilde{j}i\tilde{l}}^{(1)}(t)B_{ijl}^{(2)}(t)\Big(\exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i(t)\big\} - \exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ji}}\lambda_j\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)dsA_{jik}^{(1)}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ji}}\lambda_j\big(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ji}}\lambda_j\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta A_{jik}^{(1)}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2\\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ji}}\lambda_j\big(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2.
\end{aligned} \tag{4.7}$$

Meanwhile, by making use of mean value theorem, we can obtain that for any given positive number $\epsilon > 0$, there are

$$\begin{aligned}
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)(t)\big\} = \epsilon\exp\{\epsilon\vartheta_i^{(1)}(t)\}\big(u_i(t)-u_i^*(t)\big),\\
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)\big\} = \frac{\epsilon}{\alpha_{iik}}\exp\{\epsilon\vartheta_i^{(2)}(t)\}\\
&\qquad\qquad\times\big(\exp\big\{\alpha_{iik}u_i(t)\big\} - \exp\big\{\alpha_{iik}u_i^*(t)\big\}\big),\\
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)\big\} = \frac{\epsilon}{\beta_{iil}}\exp\{\epsilon\vartheta_i^{(3)}(t)\}
\end{aligned}$$

$$\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big). \tag{4.8}$$

Where $\vartheta_i^{(1)}(t),\ \vartheta_i^{(2)}(t),\ \vartheta_i^{(3)}(t)$ are all lie between $u_i(t)$ and $u_i^*(t)$. Thus, it follows from (4.2) and (4.8) that for any given positive number $\epsilon>0$, we have

$$\begin{aligned}
&\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)(t)\}\ge \epsilon m_{i0}^{\epsilon}\big(u_i(t)-u_i^*(t)\big),\\
&\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)(t)\}\le \epsilon M_{i0}^{\epsilon}\big(u_i(t)-u_i^*(t)\big).
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\ge{}&\frac{\epsilon}{\alpha_{iik}}m_{i0}^{\epsilon}\\
&\times\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big),\\
\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\le{}&\frac{\epsilon}{\alpha_{iik}}M_{i0}^{\epsilon}\\
&\times\Big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\Big).
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\ge{}&\frac{\epsilon}{\beta_{iil}}m_{i0}^{\epsilon}\\
&\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big),\\
\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\le{}&\frac{\epsilon}{\beta_{iil}}M_{i0}^{\epsilon}\\
&\times\Big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\Big).
\end{aligned}\tag{4.11}$$

Inequality (4.7), (4.9), (4.10) and (4.11) implies that for $t\ge T_1$

$$\begin{aligned}
D^+V(t)\le{}&\sum_{i=1}^{n}\Big\{\sum_{k=1}^{k_{ii}}-2\lambda_iA_{iik}^{(1)}(t)+\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)+\sum_{k=1}^{k_{ij}}\frac{\lambda_i}{\alpha_{iik}m_{i0}^{\alpha_{iik}}}A_{ijk}^{(1)}(t)\Big]\\
&+\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(s)ds+\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)ds\Big)\\
&+\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}l}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta+\sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}l}^{(2)}(t)\Big)\Big]\Big\}\\
&\times\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\\
&+\sum_{i=1}^{n}\Big\{-2\sum_{l=1}^{l_{ii}}\lambda_iB_{iil}^{(1)}(t)+\sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)+\sum_{l=1}^{l_{ij}}\frac{\lambda_i}{\beta_{iil}m_{i0}^{\beta_{iil}}}B_{ijl}^{(1)}(t)\Big]\\
&+\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A_{i\tilde{j}k}^{(1)}(s)ds+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{jil}^{(1)}(t)B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big)\\
&+\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\big(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\big(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big)\Big]\Big\}\\
&\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big).\\
=:{}&-\sum_{i=1}^{n}\Lambda_i(t)\mid\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\mid\\
&-\sum_{i=1}^{n}\Delta_i(t)\mid\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\mid.
\end{aligned}\tag{4.12}$$

At the same time, according to hypotheses (H_6) of Theorem 4.1, we declare that there exists a constant $\zeta>0$ such that $\Lambda_i(t),\ \Delta_i(t)>\zeta$, so it follows from (4.12) that $V(t)$ is nonincreasing, and it not difficult to see that $\dot{u}_i(t)$ are bounded for $t\ge T_1$. Hence, one can further infer that $\mid u_i(t)-u_i^*(t)\mid,\ \mid\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\mid,\ \mid\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\mid$ are

uniformly continuous on $[T_1, +\infty)$. An integration on both sides of (4.10) over time interval $[T_1, t)$ leads to

$$V(t) + \zeta\sum_{i=1}^{n}\int_{T_1}^{t}\Big[\mid \big(\exp\{\alpha_{iik}u_i(s)\} - \exp\{\alpha_{iik}u_i^*(s)\}\big)\big(u_i(s) - u_i^*(s)\big)\mid$$
$$+ \mid \big(\exp\{\beta_{iil}u_i(s)\} - \exp\{\beta_{iil}u_i^*(s)\}\big)\big(u_i(s) - u_i^*(s)\big)\mid\Big]ds \le V(T_1) < +\infty.$$

Thus

$$\limsup_{t\to+\infty}\sum_{i=1}^{n}\int_{T_1}^{t}\Big[\mid \big(\exp\{\alpha_{iik}u_i(s)\} - \exp\{\alpha_{iik}u_i^*(s)\}\big)\big(u_i(s) - u_i^*(s)\big)\mid$$
$$+ \mid \big(\exp\{\beta_{iil}u_i(s)\} - \exp\{\beta_{iil}u_i^*(s)\}\big)\big(u_i(s) - u_i^*(s)\big)\mid\Big]ds \le \frac{V(T_1)}{\zeta} < +\infty. \tag{4.13}$$

It follows from (4.13) that

$$\mid \big(\exp\{\alpha_{iik}u_i(s)\} - \exp\{\alpha_{iik}u_i^*(s)\}\big)\big(u_i(s) - u_i^*(s)\big)\mid \in L[T_1, +\infty),$$

$$\mid \big(\exp\{\beta_{iil}u_i(s)\} - \exp\{\beta_{iil}u_i^*(s)\}\big)\big(u_i(s) - u_i^*(s)\big)\mid \in L[T_1, +\infty).$$

According to Barbalat's lemma, we conclude that

$$\lim_{t\to+\infty}\mid \big(\exp\{\alpha_{iik}u_i(t)\} - \exp\{\alpha_{iik}u_i^*(t)\}\big)\big(u_i(t) - u_i^*(t)\big)\mid = 0. \tag{4.14}$$

$$\lim_{t\to+\infty}\mid \big(\exp\{\beta_{iil}u_i(t)\} - \exp\{\beta_{iil}u_i^*(t)\}\big)\big(u_i(t) - u_i^*(t)\big)\mid = 0. \tag{4.15}$$

By way of contradiction, it easy to obtain from (4.14) and (4.15) that

$$\lim_{t\to+\infty}\mid u_i(t) - u_i^*(t)\mid = 0. \tag{4.16}$$

Therefore, the positive solution $X^*(t)$ of the system (1.1) is also globally attractive. This completes the proof.

Theorem 4.2. In addition to $(H_1) - (H_5)$, we assume further that
$(H_6)'$ There exist positive constants $\lambda_i > 0$ $(i = 1, 2, ..., n)$, $\zeta > 0$ such that

$$\liminf_{t\to+\infty}\{\Lambda_i(t)\} > \zeta.$$

Where
$$\Lambda_i(t) = 2\sum_{k=1}^{k_{ii}}\lambda_i\alpha_{iik}m_{i0}^{\alpha_{iik}}A_{iik}^{(1)}(t) - \sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t) + \sum_{k=1}^{k_{ij}}\lambda_iA_{ijk}^{(1)}(t)\Big]$$
$$+2\sum_{l=1}^{l_{ii}}\lambda_i\beta_{iil}m_{i0}^{\beta_{iil}}B_{iil}^{(1)}(t) - \sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\lambda_jB_{jil}^{(1)}(t)\beta_{jil}^2M_{i0}^{2\beta_{jil}} + \sum_{l=1}^{l_{ij}}\lambda_iB_{ijl}^{(1)}(t)\Big]$$
$$-\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(s)ds + \sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}l}^{(2)}(t)\Big)$$
$$+\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{\tilde{j}i}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)ds + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta\Big)$$
$$+\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A_{i\tilde{j}k}^{(1)}(s)ds + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big)$$
$$+\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)})(t) + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)})(t)\Big)\Big].$$

Then the solution $X^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))$ of $(1.1) - (1.2)$ is globally attractive.

Proof. Let $U^*(t) = (u_1^*(t),\ ...,\ u_n^*(t)$ be the solution of system (4.1), and $U(t) = (u_1(t)\ , ...,\ u_n(t))$

be any other positive solution of system (4.1). Then for the Lyapunov functional $V(t)$ as defined in (4.6), similarly to the discuss of Theorem 4.1, one can obtain that the inequality (4.7) is true. By further making use of (4.9), (4.10) and (4.11), it follows that (4.7) implies

$$\begin{aligned}
D^+V(t) \le \sum_{i=1}^{n}\Big\{ & -2\sum_{k=1}^{k_{ii}}\lambda_i\alpha_{iik}m_{i0}^{\alpha_{iik}}A_{iik}^{(1)}(t) + \sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t) + \sum_{k=1}^{k_{ij}}\lambda_iA_{ijk}^{(1)}(t)\Big] \\
& -2\sum_{l=1}^{l_{ii}}\lambda_i\beta_{iil}m_{i0}^{\beta_{iil}}B_{iil}^{(1)}(t) + \sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\lambda_jB_{jil}^{(1)}(t)\beta_{jil}^2M_{i0}^{2\beta_{jil}} + \sum_{l=1}^{l_{ij}}\lambda_iB_{ijl}^{(1)}(t)\Big] \\
& +\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(s)ds + \sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}l}^{(2)}(t)\Big) \\
& +\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{\tilde{j}i}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)ds + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta\Big) \\
& +\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A_{i\tilde{j}k}^{(1)}(s)ds + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big) \\
& +\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)})(t) + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)})(t)\Big)\Big]\Big\}\big(u_i(t)-u_i^*(t)\big)^2. \\
& =: -\sum_{i=1}^{n}\Lambda_i(t)\big(u_i(t)-u_i^*(t)\big)^2 \qquad (4.17)
\end{aligned}$$

An integration on both sides of (4.17) over time interval $[T_1, t)$ leads to

$$V(t) + \zeta\sum_{i=1}^{n}\int_{T_1}^{t}\big(u_i(s)-u_i^*(s)\big)^{|}2ds \le V(T_1) < +\infty.$$

Thus

$$\limsup_{t\to+\infty}\sum_{i=1}^{n}\int_{T_1}^{t}\big(u_i(s)-u_i^*(s)\big)^{|}2ds \le \frac{V(T_1)}{\zeta} < +\infty. \qquad (4.18)$$

It follows from (4.18) that

$$\big(u_i(s)-u_i^*(s)\big)^2 \in L[T_1,+\infty),$$

According to Barbalat's lemma, we conclude that

$$\lim_{t\to+\infty}\big(u_i(t)-u_i^*(t)\big)^2 = 0. \qquad (4.19)$$

Taking into account the fact that for $t \ge T_1$

$$\big(x_i(t)-x_i^*(t)\big) = \exp\big\{u_i(t)\big\} - \exp\big\{u_i^*(t)\big\}$$

One infers that

$$(m_{i0}) \mid u_i(t)-u_i^*(t) \mid \le \mid x_i(t)-x_i^*(t) \mid \le (M_{i0}) \mid u_i(t)-u_i^*(t) \mid$$

So it follows that

$$\lim_{t\to+\infty} \mid x_i(t)-x_i^*(t) \mid = 0. \qquad (4.20)$$

Thus, we have verified that the positive solution $X^*(t)$ of the system (1.1) is globally attractive.

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Some approximations of the Bateman's $G-$function

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Abstract

In the paper, we presented a family $M(\mu, x)$ of approximations of the Bateman function $G(x)$. The family $M(\mu, x) = G(x)$ for a certain μ whenever x is fixed and it presented asymptotical approximation of the Bateman's $G-$function as $x \to \infty$. We studied the order of convergence of the approximations $M(\mu, x)$ of the function $G(x)$. Some properties and bounds of the error are deduced. We presented new sharp double inequality of $G(x)$ with the upper and lower bounds $M(1, x)$ and $M(\frac{4}{e^2-4}, x)$ (resp.). Also, we show that the approximations $M(\mu, x)$ are better than the approximation $\frac{1}{x} + \frac{1}{2x^2}$ for any μ in an open subinterval of $\left[1, \frac{4}{e^2-4}\right]$.

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Key Words: Bateman function, digamma function, monotonicity, sharp inequality, approximation, error.

1 Introduction.

In 1953, Erdélyi [6] defined the Bateman's $G-$function as

$$G(x) = \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad x \neq 0, -1, -2, ... \tag{1}$$

where the digamma function $\psi(x)$ is given by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x)$$

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and $\Gamma(x)$ is the ordinary gamma function defined by [3]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0.$$

The function $G(x)$ is very useful in estimating and summing certain numerical and algebraic series [18]. For more details on bounding the function $\Gamma(x)$ and its logarithmic derivatives $\psi^{(n)}(x)$, please refer to the papers [2]-[5], [7]-[23] and plenty of references therein.

The function $G(x)$ can be also defined by

$$G(x) = \frac{2}{x}\, {}_2F_1(1, x; 1 + x; -1),$$

where

$${}_rF_s(a_1, ..., a_r; b_1, ..., b_s; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k ...(a_r)_k}{(b_1)_k ...(b_s)_k} \frac{x^k}{k!}$$

is the generalized hypergeometric series [1] defined for $r, s \in \mathbb{N}$, $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} - \{0, -1, -2, ...\}$ and the Pochhammer symbol $(a)_n$ is defined by

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = \prod_{i=0}^{n-1} (a + i) = \frac{\Gamma(a + n)}{\Gamma(a)}, \qquad n \geq 1.$$

The function $G(x)$ satisfies the functional equation [6]:

$$G(1 + x) = -G(x) + \frac{2}{x} \tag{2}$$

and it has the integral representation

$$G(x) = 2 \int_0^\infty \frac{e^{-xt}}{1 + e^{-t}} dt, \qquad x > 0 \tag{3}$$

which can be deduced from the following known integral representation of the digamma [3]

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right) dt, \qquad x > 0.$$

Qiu and Vuorinen [24] deduced the inequality

$$\frac{1}{x} + \frac{4(1.5 - \log 4)}{x^2} < G(x) < \frac{1}{x} + \frac{1}{2x^2}, \qquad x > 1/2. \tag{4}$$

Mahmoud and Agarwal [9] presented the following asymptotic formula for Bateman's G-function

$$G(x) \sim \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(2^{2k} - 1) B_{2k}}{k x^{2k}}, \qquad x \to \infty \tag{5}$$

and they deduced the double inequality

$$\frac{1}{2x^2+1.5} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > 0 \tag{6}$$

which improve the lower bound of the inequality (4). Also, Mahmoud and Almuashi [11] proved that the Bateman's $G-$function satisfies the double inequality

$$\sum_{n=1}^{2m} \frac{(2^n-1)B_{2n}}{nx^{2n}} < G(x) - \frac{1}{x} < \sum_{n=1}^{2m-1} \frac{(2^n-1)B_{2n}}{nx^{2n}}, \qquad m \in \mathbb{N} \tag{7}$$

with best bounds, where $B_r{}'s$ are the Bernoulli numbers and they presented some estimates for the error term of a class of the alternating series, which improve and generalize some recent results. Mortici [13] established the inequality

$$0 < \psi(x+v) - \psi(x) \leq \psi(v) + \gamma + \frac{1}{v} - v \qquad x \geq 1;\ 0 < v < 1, \tag{8}$$

where γ is the Euler constant, which also improves the inequality (4) of Qiu and Vuorinen. Also, Alzer presented the double inequality [2]

$$\frac{1}{x} - T_n(v;x) - \rho_n(v;x) < \psi(x+v) - \psi(x) < \frac{1}{x} - T_n(v;x),$$

where $n \geq 0$ be an integer, $x > 0$, $0 < v < 1$,

$$T_n(v;x) = (1-v)\left[\frac{1}{v+n+1} + \sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+v)}\right]$$

and

$$\rho_n(v;x) = \frac{1}{x+n+v} \log \frac{(x+n)^{(x+n)(1-v)}(x+n+1)^{(x+n+1)v}}{(x+n+v)^{x+n+v}}.$$

In 2006, Muqattash and Yahdi [17] presented an infinite family of functions $I_a(x) = \psi(x)$ for a certain a when x is fixed. Local and global bounding error functions are found and new inequalities for the Digamma function are introduced. These functions are shown to approximate ψ locally and asymptotically. The approximations are compared to another approximations of the Digamma function. The technique of construct of Muqattash and Yahdi is very useful and can be updated to another functions as we will see in this paper.

In 2014, Guo and Qi improved the results of [8] and presented the two sharp inequalities

$$\ln\left(x + \frac{1}{2}\right) < \psi(x) + \frac{1}{x} < \ln\left(x + e^{-\gamma}\right), \qquad x > 0$$

where the constants $\frac{1}{2}$ and $e^{-\gamma}$ are the best possible, and

$$\ln\left(n + \frac{1}{2}\right) + \gamma < H_n(n) < \ln\left(n + e^{1-\gamma} - 1\right) + \gamma, \qquad n \in \mathbb{N}$$

where the n-th harmonic numbers are defined by

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad n \in \mathbb{N}$$

and is related to the Psi function by the relation

$$H_n = \gamma + \psi(n+1).$$

In this paper, we presented a family of functions $M(\mu, x)$ satisfies that for all $x > 0$ there exists $\mu \in [1, 2]$ such that $M(\mu, x) = G(x)$ and is asymptotically equivalent to $G(x)$ as $x \to \infty$. We proved that the approximations $M(\mu, x)$ of the function $G(x)$ are of an order of convergence of $O\left(\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right)$ for $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$. Some properties and bounds of the error are deduced. Also, we presented a new sharp double inequality of the function $G(x)$ between the lower bound $M(\frac{4}{e^2-4}, x)$ and the upper bound $M(1, x)$. We proved that the approximations $M(\mu, x)$ are better than the approximation $\frac{1}{x} + \frac{1}{2x^2}$ for any μ in an open subinterval of $\left[1, \frac{4}{e^2-4}\right]$.

2 Main Results

Lemma 2.1. *For $x > 0$, we have*

$$\ln\left(1 + \frac{1}{x+2}\right) + \frac{2}{x(x+1)} \leq G(x) \leq \ln\left(1 + \frac{1}{x+1}\right) + \frac{2}{x(x+1)}. \tag{9}$$

Proof. Consider the function

$$H_\mu(x) = \ln\left(1 + \frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} - G(x), \qquad x > 0;\ \mu > 0$$

which can be represented using (3) by the integral formula

$$H_\mu(x) = \int_0^\infty \frac{e^{-(\mu+1)t}[e^{2t} - 1 - 2te^{\mu t}]}{t(1+e^t)} e^{-xt} dt.$$

The function $m_1(t) = e^{2t} - 1 - 2te^t$ is strictly increasing pass through the origin, then $H_1(x) > 0$, that is

$$\ln\left(1 + \frac{1}{x+1}\right) + \frac{2}{x(x+1)} > G(x).$$

Also, $m_2(t) = e^{2t} - 1 - 2te^{2t}$ is strictly decreasing function pass through the origin, then $H_2(x) < 0$, that is

$$\ln\left(1 + \frac{1}{x+2}\right) + \frac{2}{x(x+1)} < G(x).$$

□

The double inequality (9) show that the function $G(x)$ lies between two functions of the following family of functions

$$M(\mu, x) = \ln\left(1 + \frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} \qquad x > 0;\ \mu > 0. \tag{10}$$

and hence we can conclude the following result:

Theorem 1. *For every $x > 0$, there exists $\mu \in [1, 2]$ such that*

$$M(\mu, x) = G(x).$$

Proof. For a positive fixed x, consider the function $M_2(\mu) = M(\mu, x)$ with $1 \le \mu \le 2$ and $G(x) = \lambda$. $M_2(\mu)$ is a continuous on $[1, 2]$ and using the inequality (9), we obtain

$$M_2(2) \le \lambda \le M_2(1).$$

Then by the Intermediate Value Theorem, there exists $\mu \in [1, 2]$ such that $M_2(\mu) = \lambda$. □

Also, by using the relations

$$\frac{\partial M(\mu, x)}{\partial x} = -\frac{2\mu + 2\mu^2 + 2x + 8\mu x + 4\mu^2 x + 7x^2 + 8\mu x^2 + 6x^3 + x^4}{x^2(1+x)^2(\mu + \mu^2 + x + 2\mu x + x^2)} < 0$$

and

$$\frac{\partial M(\mu, x)}{\partial \mu} = \frac{-1}{(x+\mu+1)(x+\mu)} < 0,$$

we obtain the following properties of the family $M(\mu, x)$.

Lemma 2.2.

1. *$M_1(x) = M(\mu, x)$ is a positive and strictly decreasing as a function of x, $x > 0$.*
2. *$M_2(\mu) = M(\mu, x)$ is strictly decreasing as a function of μ, $1 \le \mu \le 2$*

and hence

$$0 < M(2, x) \le M(\mu, x) \le M(1, x), \qquad x > 0;\ \mu \in [1, 2]. \tag{11}$$

Now, we will show that the family $M(\mu, x)$ presented asymptotical approximation of the Bateman's $G-$function for all $\mu \in [1, 2]$.

Theorem 2. *For all $\mu \in [1, 2]$, the Bateman's $G-$function and the family $M(\mu, x)$ are asymptotically equivalent as $x \to \infty$, that is*

$$\lim_{x\to\infty} \frac{G(x)}{M(\mu, x)} = 1$$

and this is written symbolically as $G(x) \sim M(\mu, x)$.

Proof. Using the inequality (9), we get

$$M(2,x) \leq G(x) \leq M(1,x) \tag{12}$$

and hence

$$\frac{M(2,x)}{M(1,x)} \leq \frac{G(x)}{M(1,x)} \leq 1.$$

But

$$\lim_{x\to\infty} \frac{M(2,x)}{M(1,x)} = \frac{12+34x+23x^2+6x^3+x^4}{(3+x)(4+10x+5x^2+x^3)} = 1$$

and then

$$\lim_{x\to\infty} \frac{G(x)}{M(1,x)} = 1. \tag{13}$$

Similarly, we have

$$\lim_{x\to\infty} \frac{G(x)}{M(2,x)} = 1. \tag{14}$$

Using the inequality (11), we obtain

$$\frac{G(x)}{M(1,x)} \leq \frac{G(x)}{M(\mu,x)} \leq \frac{G(x)}{M(2,x)}. \tag{15}$$

From (13), (14) and (15), we get

$$1 \leq \lim_{x\to\infty} \frac{G(x)}{M(\mu,x)} \leq 1.$$

□

Now, we will study the error of the approximation $M(\mu,x)$ of the function $G(x)$.

Theorem 3. *For any $\mu \in [1,2]$, the error*

$$e_\mu(x) = G(x) - M(\mu,x)$$

approaches zero as $x \to \infty$ and

$$G(x) = \ln\left(1+\frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} + \mathrm{O}\left(\ln\left(1+\frac{1}{(x+1)(x+3)}\right)\right). \tag{16}$$

Proof. From inequality (12), we have

$$M(2,x) - M(\mu,x) \leq G(x) - M(\mu,x) \leq M(1,x) - M(\mu,x)$$

and using (11), we get

$$M(2,x) - M(1,x) \leq M(2,x) - M(\mu,x).$$

Hence

$$0 \leq |G(x) - M(\mu,x)| \leq M(1,x) - M(2,x) \tag{17}$$

or

$$0 \le |e_\mu(x)| \le \ln\left(1 + \frac{1}{(x+1)(x+3)}\right). \tag{18}$$

Then

$$G(x) = M(\mu, x) + \mathrm{O}\left(\ln\left(1 + \frac{1}{(x+1)(x+3)}\right)\right)$$

and

$$\lim_{x\to\infty} e_\mu(x) = 0.$$

□

As a consequence of the above result, we obtain some bounds of the error $e_\mu(x)$.

Corollary 2.3. *The error $e_\mu(x)$ is uniformly bounded by $\pm\ln\left(1+\frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$ $\forall\ x > \varepsilon > 0$ and $\forall\ \mu \in [1,2]$.*

Proof. Using the inequality (18), we obtain

$$\sup_{0<x<\infty} |e_\mu(x)| \le \ln\left(1 + \frac{1}{(x+1)(x+3)}\right).$$

Also, the function $g(x) = \ln\left(1+\frac{1}{(x+1)(x+3)}\right)$ for $x > 0$ is decreasing. Then the errors $e_\mu(x)$ are uniformly bounded between $-\ln\left(1+\frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$ and $\ln\left(1+\frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$. □

3 The best bounds of the double inequality (9).

Firstly, we will prove the following auxiliary results:

Lemma 3.1.

$$\lim_{x\to\infty}\left(\frac{1}{e^{G(x+2)}-1} - x\right) = 1 \tag{19}$$

and

$$\lim_{x\to\infty} \frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)}-1)^2} = -1. \tag{20}$$

Proof. Using the double inequality (6) with

$$\beta(x) = \frac{1}{x} + \frac{1}{2x^2+3/2} \quad \text{and} \quad \alpha(x) = \frac{1}{x} + \frac{1}{2x^2},$$

we get

$$\lim_{x\to\infty}\left(\frac{1}{e^{\alpha(x+2)}-1} - x\right) \le \lim_{x\to\infty}\left(\frac{1}{e^{G(x+2)}-1} - x\right) \le \lim_{x\to\infty}\left(\frac{1}{e^{\beta(x+2)}-1} - x\right).$$

But

$$\lim_{x\to\infty}\left(\frac{1}{e^{\alpha(x+2)}-1} - x\right) = \lim_{x\to\infty}\left(\frac{1}{\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{5}{12x^4}-O\left(\frac{1}{x^5}\right)\right]-1} - x\right) = 1$$

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and

$$\lim_{x\to\infty}\left(\frac{1}{e^{\beta(x+2)}-1}-x\right)=\lim_{x\to\infty}\left(\frac{1}{\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]-1}-x\right)=1.$$

Also, using the double inequality (6), we have

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\alpha(x+2)}}{(e^{\beta(x+2)}-1)^2}\le\lim_{x\to\infty}\frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)}-1)^2}\le\lim_{x\to\infty}\frac{G'(x+2)e^{\beta(x+2)}}{(e^{\alpha(x+2)}-1)^2}.$$

Now, using the asymptotic formula for Bateman's G-function (5), we obtain

$$G'(x)=\frac{-1}{x^2}-O\left(\frac{1}{x^3}\right).$$

Then

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\alpha(x+2)}}{(e^{\beta(x+2)}-1)^2}=\lim_{x\to\infty}\frac{\left[\frac{-1}{(x+2)^2}-O\left(\frac{1}{x^3}\right)\right]\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{5}{12x^4}-O\left(\frac{1}{x^5}\right)\right]}{\left(\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]-1\right)^2}=-1$$

and

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\beta(x+2)}}{(e^{\alpha(x+2)}-1)^2}=\lim_{x\to\infty}\frac{\left[\frac{-1}{(x+2)^2}-O\left(\frac{1}{x^3}\right)\right]\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]}{\left(\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{5}{12x^4}-O\left(\frac{1}{x^5}\right)\right]-1\right)^2}=-1$$

□

Now, we will present the sharp bounds of the double inequality (9).

Theorem 4. *For all* $x\in(0,\infty)$

$$\ln\left(1+\frac{1}{x+\frac{4}{e^2-4}}\right)+\frac{2}{x(x+1)}<G(x)<\ln\left(1+\frac{1}{x+1}\right)+\frac{2}{x(x+1)},\tag{21}$$

where the constants 1 *and* $\frac{4}{e^2-4}$ *are the best possible.*

Proof. Using the inequality (9) and functional equation (2), we get

$$0<\frac{1}{e^{G(x+2)}-1}-x<2.$$

Now consider the two functions

$$f(x)=e^{G(x+2)}-1,\qquad x>0$$

and

$$q(x)=\frac{1}{f(x)}-x,\qquad x>0.$$

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Then $f'(x) = G'(x+2)e^{G(x+2)} < 0$ and $f(x)$ is strictly decreasing function. Hence $\frac{1}{f(x)}$ is strictly increasing function. Since $\frac{d}{dx}\frac{1}{f(x)}|_{x=0} \simeq 0.91$, and $\frac{d}{dx}\frac{1}{f(x)}|_{x=1} \simeq 0.96$. Then the function $\frac{1}{f(x)}$ is convex and $\frac{d}{dx}\frac{1}{f(x)}$ is increasing function. Thus we get

$$\frac{d}{dx}\frac{1}{f(x)} < \lim_{x\to\infty}\frac{d}{dx}\frac{1}{f(x)} = -\lim_{x\to\infty}\frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)}-1)^2}.$$

Using the limit (20), we obtain

$$\frac{d}{dx}\frac{1}{f(x)} < 1, \qquad x > 0.$$

Then $q(x)$ is strictly decreasing function for all $x > 0$, where $\frac{dq(x)}{dx} = \frac{d}{dx}\frac{1}{f(x)} - 1 < 0$. Hence

$$\lim_{x\to\infty} q(x) < q(x) < \lim_{x\to 0^+} q(x)$$

and using the limit (19) and $G(2) = 2 - \ln 4$, we have

$$1 < q(x) < \frac{4}{e^2-4}. \tag{22}$$

with best bounds. □

In the proof of theorem (4), we proved that the function $\frac{1}{f(x)}$ is convex. Also, the second derivatives of the functions $q(x)$ and $\frac{1}{f(x)}$ have the same sign, then we get the following resuts:

Corollary 3.2. *The function $q(x)$ is strictly decreasing and convex for all $x > 0$.*

Corollary 3.3. *For every $x > 0$ there exists a unique number $\mu \in \left(1, \frac{4}{e^2-4}\right)$ such that $G(x) = M(\mu, x)$. Conversely for every $\mu \in \left(1, \frac{4}{e^2-4}\right)$ there exists a unique number $x > 0$ such that $M(\mu, x) = G(x)$.*

Proof. The function $q(x)$ is strictly decreasing from $(0,\infty)$ onto $\left(1, \frac{4}{e^2-4}\right)$ then the mapping $q(x) : (0,\infty) \to \left(1, \frac{4}{e^2-4}\right)$ is bijective and the proof is easy consequence of this result. □

Corollary 3.4. *For $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$ we have*

1) the errors $e_\mu(x)$ are uniformly bounded by $\pm \ln\left(\frac{4(2e^2-4)}{3(3e^2-8)}\right)$.

2) $G(x) = M(\mu, x) + O\left(\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right)$.

Proof. Analogues to inequality (17), we can deduce for all $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$ that

$$0 \le |G(x) - M(\mu, x)| \le \left| M(1, x) - M\left(\frac{4}{e^2-4}, x\right)\right|$$

which is equivalent to

$$0 \le |e_\mu(x)| \le \left|\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right| \le \left|\ln\left(\frac{4(2e^2-4)}{3(3e^2-8)}\right)\right|.$$

□

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4 Comparing approximations

Firstly, we will prove the following one side inequality the function $G(x)$ which proves a special case of a conjecture posed in [9] and proved in [11] about the best bounds of the Bateman's function but with different proof.

Lemma 4.1. *For all $x > 0$, we have*

$$G(x) - \frac{1}{x} > \frac{1}{2x^2} - \frac{1}{4x^4}. \tag{23}$$

Proof. Consider the function

$$K(x) = G(x) - \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{4x^4}, \qquad x > 0.$$

Using the integral representation (3) of $G(x)$ and the formula

$$\frac{1}{x^r} = \frac{1}{(r-1)!}\int_0^\infty t^{r-1}e^{-xt}dt, \qquad r \in N$$

we get

$$K(x) = \int_0^\infty \varphi(t)\frac{e^{-xt}}{1+e^t}dt,$$

where

$$\varphi(t) = e^t - 1 - \frac{1}{2}t(1+e^t) + \frac{1}{24}t^3(1+e^t).$$

But

$$\begin{aligned}
\varphi(t) &= \sum_{k=4}^{\infty}\frac{t^k}{k!} - \frac{1}{2}\sum_{k=3}^{\infty}\frac{t^{k+1}}{k!} + \frac{1}{24}\sum_{k=1}^{\infty}\frac{t^{k+3}}{k!} \\
&= \sum_{k=0}^{\infty}\frac{t^{(k+4)}}{(k+4)!}(1 + \frac{1}{24}(k+4)[(k+3)(k+2) - 12]) \\
&= \sum_{k=0}^{\infty}\frac{t^{(k+5)}}{(k+5)!}(1 + \frac{1}{24}k(k+5)(k+7)) > 0.
\end{aligned}$$

Hence $\varphi(x) > 0$ and then $K(x) > 0$. □

As by-product of the the inequalities (6) and (23), we obtain the following double inequality.

Corollary 4.2. *For all $x > 1$, we have*

$$0 < \frac{(2x+1)(x-1)(x^2+1)}{2x^4(x+1)} < 2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \frac{2x^2 - x + 1}{2x^2(x+1)}. \tag{24}$$

Now, we will prove the following auxiliary results:

Lemma 4.3. *For all $x > x_0 \approx 2.5315129$, we have*

$$\frac{1}{e^{2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2}} - 1} - x > \frac{1}{e^{\frac{2x^2 - x + 1}{2x^2(x+1)}} - 1} - x > 1. \tag{25}$$

Proof. Using the inequality (24), we have

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{x+2}{x+1}\right) < u(x)$$

where

$$u(x) = \frac{2x^2 - x + 1}{2x^2(x+1)} - \ln\left(\frac{x+2}{x+1}\right), \qquad x > 0.$$

Then

$$u'(x) = \frac{(x - \frac{3+\sqrt{17}}{2})(x - \frac{3-\sqrt{17}}{2})}{x^3(x+1)^2},$$

and the function $u(x)$ has only one positive critical point at $x_m = \frac{3+\sqrt{17}}{2}$. Now,

$$u(x_m) = \frac{10}{(3+\sqrt{17})^2} - \ln\frac{7+\sqrt{17}}{5+\sqrt{17}} \approx -0.00113 < 0,$$

$$\lim_{x\to\infty} u(x) = 0$$

and

$$\lim_{x\to 0^-} u(x) = \infty.$$

Hence $u(x)$ has only one positive root $x_0 \approx 2.5315129$ and

$$u(x) < 0, \qquad \forall x > x_0.$$

Then

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \ln\left(\frac{x+2}{x+1}\right), \qquad \forall x > x_0.$$

□

Lemma 4.4. *For all $x > x_1 \approx 2.6925094$, we have*

$$\frac{1}{e^{2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2}} - 1} - x < \frac{4}{e^2 - 4}. \tag{26}$$

Proof. Using the inequality (24), we have

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{e^2 + (e^2-4)x)}{4 + (e^2-4)x}\right) > v(x),$$

where

$$v(x) = \frac{(2x+1)(x-1)(x^2+1)}{2x^4(x+1)} - \ln\left(\frac{e^2 + (e^2-4)x)}{4 + (e^2-4)x}\right), \qquad x > 1.$$

Hence

$$v'(x) = \frac{L(x)}{S(x)},$$

where

$$L(x) = 8e^2 + (-32 + 16e^2 + 2e^4)x + (-32 - 12e^2 + 6e^4)x^2 + (48 - 36e^2 + 5e^4)x^3 + (32 - 4e^2)x^4 \\ +(-16 - 4e^2 + e^4)x^5 + (64 - 24e^2 + 2e^4)x^6$$

and

$$S(x) = x^5(x+1)^2(4e^2 + (e^4 - 16)x + (16 - 8e^2 + e^4)x^2 > 0, \qquad x > 0.$$

The function $L''(x)$ is a polynomial of fourth degree has one positive root at $x_I \approx 2.31866$ with $L''(3) < 0$, then $L(x)$ is concave function on (x_I, ∞). Also, $L(x_I) > 0$ and $\lim_{x\to\infty} L(x) = -\infty$. Hence, the function $L(x)$ has only one root on (x_I, ∞) at $x_3 \approx 4.0635204$, where $L(4.063) > 0$ and $L(4.064) < 0$. Then $L(x) > 0$ on $[x_I, x_3)$ and $L(x) < 0$ for all $x > x_3$. Hence $v(x)$ is increasing on (x_I, x_3) and decreasing function on (x_3, ∞) and it has a maximum point at x_3. But $v(2.69) < 0$ and $v(2.7) > 0$ and then $v(x)$ has a root $x_1 \approx 2.6925094 \in (x_I, x_3)$. Also, $\lim_{x\to\infty} v(x) = 0$, then we have

$$v(x) > 0, \qquad x > x_1$$

and hence

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{e^2 + (e^2-4)x}{4 + (e^2-4)x}\right) > 0, \qquad x > x_1.$$

□

Theorem 5. *For a fixed $x > x_1$, consider I_x be the nonempty open interval of $\left[1, \frac{4}{e^2-4}\right]$ defined by*

$$I_x = \left(\frac{1}{e^{-\frac{2}{x(x+1)}+\frac{1}{x}+\frac{1}{2x^2}} - 1} - x, \frac{1}{e^{2G(x)-\frac{2}{x(x+1)}-\frac{1}{x}-\frac{1}{2x^2}} - 1} - x\right).$$

For any $\mu \in I_x$, we have

$$|e_\mu(x)| < \left|G(x) - \left(\frac{1}{x} + \frac{1}{2x^2}\right)\right|.$$

Proof. Using the inequalities (25) and (26), we obtain

$$I_x \subset \left[1, \frac{4}{e^2-4}\right].$$

For any positive real number μ,

$$\frac{1}{e^{-\frac{2}{x(x+1)}+\frac{1}{x}+\frac{1}{2x^2}} - 1} - x < \mu \ \text{ iff } \ -M(\mu, x) > -\frac{1}{x} - \frac{1}{2x^2}$$

and hence

$$\frac{1}{e^{-\frac{2}{x(x+1)}+\frac{1}{x}+\frac{1}{2x^2}} - 1} - x < \mu \ \text{ iff } \ G(x) - M(\mu, x) > G(x) - \frac{1}{x} - \frac{1}{2x^2}. \tag{27}$$

Also,

$$\frac{1}{e^{2G(x)-\frac{2}{x(x+1)}-\frac{1}{x}-\frac{1}{2x^2}} - 1} - x > \mu \ \text{ iff } \ 2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \ln\left(1 + \frac{1}{x+\mu}\right)$$

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and hence

$$\frac{1}{e^{2G(x)-\frac{2}{x(x+1)}-\frac{1}{x}-\frac{1}{2x^2}}-1}-x>\mu \text{ iff } G(x)-M(\mu,x)<-G(x)+\frac{1}{x}+\frac{1}{2x^2}. \tag{28}$$

From the inequalities (27) and (28) we have

$$G(x)-\frac{1}{x}-\frac{1}{2x^2}<G(x)-M(\mu,x)<-G(x)+\frac{1}{x}+\frac{1}{2x^2}, \qquad \forall \mu \in I_x.$$

Thus

$$|G(x)-M(\mu,x)|<\left|G(x)-\left(\frac{1}{x}+\frac{1}{2x^2}\right)\right|, \qquad \forall \mu \in I_x. \tag{29}$$

□

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DIFFERENTIAL EQUATIONS ASSOCIATED WITH MODIFIED DEGENERATE BERNOULLI AND EULER NUMBERS

TAEKYUN KIM, DAE SAN KIM, HYUCK IN KWON, AND JONG JIN SEO

Abstract. In this paper, we consider some ordinary differential equations associated with modified degenerate Euler and Bernoulli numbers and give some new identities for these numbers arising from our differential equations.

1. Introduction

As is well known, Bernoulli numbers are defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad (\text{see [1–12]})\,,\tag{1.1}$$

and the Euler numbers are given by generating function

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad (\text{see [7, 8]})\,.\tag{1.2}$$

In [2], L. Carlitz considered the degenerate Bernoulli and Euler numbers which are defined by the generating functions

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}=\sum_{n=0}^{\infty}\beta_n(\lambda)\frac{t^n}{n!},\tag{1.3}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\frac{t^n}{n!}.\tag{1.4}$$

Note that $\lim_{\lambda\to 0}\beta_n(\lambda)=B_n$ and $\lim_{\lambda\to 0}\mathcal{E}_n(\lambda)=E_n$, $(n\geq 0)$.

Now, we define the modified degenerate Bernoulli and Euler numbers which are slightly different from the Carlitz degenerate Bernoulli and Euler numbers as follows:

$$\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}=\sum_{n=0}^{\infty}\tilde{\beta}_n(\lambda)\frac{t^n}{n!},\quad (\text{see [3]})\,,\tag{1.5}$$

and

$$\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_n(\lambda)\frac{t^n}{n!},\quad (\text{see [9]})\,.\tag{1.6}$$

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From (1.5) and (1.4), we easily note that

$$\lim_{\lambda\to 0}\tilde{\beta}_n(\lambda)=B_n \quad \text{and} \quad \lim_{\lambda\to 0}\tilde{\mathcal{E}}_n(\lambda)=E_n, \quad (n\geq 0). \tag{1.7}$$

For $r\in\mathbb{N}$, the higher-order modified Bernoulli and Euler numbers are also defined by the generating functions

$$\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^r=\sum_{n=0}^{\infty}\tilde{\beta}_n^{(r)}(\lambda)\frac{t^n}{n!}, \tag{1.8}$$

and

$$\left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right)^r=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_n^{(r)}(\lambda)\frac{t^n}{n!}. \tag{1.9}$$

Recall that the higher order Bernoulli and Euler numbers are given by the generating functions

$$\left(\frac{t}{e^t-1}\right)^r=\sum_{n=0}^{\infty}B_n^{(r)}\frac{t^n}{n!}, \tag{1.10}$$

and

$$\left(\frac{2}{e^t+1}\right)^r=\sum_{n=0}^{\infty}E_n^{(r)}\frac{t^n}{n!}, \quad (\text{see [6, 11]}). \tag{1.11}$$

From (1.8), (1.9), (1.10) and (1.11), we note that

$$\lim_{\lambda\to 0}\tilde{\beta}_n^{(r)}(\lambda)=B_n^{(r)} \quad \text{and} \quad \lim_{\lambda\to 0}\tilde{\mathcal{E}}_n^{(r)}(\lambda)=E_n^{(r)}.$$

In [1], Bayad-Kim studied the following nonlinear differential equations:

$$F_q^N=\frac{1}{(N-1)!}\sum_{k=1}^{N}a_k(N)F_q^{(k-1)}, \quad (N\in\mathbb{N}), \tag{1.12}$$

where $F^{(k)}=F^{(k)}(t)=\left(\frac{d}{dt}\right)^kF$.

For $F_q(t)=\frac{1}{qe^t\pm 1}$, Bayad-Kim gave explicit formulae for Apostol-Bernoulli and Apostol-Euler numbers and polynomials which are derived from (1.12).

In [4], Guo-Qi obtained the following results

$$\left(\frac{d}{dt}\right)^k\left(\frac{1}{\lambda e^{\alpha t}-1}\right)=(-1)^k\alpha^k\sum_{m=1}^{k+1}(m-1)!S_2(k+1,m)\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^m, \tag{1.13}$$

and

$$\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^k=\frac{1}{(k-1)!}\sum_{m=1}^{k}\frac{(-1)^{m-1}}{\alpha^{m-1}}S_1(k,m)\left(\frac{d}{dt}\right)^{m-1}\left(\frac{1}{\lambda e^{\alpha t}-1}\right), \tag{1.14}$$

where $k\in\mathbb{N}$, and $S_1(k,m)$ and $S_2(k,m)$ are respectively the Stirling numbers of the first kind and of the second kind (see [4, 10]). However, the results of Guo-Qi are immediately obtained from the paper of Bayad-Kim in [1] by replacing q by λ and t by αt (α =constnat).

Recently, Kim-Kim studied the nonlinear differential equations given by

$$\left(\frac{d}{dt}\right)^N\left(\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}\pm 1}\right)=\frac{(-1)^N}{(1+\lambda t)^N}\sum_{i=1}^{N+1}a_i(N,\lambda)F^i, \tag{1.15}$$

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where

$$F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} \pm 1} \quad (\text{see } [7]).$$

From (1.15), we derived some new identities involving degenerate Euler and Bernoulli polynomials.

In this paper, along the same line as [7] we study some ordinary differential equations arising from the generating functions of the modified degenerate Bernoulli and Euler numbers. From those equations, we derive some new identities for the modified degenerate Bernoulli and Euler numbers.

2. Differential equations associated with modified degenerate Bernoulli and Euler numbers

Let

$$F = F(t) = \left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-1}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} F^{(1)} = \frac{dF}{dt} &= -\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-2} (1+\lambda)^{\frac{t}{\lambda}} \frac{1}{\lambda}\log(1+\lambda) \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-2}\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1 \mp 1\right) \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left(F \mp F^2\right), \end{aligned} \tag{2.2}$$

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left(F^{(1)} \mp 2FF^{(1)}\right) \\ &= -\frac{1}{\lambda}\log(1+\lambda)(1 \mp 2F)F^{(1)} \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^2 (1 \mp 2F)\left(F \mp F^2\right) \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^2 \left(F \mp 3F^2 + 2F^3\right). \end{aligned} \tag{2.3}$$

Thus we are led to put

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{\pm}(N) F^i, \quad (N = 0, 1, 2, \dots), \end{aligned} \tag{2.4}$$

where $a_{i-1}^{+}(N)$ corresponds to $\left((1+\lambda)^{\frac{t}{\lambda}} + 1\right)^{-1}$ and $a_{i-1}^{+}(N)$ does to $\left((1+\lambda)^{\frac{t}{\lambda}} - 1\right)^{-1}$.

Now, from (2.4), we have

$$\begin{aligned} &F^{(N+1)} \\ &= \frac{d}{dt}F^{(N)} \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1} a_{i-1}^{\pm}(N)\, iF^{i-1}F^{(1)} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\left\{\sum_{i=1}^{N+1} ia_{i-1}^{\pm}(N)\,F^{i} \mp \sum_{i=2}^{N+2}(i-1)\,a_{i-2}^{\pm}(N)\,F^{i}\right\} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\Big\{a_0^{\pm}(N)\,F \mp (N+1)\,a_N^{\pm}(N)\,F^{N+2} \\
&\quad + \sum_{i=2}^{N+1}\left(ia_{i-1}^{\pm}(N) \mp (i-1)\,a_{i-2}^{\pm}(N)\right)F^{i}\Big\}.
\end{aligned}
$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\sum_{i=1}^{N+2} a_{i-1}^{\pm}(N+1)\,F^{i}. \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain

$$a_0^{\pm}(N+1) = a_0^{\pm}(N), \tag{2.7}$$

$$a_{N+1}^{\pm}(N+1) = \mp(N+1)\,a_N^{\pm}(N), \tag{2.8}$$

and

$$a_{i-1}^{\pm}(N+1) = ia_{i-1}^{\pm}(N) \mp (i-1)\,a_{i-2}^{\pm}(N), \tag{2.9}$$

for $2 \le i \le N+1$.

Also, by (1.12), we get

$$F = F^{(0)} = a_0^{\pm}(0)\,F. \tag{2.10}$$

Thus, by (2.10), we see that

$$a_0^{\pm}(0) = 1. \tag{2.11}$$

It is easy to show that

$$
\begin{aligned}
F^{(1)} &= -\frac{1}{\lambda}\log(1+\lambda)\sum_{i=1}^{2} a_{i-1}^{\pm}(1)\,F^{i} \\
&= -\frac{1}{\lambda}\log(1+\lambda)\left(a_0^{\pm}(1)\,F + a_1^{\pm}(1)\,F^2\right) \\
&= -\frac{1}{\lambda}\log(1+\lambda)\left(F \mp F^2\right).
\end{aligned}
\tag{2.12}
$$

Thus, by comparing the coefficients on both sides of (2.12), we have

$$a_0^{\pm}(1) = 1, \quad a_1^{\pm}(1) = \mp 1. \tag{2.13}$$

From (2.7) and (2.8), we note that

$$a_0^{\pm}(N+1) = a_0^{\pm}(N) = \cdots = a_0^{\pm}(0) = 1, \tag{2.14}$$

and

$$
\begin{aligned}
a_{N+1}^{+}(N+1) &= -(N+1)\,a_N^{+}(N) \\
&= (-1)^2(N+1)\,Na_{N-1}^{+}(N-1) \\
&\;\;\vdots
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&= (-1)^{N+1} (N+1)! a_0^+ (0) \\
&= (-1)^{N+1} (N+1)!,
\end{aligned}
$$

$$
\begin{aligned}
a_{N+1}^- (N+1) &= (N+1)\, a_N^- (N) \\
&= (N+1)\, N a_{N-1}^- (N-1) \\
&\vdots \\
&= (N+1)! a_0^- (0) \\
&= (N+1)!.
\end{aligned} \tag{2.16}
$$

By (2.15) and (2.16), we easily get

$$
a_{N+1}^{\pm} (N+1) = (\mp 1)^{N+1} (N+1)!. \tag{2.17}
$$

Observe also that the matrix $\left(a_i^+ (j)\right)_{0\le i,j\le N}$ and $\left(a_i^- (j)\right)_{0\le i,j\le N}$ are as follows:

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ \\ N
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & & N \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
 & (-1)1! & & & & \\
 & & (-1)^2 2! & & & \\
 & & & (-1)^3 3! & & \\
 & & 0 & & \ddots & \\
 & & & & & (-1)^N N!
\end{bmatrix}
\end{array}
= \left(a_i^+ (j)\right)_{0\le i,j\le N}
$$

and

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ \\ N
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & & N \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
 & 1! & & & & \\
 & & 2! & & & \\
 & & & 3! & & \\
 & & 0 & & \ddots & \\
 & & & & & N!
\end{bmatrix}
\end{array}
= \left(a_i^- (j)\right)_{0\le i,j\le N}
$$

For $i=2$ in (2.9), we have

$$
\begin{aligned}
& a_1^{\pm} (N+1) \\
&= \mp a_0^{\pm} (N) + 2 a_1^{\pm} (N) \\
&= \mp a_0^{\pm} (N) + 2 \left(\mp a_0^{\pm} (N-1) + 2 a_1^{\pm} (N-1)\right) \\
&= \mp \left(a_0^{\pm} (N) + 2 a_0^{\pm} (N-1)\right) + 2^2 a_1^{\pm} (N-1) \\
&= \mp \left(a_0^{\pm} (N) + 2 a_0^{\pm} (N-1)\right) + 2^2 \left(\mp a_0^{\pm} (N-2) + 2 a_1^{\pm} (N-2)\right) \\
&= \mp \left(a_0^{\pm} (N) + 2 a_0^{\pm} (N-1) + 2^2 a_0^{\pm} (N-2)\right) + 2^3 a_1^{\pm} (N-2) \\
&\vdots
\end{aligned} \tag{2.18}
$$

$$= \mp \sum_{i=0}^{N-1} 2^i a_0^{\pm}(N-i) + 2^N a_1^{\pm}(1) = \mp \sum_{i=0}^{N} 2^i a_0^{\pm}(N-i).$$

Let us take $i=3$ in (2.9). Then, we note that

$$\begin{aligned}
&a_2^{\pm}(N+1) \\
&= \mp 2a_1^{\pm}(N) + 3a_2^{\pm}(N) \\
&= \mp 2a_1^{\pm}(N) + 3\left(\mp 2a_1^{\pm}(N-1) + 3a_2^{\pm}(N-1)\right) \\
&= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1)\right) + 3^2 a_2^{\pm}(N-1) \\
&= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1)\right) + 3^2\left(\mp 2a_1^{\pm}(N-2) + 3a_2^{\pm}(N-2)\right) \\
&= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1) + 3^2 a_1^{\pm}(N-2)\right) + 3^3 a_2^{\pm}(N-2) \\
&\vdots \\
&= \mp 2\sum_{i=0}^{N-2} 3^i a_1^{\pm}(N-i) + 3^{N-1} a_2^{\pm}(2) \\
&= \mp 2\sum_{i=0}^{N-1} 3^i a_1^{\pm}(N-i).
\end{aligned} \tag{2.19}$$

For $i=4$ in (2.9), we have

$$\begin{aligned}
&a_3^{\pm}(N+1) \\
&= \mp 3a_2^{\pm}(N) + 4a_3^{\pm}(N) \\
&= \mp 3a_2^{\pm}(N) + 4\left(\mp 3a_2^{\pm}(N-1) + 4a_3^{\pm}(N-1)\right) \\
&= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1)\right) + 4^2 a_3^{\pm}(N-1) \\
&= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1)\right) + 4^2\left(\mp 3a_2^{\pm}(N-2) + 4a_3^{\pm}(N-2)\right) \\
&= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1) + 4^2 a_2^{\pm}(N-2)\right) + 4^3 a_3^{\pm}(N-2) \\
&\vdots \\
&= \mp 3\sum_{i=0}^{N-3} 4^i a_2^{\pm}(N-i) + 4^{N-2} a_3^{\pm}(3) \\
&= \mp \sum_{i=0}^{N-2} 4^i a_2^{\pm}(N-i).
\end{aligned} \tag{2.20}$$

Continuing this process, we can deduce that

$$a_j^{\pm}(N+1) = \mp j \sum_{i=0}^{N-j+1} (j+1)^i a_{j-1}^{\pm}(N-i), \tag{2.21}$$

for $1 \le j \le N$.

Now, we give explicit expression for $a_j^{\pm}(N+1)$, $(1 \le j \le N)$.

$$a_1^{\pm}(N+1) = \mp \sum_{i_1=0}^{N} 2^{i_1}, \tag{2.22}$$

$$\begin{aligned} a_2^{\pm}(N+1) &= \mp 2 \sum_{i_2=0}^{N-1} 3^{i_2} a_1^{\pm}(N-i_2) \\ &= \mp 2 \sum_{i_2=0}^{N-1} 3^{i_2} (\mp 1) \sum_{i_1=0}^{N-i_2-1} 2^{i_1} \\ &= (\mp 1)^2\, 2! \sum_{i_2=0}^{N-1} \sum_{i_1=0}^{N-1-i_2} 3^{i_2} 2^{i_1}, \end{aligned} \tag{2.23}$$

and, by (2.23), we get

$$\begin{aligned} & a_3^{\pm}(N+1) \\ &= \mp 3 \sum_{i_3=0}^{N-2} 4^{i_3} a_2^{\pm}(N-i_3) \\ &= \mp 3 \sum_{i_3=0}^{N-2} 4^{i_3} (\mp 1)^2\, 2! \sum_{i_2=0}^{N-i_3-2} \sum_{i_1=0}^{N-i_3-i_2-2} 3^{i_2} 2^{i_1} \\ &= (\mp 1)^3\, 3! \sum_{i_3=0}^{N-2} \sum_{i_2=0}^{N-2-i_3} \sum_{i_1=0}^{N-2-i_3-i_2} 4^{i_3} 3^{i_2} 2^{i_1}. \end{aligned} \tag{2.24}$$

So, we can deduce that

$$a_j^{\pm}(N+1) = (\mp 1)^j\, j! \sum_{i_j=0}^{N-j+1} \sum_{i_{j-1}=0}^{N-j+1-i_j} \cdots \sum_{i_1=0}^{N-j+1-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1}, \tag{2.25}$$

where $1 \le j \le N$.

Remark. Observe that $a_{N+1}^{\pm}(N+1) = (\mp 1)^{N+1}(N+1)!$ is the same as the above expression with $j = N+1$. Therefore, by (2.4) and (2.25), we obtain the following theorem.

Theorem 1. *The ordinary differential equations*

$$F^{(N)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{-}(N) F^i, \quad (N = 0, 1, 2, \dots),$$

have a solution $F = F(t) = \frac{1}{(1+\lambda)^{\frac{t}{\lambda}} - 1}$, *where* $a_0^{-}(N) = 1$,

$$a_j^{-}(N) = j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1},$$

for $1 \le j \le N$.

Theorem 2. *The ordinary differential equations*

$$F^{(N)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{+}(N) F^i, \quad (N = 0, 1, 2, \dots),$$

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have a solution $F = F(t) = \frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1}$, *where* $a_0^+(N) = 1$,

$$a_j^+(N) = (-1)^j j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1},$$

for $1 \le j \le N$.

Now, we observe that

$$\begin{aligned}(2.26)\qquad & \sum_{k=0}^{\infty} \tilde{\mathcal{E}}_{k+N}(\lambda) \frac{t^k}{k!} \\ &= \left(\sum_{k=0}^{\infty} \tilde{\mathcal{E}}_k(\lambda) \frac{t^k}{k!} \right)^{(N)} \\ &= 2\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^{(N)} \\ &= 2\left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} a_{i-1}^+(N) \left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^i \\ &= \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} a_{i-1}^+(N) 2^{1-i} \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^i \\ &= \sum_{k=0}^{\infty} \left(\left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N) \mathcal{E}_k^{(i)}(\lambda) \right) \frac{t^k}{k!}.\end{aligned}$$

Thus, by comparing the coefficients on both sides of (2.26), we get

$$\tilde{\mathcal{E}}_{k+N}(\lambda) = \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N) \tilde{\mathcal{E}}_k^{(i)}(\lambda), \tag{2.27}$$

for $k, N = 0, 1, 2, \ldots$.

Therefore, by (2.27), we obtain the following theorem.

Theorem 3. *For* $k, N = 0, 1, 2, \ldots,$ *we have*

$$\tilde{\mathcal{E}}_{k+N}(\lambda) = \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N) \tilde{\mathcal{E}}_k^{(i)}(\lambda),$$

where $a_0^+(N) = 1$,

$$a_j^+(N) = (-1)^j j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1}, \tag{2.28}$$

where $1 \le j \le N$.

Corollary 4. $\tilde{\mathcal{E}}_N(x) = \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N)$.

Replacing t by $\frac{t}{\lambda} \log(1+\lambda)$ in (1.11), we obtain

$$\sum_{n=0}^{\infty} \tilde{\mathcal{E}}_n^{(r)}(\lambda) \frac{t^n}{n!} = \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^r \tag{2.29}$$

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$$= \sum_{n=0}^{\infty} E_n^{(r)} \frac{\left(\frac{1}{\lambda}\log(1+\lambda)\,t\right)^n}{n!}.$$

Thus, by (2.29), we get

$$\tilde{\mathcal{E}}_n^{(r)}(\lambda) = \left(\frac{1}{\lambda}\log(1+\lambda)\right)^n E_n^{(r)}, \quad (n \geq 0). \tag{2.30}$$

From (2.30), we obtain the following corollary.

Corollary 5. *For $k, N = 0, 1, 2, \ldots,$ we have*

$$E_{k+N} = (-1)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^{+}(N)\, E_k^{(i)},$$

where $a_j^{+}(N)(0 \leq j \leq N)$ are as in (2.28).

From (1.3), we note that

$$\begin{aligned} &\frac{1}{(1+\lambda)^{\frac{t}{\lambda}} - 1} \\ &= \sum_{k=0}^{\infty} \tilde{\beta}_k(\lambda) \frac{t^{k-1}}{k!} \\ &= \sum_{k=1}^{\infty} \tilde{\beta}_k(\lambda) \frac{t^{k-1}}{k!} + \tilde{\beta}_0(\lambda)\frac{1}{t} \\ &= \sum_{k=0}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{t^k}{(k+1)!} + \frac{\lambda}{\log(1+\lambda)} t^{-1}. \end{aligned} \tag{2.31}$$

Thus, by (2.31), we get

$$\begin{aligned} &\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}} - 1}\right)^{(N)} \\ &= \sum_{k=N}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{(k)_N}{(k+1)!} t^{k-N} \\ &\quad + (-1)^N N! \frac{\lambda}{\log(1+\lambda)} t^{-N-1}. \end{aligned} \tag{2.32}$$

From (2.32), we note that

$$\begin{aligned} &t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}} - 1}\right)^{(N)} \\ &= \sum_{k=N}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{(k)_N}{(k+1)!} t^{k+1} + (-1)^N N! \frac{\lambda}{\log(1+\lambda)} \\ &= \sum_{k=N+1}^{\infty} \tilde{\beta}_k(\lambda)(k-1)_N \frac{t^k}{k!} + (-1)^N N! \frac{\lambda}{\log(1+\lambda)}. \end{aligned} \tag{2.33}$$

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On the other hand, by Theorem 1, we get

$$
\begin{aligned}
(2.34)\qquad & t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{(N)} \\
&= t^{N+1}\left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1}a_{i-1}^{-}(N)\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{i} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1}a_{i-1}^{-}(N)\,t^{N+1-i}\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{i} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1}a_{i-1}^{-}(N)\,t^{N+1-i}\sum_{l=0}^{\infty}\tilde{\beta}_{l}^{(i)}(\lambda)\frac{t^{l}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}a_{N-i}^{-}(N)\sum_{l=0}^{\infty}\tilde{\beta}_{l}^{(N+1-i)}(\lambda)\frac{t^{l+i}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}\sum_{l=0}^{\infty}a_{N-i}^{-}(N)\,\tilde{\beta}_{l}^{(N+1-i)}(\lambda)\frac{t^{l+i}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}\sum_{k=i}^{\infty}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)\frac{t^{k}}{(k-i)!}
\end{aligned}
$$

From (2.34), we have

$$
\begin{aligned}
(2.35)\qquad & t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{(N)} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N} \\
&\times\left\{\sum_{k=0}^{N}\sum_{i=0}^{k}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)(k)_{i}\frac{t^{k}}{k!}\right. \\
&\left.+\sum_{k=N+1}^{\infty}\sum_{i=0}^{N}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)(k)_{i}\frac{t^{k}}{k!}\right\}.
\end{aligned}
$$

Comparing (2.33) and (2.35), we obtain the following theorem.

Theorem 6. *Let N be a positive integer. Then*

(i) $\tilde{\beta}_{k}(\lambda)=\frac{1}{(k-1)_{N}}\left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)(k)_{i}$, *where* $k\geq N+1$, $(k)_{N}=k(k-1)\cdots(k-N+1)$ *for* $N\geq 1$, *and* $(k)_{0}=1$.

(ii) *For* $1\leq k\leq N$, *we have*

$$\sum_{i=0}^{k}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)(k)_{i}=0,$$

where $a_{0}^{-}(N)=1$,

$$(2.36)\qquad a_{j}^{-}(N)=j!\sum_{i_j=0}^{N-j}\sum_{i_{j-1}=0}^{N-j-i_j}\cdots\sum_{i_1=0}^{N-j-i_j-\cdots i_2}(j+1)^{i_j}j^{i_{j-1}}\cdots 2^{i_1},$$

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$(1 \leq j \leq N)$.

Replacing t by $\frac{t}{\lambda}\log(1+\lambda)$ in (1.10), we get

$$\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \left(\frac{1}{\lambda}\log(1+\lambda)\right)^{n-r} \frac{t^n}{n!}. \tag{2.37}$$

Thus, from (2.37), we have

$$\tilde{\beta}_n^{(r)}(\lambda) = \left(\frac{1}{\lambda}\log(1+\lambda)\right)^{n-r} B_n^{(r)}, \quad \text{for } n \geq 0. \tag{2.38}$$

From (2.38), we obtain the following corollary.

Corollary 7. *Let N be any positive integer. Then*

(i) $B_k = \frac{(-1)^N}{(k-1)_N} \sum_{i=0}^{N} a_{N-i}^{-}(N) B_{k-i}^{(N+1-i)}(k)_i$, *for* $k \geq N+1$,

(ii) $\sum_{i=0}^{k} a_{N-i}^{-}(N) B_{k-i}^{(N+1-i)}(k)_i = 0$, *for* $1 \leq k \leq N$, *where* $a_j^{-}(N)$ $(0 \leq j \leq N)$ *are as in (2.36).*

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ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN BANACH SPACES

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Abstract. Let

$$\begin{aligned} M_1 f(x,y): &= \frac{3}{4} f(x+y) - \frac{1}{4} f(-x-y) \\ &\quad + \frac{1}{4} f(x-y) + \frac{1}{4} f(y-x) - f(x) - f(y), \end{aligned}$$

$$M_2 f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic ρ-functional inequalities

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$ and

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [22] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 7, 10, 13, 14, 16, 17, 18, 19, 20, 21, 24, 25]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in Banach spaces.

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In Section 3, we solve the additive-quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in Banach spaces.

In this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Additive-quadratic ρ-functional inequality (0.1) in Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.1) in normed spaces.

Lemma 2.1.
(i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x,y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*
(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x,y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof. (i)

$$M_1 f_o(x,y) = f_o(x+y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So f_o is the Cauchy additive mapping.

$$M_1 f_e(x,y) = \frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So f_o is the quadratic mapping.

(ii)

$$M_2 f_o(x,y) = 2 f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.

$$M_2 f_e(x,y) = 2 f_e\left(\frac{x+y}{2}\right) + 2 f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

Lemma 2.2.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies* (2.1), *then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \le 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y)-f(x)-f(y)\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= |\rho|\|f(x+y)-f(x)-f(y)\| \end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get

$$\|f(0)\| \le \|2\rho f(0)\|.$$

So $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|\frac{1}{2}f(2x)-2f(x)\| \le 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{4}f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned} &\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \\ &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces for an odd mapping case.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an odd mapping such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{2.4}$$

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| + \varphi(x,y) \tag{2.5}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{1}{2}\Psi(x,x) \tag{2.6}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x, x) \tag{2.7}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.6).

It follows from (2.4) and (2.5) that

$$\begin{aligned}\|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} \left\|2^n \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &+ \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y) - A(x) - A(y)\| \le \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.6). Then we have

$$\begin{aligned}\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A, as desired. □

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Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| + \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 2.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying (2.5) and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x,x) \tag{2.11}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi(2^j x, 2^j x)\end{aligned} \tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^r}\|x\|^r \tag{2.13}$$

for all $x \in X$.

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces for an even mapping case.

Theorem 2.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (2.5) and*

$$\Psi(x,y) : \;=\; \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.14}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2}\Psi(x,x) \tag{2.15}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\left\|\frac{1}{2}f(2x) - 2f(x)\right\| \le \varphi(x,x) \tag{2.16}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} \frac{4^{j+1}}{2}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned}\tag{2.17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.17) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.17), we get (2.15).

It follows from (2.5) and (2.14) that

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right\| \\ &= \lim_{n\to\infty}\left\|4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}\left\|4^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| + \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\|\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right\| \\ &\quad\le \left\|\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.15). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 4^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q, as desired. □

Corollary 2.8. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{4\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 2.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (2.5) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty \tag{2.18}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2}\Psi(x, x) \tag{2.19}$$

for all $x \in X$.

Proof. It follows from (2.16) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{2}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \quad &\le \quad \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \quad \sum_{j=l}^{m-1}\frac{1}{2 \cdot 4^j}\varphi(2^j x, 2^j x)\end{aligned} \tag{2.20}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.20) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.20), we get (2.19).

The rest of the proof is similar to the proof of Theorem 2.7. □

Corollary 2.10. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{4\theta}{4 - 2^r}\|x\|^r \tag{2.21}$$

for all $x \in X$.

Remark 2.11. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive-quadratic ρ-functional inequality (0.2) in complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.2) in complex normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_2 f(x, y)\| \leq \|\rho M_1 f(x, y)\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (3.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x + y) - f(x) - f(y)\| &= \left\|2f\left(\frac{x + y}{2}\right) - f(x) - f(y)\right\| \\ &\leq |\rho|\|f(x + y) - f(x) - f(y)\|\end{aligned}$$

and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}&\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\\&=\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\|\\&\leq|\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. $\square$

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in complex Banach spaces for an odd mapping case.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned}\Psi(x,y) :&= \sum_{j=0}^{\infty}2^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right)<\infty,\\ \|M_2f(x,y)\| &\leq \|\rho M_1f(x,y)\|+\varphi(x,y) \qquad (3.4)\end{aligned}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\|\leq\Psi(x,0) \qquad (3.5)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|=\left\|2f\left(\frac{x}{2}\right)-f(x)\right\|\leq\varphi(x,0) \qquad (3.6)$$

for all $x \in X$. So

$$\begin{aligned}\left\|2^lf\left(\frac{x}{2^l}\right)-2^mf\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|2^jf\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|\\&\leq\sum_{j=l}^{m-1}2^j\varphi\left(\frac{x}{2^j},0\right) \qquad (3.7)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^kf(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^kf(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x):=\lim_{k\to\infty}2^kf\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.3. $\square$

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\| + \theta(\|x\|^r + \|y\|^r) \tag{3.8}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying* (3.4) *and*

$$\Psi(x,y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi(x, 0) \tag{3.9}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l+1}^{m}\frac{1}{2^j}\varphi(2^j x, 0) \end{aligned} \tag{3.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.9).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (3.8)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in complex Banach spaces for an even mapping case.

Theorem 3.6. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.4) and*

$$\Psi(x,y) : \quad = \quad \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \Psi(x,0) \tag{3.11}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x,0) \tag{3.12}$$

for all $x \in X$. So

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \quad &\le \quad \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \quad \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned}\tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.7. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.8). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 3.8. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.4) and*

$$\Psi(x,y) := \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \Psi(x,0) \tag{3.14}$$

for all $x \in X$.

Proof. It follows from (3.12) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\varphi(2x,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^lx)-\frac{1}{4^m}f(2^mx)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^jx\right)-\frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq \sum_{j=l+1}^{m}\frac{1}{4^j}\varphi(2^jx,0) \qquad (3.15)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.15) that the sequence $\{\frac{1}{4^n}f(2^nx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^nx)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.15), we get (3.14).

The rest of the proof is similar to the proof of Theorem 2.3. ☐

Corollary 3.9. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.8). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \leq \frac{2^r\theta}{4-2^r}\|x\|^r$$

for all $x \in X$.

Remark 3.10. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

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STABILITY OF ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN BANACH SPACES

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Abstract. Let

$$\begin{aligned} M_1 f(x,y): &= \frac{3}{4}f(x+y)-\frac{1}{4}f(-x-y) \\ &+\frac{1}{4}f(x-y)+\frac{1}{4}f(y-x)-f(x)-f(y), \end{aligned}$$

$$M_2 f(x,y) := 2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)-f(x)-f(y).$$

We solve the additive-quadratic ρ-functional inequalities

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$ and

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [31] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [30] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have

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been extensively investigated by a number of authors (see [1, 3, 7, 10, 17, 18, 19, 20, 21, 24, 25, 26, 27, 28, 29, 32, 33]).

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 9] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 15, 16, 22]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 3, we solve the additive-quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in Banach spaces by using the fixed point method.

In this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Additive-quadratic ρ-functional inequality (0.1) in Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.1) in complex normed spaces.

Lemma 2.1.
(i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*
(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof. (i)

$$M_1 f_o(x, y) = f_o(x + y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So f_o is the Cauchy additive mapping.

$$M_1 f_e(x, y) = \frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So f_o is the quadratic mapping.

(ii)

$$M_2 f_o(x,y) = 2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.

$$M_2 f_e(x,y) = 2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

Lemma 2.2.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies* (2.1), *then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \\ &= |\rho|\|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|f(0)\| \le \|2\rho f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|\frac{1}{2} f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned}\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \le \frac{L}{2}\varphi(x,y) \tag{2.4}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|M_1f(x,y)-\rho M_2f(x,y)\| \le \varphi(x,y) \tag{2.5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{L}{2(1-L)}\varphi(x,x) \tag{2.6}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\|f(2x)-2f(x)\| \le \varphi(x,x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x)-h(x)\| \le \mu\varphi(x,x),\ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [14]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x)-h(x)\| \le \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le 2\varepsilon\frac{L}{2}\varphi(x,x) = L\varepsilon\varphi(x,x) \end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{2}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \le \mu\varphi(x,x)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)}\varphi(x,x)$$

for all $x \in X$.

It follows from (2.4) and (2.5) that

$$\begin{aligned} &\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\ &= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right. \\ &\qquad \left. -2^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive. □

Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_1 f(x,y) - \rho M_2 f(x,y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f_o(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y) \tag{2.10}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_e(x) - Q(x)\| \le \frac{L}{2(1-L)}\varphi(x,x) \tag{2.11}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5) for f_e, we get

$$\left\|\frac{1}{2}f(2x) - 2f(x)\right\| \le \varphi(x,x) \tag{2.12}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le 4\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le 4\varepsilon\frac{L}{4}\varphi(x,x) = L\varepsilon\varphi(x,x)\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.12) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi(x, x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{2.13}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.13) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \quad \leq \quad \mu\varphi(x, x)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - Q(x)\| \leq \frac{L}{2(1-L)}\varphi(x, x)$$

for all $x \in X$.

It follows from (2.4) and (2.5) that

$$\begin{aligned}
&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right. \\
&\left. - \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&= \lim_{n\to\infty}\left\|4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right. \\
&\left. - 4^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\
&\leq \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) \\
&\qquad = \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic. $\square$

Corollary 2.6. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{4\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{2-r}$ and we get the desired result. □

Theorem 2.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(1 - L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.7 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

Theorem 2.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2(1 - L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.
It follows from (2.12) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{2} \varphi(x, x)$$

for all $x \in X$.
Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g\,(2x)$$

for all $x \in X$.
The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 2.10. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \quad \leq \quad \frac{4\theta}{4 - 2^r} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.9 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

Remark 2.11. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive-quadratic ρ-functional inequality (0.2) in complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.
We solve and investigate the additive-quadratic ρ-functional inequality (0.2) in complex normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_2 f(x, y)\| \leq \|\rho M_1 f(x, y)\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (3.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (3.1).
Letting $y = 0$ in (3.1), we get

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2} f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \\ &\leq |\rho|\|f(x+y)-f(x)-f(y)\|\end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|4f\left(\frac{x}{2}\right)-f(x)\right\| \leq 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}&\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \\ &= \left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\ &\leq |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in complex Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leq \frac{L}{2}\varphi(x,y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|M_2f(x,y)-\rho M_1f(x,y)\| \leq \varphi(x,y) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ a such that

$$\|f(x)-A(x)\| \leq \frac{1}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \leq \varphi(x,0) \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x,0), \ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [14]).

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_2 f(x,y) - \rho M_1 f(x,y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{3.6}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 3.2.

Letting $y = 0$ in (3.4), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x,0) \tag{3.7}$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 3.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{2-r}$ and we get desired result. □

Theorem 3.6. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (3.4). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{1-L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}\varphi(2x, 0) \leq L\varphi(x, 0)$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.7. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying (3.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.6 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

Theorem 3.8. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{1-L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.7) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{4}\varphi(2x, 0) \leq L\varphi(x, 0)$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 3.9. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r \theta}{4 - 2^r} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.8 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

Remark 3.10. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

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Global Attractivity and the Periodic Nature of Third Order Rational Difference Equation

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ABSTRACT

The main target of our study to cover the solutions behavior of the following difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}}, \quad n = 0, 1, ...,$$

where the parameters $a,\ b,\ c,\ d,\ e$ and f are positive real numbers and the initial conditions $x_{-2},\ x_{-1}$ and x_0 are positive real numbers.

Keywords: stability, boundedness, periodicity, global attractor, difference equations.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Our objective in this research is to study character of global stability and the periodicity of the solutions of the recursive sequence

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}}, \tag{1}$$

where the following parameters $a,\ b,\ c,\ d,\ e$ and f are defined as positive real numbers and the initial conditions $x_{-2},\ x_{-1}$ and x_0 are also defined as positive real numbers.

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have appeared recently in many areas. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics, probability theory, genetics, psychology and resource management [12]. It is very interesting to investigate the behavior of solutions of a higher-order rational difference equation and to discuss the local asymptotic stability of its equilibrium points. Rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such equations. For more results for the rational difference equations, we refer the interested reader to [1–30].

The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order, recently, many researchers have investigated the behavior

of the solution of difference equations for example: Abo-Zeid and Al-Shabi [1] investigated the global stability, and periodic nature of the positive solutions of the difference equation

$$x_{n+1} = \frac{A+Bx_n}{C+Dx_nx_{n-2}}.$$

Belhannache et al. [5] studied the global behavior of positive solutions of the following third order difference equation

$$x_{n+1} = \frac{A+Bx_{n-1}}{C+Dx_n^p x_{n-2}^q}.$$

Dehghan and Rastegar [11], deal with the qualitative behavior of solutions of the higher-order non-linear difference equation

$$x_{n+1} = \frac{p+qx_n+rx_{n-k}}{1+x_{n-k}}.$$

Din [14] investigated the local asymptotic stability, global stability, the periodic character, semicycle analysis and the boundedness nature of the following rational difference equation

$$x_{n+1} = \frac{A+Bx_n+Cx_{n-k}}{1+x_n+x_{n-k}}.$$

In [16] Elabbasy et al. investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}}.$$

Elsayed [22] investigated the local and global stability, boundedness character and obtained the solution of some special cases of the following recursive sequence

$$x_{n+1} = ax_{n-1} + \frac{bx_nx_{n-1}}{cx_n+dx_{n-2}}.$$

A. El-Moneam, and Zayed [20]-[21] studied the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equations

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k}-ex_{n-l}}. \\ x_{n+1} &= Ax_n + Bx_{n-k} + Cx_{n-l} + +Dx_{n-\sigma} + \frac{bx_{n-k}+hx_{n-l}}{dx_{n-k}+ex_{n-l}}. \end{aligned}$$

Su and Li [52] studied the global asymptotic stability of the nonlinear difference equation

$$x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}.$$

Yalçınkaya et al. [54] considered the dynamics of the difference equation

$$x_{n+1} = \frac{ax_{n-k}}{b+cx_n^p}.$$

For some related work see [31–57].

2. SOME BASIC PROPERTIES AND DEFINITIONS

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let $F : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k},\ x_{-k+1},\ ...,\ x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point) A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2. (Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \tfrac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A. [47] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$, is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark: Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \tag{4}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq. (4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Theorem B. [48] Let $g : [a, b]^{k+1} \to [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{5}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k+1$; the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{k+1}), \qquad M = g(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

$$m_i = \left\{ \begin{matrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{matrix} \right\},$$
$$Mi = \left\{ \begin{matrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{matrix} \right\}.$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation (5), and every solution of Equation (5) converges to $\bar{x}$.

3. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ.(1)

This section deals with study the local stability character of the equilibrium point of Eq.(1)

Eq.(1) has equilibrium point and is given by

$$\overline{x} = a\overline{x} + b\overline{x} + \frac{c + d\overline{x}}{e + f\overline{x}} \qquad \Rightarrow \qquad \overline{x}(1 - a - b) = \frac{c + d\overline{x}}{e + f\overline{x}},$$

$$f(1 - a - b)\overline{x}^2 + [e(1 - a - b) - d]\,\overline{x} - c = 0$$

If $d > e(1 - a - b) > 0$, then the only positive equilibrium point of Eq.(1) is given by

$$\overline{x} = \frac{[d - e(1-a-b)] + \sqrt{[d - e(1-a-b)]^2 + 4fc(1-a-b)}}{2f(1-a-b)}.$$

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a continuous function defined by

$$f(u, v, w) = au + bv + \frac{c + dw}{e + fw}. \tag{6}$$

Therefore it follows that

$$\frac{\partial f(u, v, w)}{\partial u} = a, \quad \frac{\partial f(u, v, w)}{\partial v} = b, \quad \frac{\partial f(u, v, w)}{\partial w} = \frac{(de - fc)}{(e + fw)^2}.$$

Then we see that

$$\frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial u} = a = -a_2, \quad \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial v} = b = -a_1, \quad \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial w} = \frac{de - fc}{(e + f\overline{x})^2} = -a_0.$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + a_2 y_n + a_1 y_{n-1} + a_0 y_{n-2} = 0, \tag{7}$$

whose characteristic equation is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{8}$$

Theorem 1. Assume that

$$\frac{|de - fc|}{(e + f\overline{x})^2} < 1 - a - b.$$

Then the positive equilibrium point of Eq.(1) is locally asymptotically stable.

Proof: It follows by Theorem A that, Eq.(7) is asymptotically stable if all roots of Eq.(8) lie in the open disc $|\lambda| < 1$ that is if

$$|a_2| + |a_1| + |a_0| < 1 \qquad \Rightarrow \qquad |a| + |b| + \left| \frac{de - fc}{(e + f\overline{x})^2} \right| < 1,$$

and so

$$a + b + \frac{|de - fc|}{(e - f\overline{x})^2} < 1,$$

or

$$\frac{|de - fc|}{(e + f\overline{x})^2} < 1 - a - b.$$

The proof is complete.

4. BOUNDEDNESS OF SOLUTIONS OF EQ.(1)

Here we study the boundedness nature of solutions of Eq.(1).

Theorem 2. Every solution of Eq.(1) is bounded if $a + b + \dfrac{d}{e} < 1$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}} \leq ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e}.$$

Then

$$x_{n+1} \leq ax_n + bx_{n-1} + \frac{d}{e}x_{n-2} + \frac{c}{e} \quad \text{for all} \quad n \geq 1.$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1} = ay_n + by_{n-1} + \frac{d}{e}y_{n-2} + \frac{c}{e},$$

and this equation is locally asymptotically stable if $a + b + \frac{d}{e} < 1$, and converges to the equilibrium point $\overline{y} = \frac{c}{e\left(1-a-b-\frac{d}{e}\right)}$. Therefore

$$\limsup_{n\to\infty} x_n \leq \frac{c}{e\left(1-a-b-\frac{d}{e}\right)}.$$

Thus the solution is bounded.

Theorem 3. Every solution of Eq.(1) is unbounded if $a > 1$ (or $b > 1$).

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). Then from Eq.(1) we see that

$$x_{n+1} = ax_n + bx_{n-1} + \tfrac{c+dx_{n-2}}{e+fx_{n-2}} > ax_n \quad \text{for all} \quad n \geq 1.$$

We see that the right hand side can write as follows

$$y_{n+1} = ay_n \quad \Rightarrow \quad y_n = a^n y_0,$$

and this equation is unstable because $a > 1$, and $\lim\limits_{n\to\infty} y_n = \infty$. Then by using ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above (when $b > 1$ is similar).

5. EXISTENCE OF PERIOD TWO SOLUTIONS

In this section we study the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 4. Eq.(1) has positive prime period two solutions if and only if

$$(i)\ (eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0,\quad B=b-a-1.$$

Proof: First suppose that there exists a prime period two solution $..., p, q, p, q, ...,$ of Eq.(1). We will prove that Condition (i) holds. We see from Eq.(1) that

$$\begin{aligned} p &= aq+bp+\frac{c+dq}{e+fq}, \qquad q=ap+bq+\frac{c+dp}{e+fp}. \\ p(1-b)-aq &= \frac{c+dq}{e+fq}, \qquad q(1-b)-ap=\frac{c+dp}{e+fp}. \end{aligned}$$

Then

$$ep(1-b)+pqf(1-b)-aeq-afq^2=c+dq,$$

and

$$eq(1-b)+pqf(1-b)-aep-afp^2=c+dp.$$

Then

$$ep(1-b)+pqf(1-b)-afq^2=c+(d+ae)q, \tag{9}$$

and

$$eq(1-b)+pqf(1-b)-afp^2=c+(d+ae)p. \tag{10}$$

Subtracting (9) from (10) gives

$$e(1-b)(p-q)+af(p-q)(p+q)=-(d+ae)(p-q).$$

Since $p\neq q$, it follows that

$$\begin{aligned} e(1-b)+af(p+q) &= -(d+ae), \\ p+q &= \frac{e(b-1-a)-d}{af}. \end{aligned}$$

or

$$p+q=\frac{eB-d}{af},\qquad B=b-a-1. \tag{11}$$

Again, adding (9) and (10) yields

$$e(1-b)(p+q)+2pqf(1-b)-af(p^2+q^2)=2c+(d+ae)(p+q),$$

$$2pqf(1-b)-af((p+q)^2-2pq)=2c+(p+q)(d+ae-e(1-b)). \tag{12}$$

It follows by (11), (12) and the relation

$$p^2+q^2=(p+q)^2-2pq \quad \text{for all } p,q\in R,$$

that

$$2pqf(1-b)+2afpq=af(p+q)^2+2c+(p+q)(d+e(a-1+b)).$$

and

$$2pqf((1-b)+a)=2c+(p+q)\left\{d+e(a-1+b)+af(p+q)\right\}.$$

From Eq. (11) we have

$$\begin{aligned} 2pqf((1-b)+a) &= 2c+(p+q)\left\{d+e(a-1+b)+e(b-1-a)-d\right\}, \\ 2pqf((1-b+a)) &= 2c+(p+q)\left\{-2e+2eb\right\}, \end{aligned}$$

$$\begin{aligned} pqf(-B) &= c+(p+q)e(b-1) \\ pqfB &= e(1-b)\left(\frac{eB-d}{af}\right)-c. \end{aligned}$$

Thus

$$pq=\frac{e^2(1-b)B-ed(1-b)-af}{aBf^2}. \tag{13}$$

Now it is clear from Eq.(11) and Eq.(13) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned} t^2-\left(\tfrac{eB-d}{af}\right)t+\left(\tfrac{e^2(1-b)B-ed(1-b)-acf}{aBf^2}\right) &= 0, \\ aBf^2t^2-(eB-d)Bft+\left(e^2(1-b)B-ed(1-b)-acf\right) &= 0, \end{aligned} \tag{14}$$

and so

$$(eB-d)^2B^2f^2>4aBf^2\left(e^2(1-b)B-ed(1-b)-acf\right),$$

or

$$(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0.$$

Therefore Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Eq.(1) has a prime period two solution. Assume that

$$p=\frac{(eB-d)Bf+\sqrt{\zeta}}{2aBf^2}, \quad q=\frac{(eB-d)Bf-\sqrt{\zeta}}{2aBf^2},$$

where $\zeta=(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)$.
We see from Inequality (i) that

$$(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0,$$

which equivalents to

$$(eB-d)^2B^2f^2>4aBf^2\left(e^2(1-b)B-ed(1-b)-acf\right).$$

Therefore p and q are distinct real numbers. Set $x_{-2}=p$, $x_{-1}=q$ and $x_0=p$. We wish to show that $x_1=x_{-1}=q$ and $x_2=x_0=p$. It follows from Eq.(1) that

$$x_1=ap+bq+\frac{c+dp}{e+fp}=\tfrac{a(eB-d)Bf+a\sqrt{\zeta}}{2aBf^2}+\tfrac{b(eB-d)Bf-b\sqrt{\zeta}}{2aBf^2}+\frac{c+\left(\frac{d(eB-d)Bf+d\sqrt{\zeta}}{2aBf^2}\right)}{e+\left(\frac{(eB-d)Bf^2+f\sqrt{\zeta}}{2aBf^2}\right)}.$$

Multiplying the denominator and numerator by $2aBf^2$ gives

$$x_1=a(eB-d)Bf+a\sqrt{\zeta}+b(eB-d)Bf-b\sqrt{\zeta}+\frac{2acBf^2+\left(d(eB-d)Bf+d\sqrt{\zeta}\right)}{2aeBf^2+\left((eB-d)Bf^2+f\sqrt{\zeta}\right)}.$$

By simple computations we can see that

$$x_1=\frac{(eB-d)Bf+\sqrt{\zeta}}{2aBf^2}=q.$$

Similarly as before one can easily show that $x_2=p$. Then it follows by induction that

$$x_{2n}=p \quad \text{and} \quad x_{2n+1}=q \quad \text{for all} \quad n\geq -2.$$

Thus Eq.(1) has the prime period two solution ...,p,q,p,q,..., where p and q are the distinct roots of the quadratic equation (14) and the proof is complete.

6. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we investigate the global asymptotic stability of Eq.(1).

Theorem 5. The equilibrium point $\overline{x}$ is a global attractor of Eq.(1) if one of the following statements holds

$$de \geq fc \text{ and } (1-a-b)\,e \geq d. \tag{15}$$
$$de < fc \text{ and } (1-a-b) \geq 0. \tag{16}$$

Proof: Let α and β be a real numbers and assume that $g : [\alpha, \beta]^3 \longrightarrow [\alpha, \beta]$ be a function defined by

$$g(u, v, w) = au + bv + \frac{c + dw}{e + fw}.$$

Then

$$\frac{\partial g(u, v, w)}{\partial u} = a, \quad \frac{\partial g(u, v, w)}{\partial v} = b, \quad \frac{\partial g(u, v, w)}{\partial w} = \frac{de - fc}{(e + fw)^2}.$$

We consider the two cases:-

Case (1): Assume that (15) is true, then we can easily see that the function $g(u, v, w)$ increasing in u, v and w.

Suppose that (m, M) is a solution of the system $M = g(M, M, M)$ *and* $m = g(m, m, m)$. Then from Eq.(1), we see that

$$\begin{aligned} M &= aM + bM + \frac{c + dM}{de + fM}, \quad m = am + bm + \frac{c + dm}{e + fm}, \\ M(1 - a - b) &= \frac{c + dM}{e + fM}, \quad m(1 - a - b) = \frac{c + dm}{e + fm}, \end{aligned}$$

then

$$MAe + AfM^2 = c + dM, \quad mAe + Afm^2 = c + dm, \quad A = 1 - a - b.$$

Subtracting this two equations we obtain

$$(M - m)\{Ae + Af(M + m) - d\} = 0,$$

under the conditions $Ae \geq d,\ a < 1$, we see that $M = m$. It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

Case (2): Assume that (16) is true, then we can easily see that the function $g(u, v, w)$ increasing in u, v and decreasing in w.

Suppose that (m, M) is a solution of the system $M = g(M, M, m)$ *and* $m = g(m, m, M)$.Then from Eq.(1), we see that

$$\begin{aligned} M &= aM + bM + \frac{c + dm}{e + fm}, \quad m = am + bm + \frac{c + dM}{e + fM}, \\ MA &= \frac{c + dm}{e + fm}, \quad mA = \frac{c + dM}{e + fM}, \end{aligned}$$

then

$$MAe + MAfm = c + dm, \quad mAe + fMmA = c + dM.$$

Subtracting we obtain

$$(M - m)\,(Ae + d) = 0,$$

under the conditions $(1 - a - b) > 0$, we see that $M = m$. Also, from Theorem B, we see that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

7. NUMERICAL EXAMPLES

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We assume $x_{-2} = .5,\ x_{-1} = 3,\ x_0 = 9,\ a = .2,\ b = .7,\ c = .2,\ d = .6,\ e = 1.3,\ f = 5.3$**.** See Fig. 1.

Example 2. See Fig. 2, since $x_{-2} = .5,\ x_{-1} = 3,\ x_0 = 9,\ a = .4,\ b = .6,\ c = .2,\ d = .6,\ e = 1.3,\ f = 5.3.$

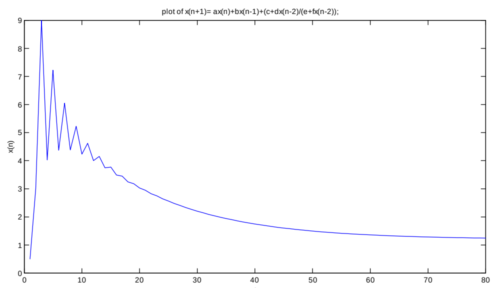

Figure 1.

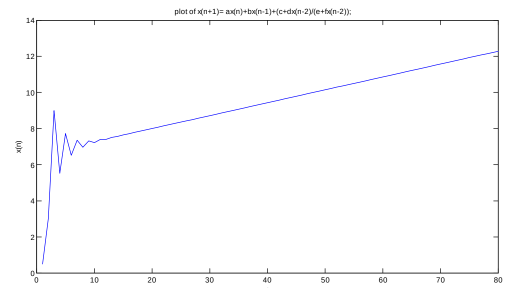

Figure 2.

Example 3. We consider $x_{-2} = 2.5,\ x_{-1} = 3,\ x_0 = 9,\ a = .4,\ b = .5,\ c = 2,\ d = 6,\ e = 3,\ f = 5$. See Fig. 3.

Example 4. See Fig. 4, since $x_{-2} = 2.5,\ x_{-1} = 3,\ x_0 = 9,\ a = 1,\ b = .5,\ c = 2,\ d = 6,\ e = 3,\ f = 5.$

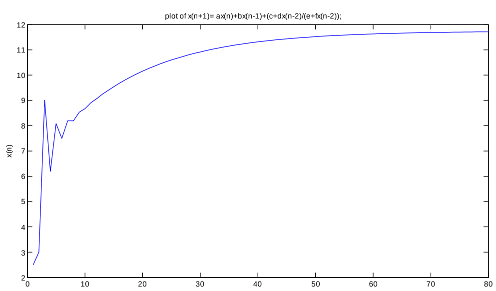

Figure 3.

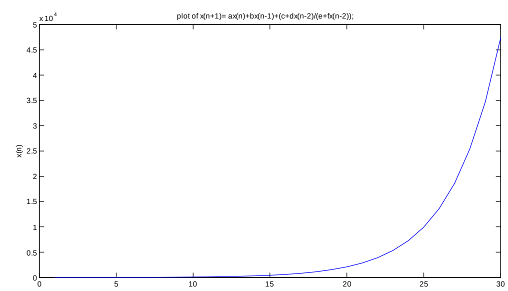

Figure 4.

Example 5. Fig. 5. shows the solutions when $a = .7,\ b = .5,\ c = .2,\ d = .1,\ e = .3,\ f = .5,\ x_{-2} = 2.5,\ x_{-1} = .3,\ x_0 = .9.$

Example 6. Fig. 6. shows the period two solutions when $a = .6,\ b = .5,\ c = .82,\ d = .7,\ e = .3,\ f = .5,\ x_{-2} = p,\ x_{-1} = q,\ x_0 = p.$ $\left(\text{Since } p, q = \frac{(eB-d)Bf \pm \sqrt{\zeta}}{2aBf^2}\right).$

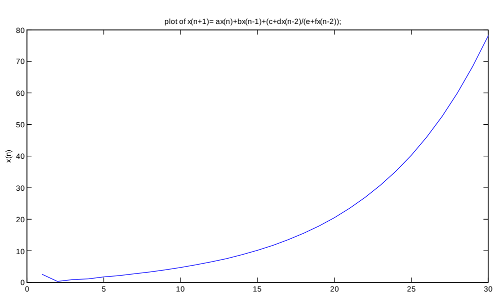

Figure 5.

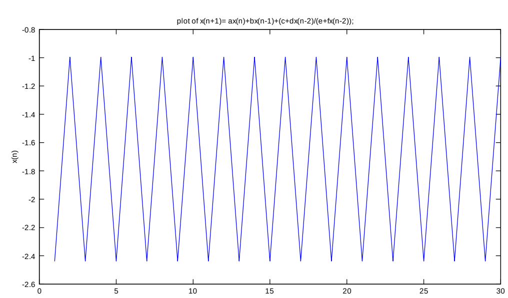

Figure 6.

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Asymptotically stability of solutions of fuzzy differential equations in the quotient space of fuzzy numbers

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Abstract

In this paper, we investigate essentially stability theory for the fuzzy differential equations in the quotient space of fuzzy numbers by Lyapunov-like functions. By using the differential inequalities and the comparison principle for Lyapunov-like functions, we give some sufficient criterias for the asymptotically stability, equi-asymptotically stability and uniformly asymptotically stability of the trivial solution of the fuzzy differential equations.

Keywords: Fuzzy number; Quotient space; Fuzzy differential equation; Asymptotically stability

1 Introduction

Recently, the study of fuzzy differential equations has been gained importance due to its application. Subsequently, the existence and uniqueness of solutions of the initial value ptoblems for fuzzy differential equations under kinds of conditions were studied in [8, 9, 11, 14, 18, 24] and the relationship between a solution and its approximate solutions to fuzzy differential equations were established in [19, 25, 26]. Further, the essentially stability theory for fuzzy differential equations by Lyapunov-like functions were investigated in [2, 12, 28]. In particular, Hien [4] researched the asymptotic stability of solutions of fuzzy differential equations by Lyapunovs second method.

The above these results of fuzzy differential equations based on well known and widely used Hukuhara difference [6] and the H-differentiability of Puri and Ralescu [20]. But in many applications the Hukuhara difference appears to have several limitations and to be very restrictive [1, 8]. In [15, 16], Mareš presented a natural equivalence relation between fuzzy quantities. This equivalence relation can be used to partition of the set of fuzzy quantities into equivalence classes having the desired group properties for the addition operation [7, 17, 27]. Hong and Do [5] defined a more refined equivalence relation than Mareš [15] and improved Mareš's results. In [21], Qiu et al. showed that the method of finding the inverse operation of fuzzy numbers in the sense of Mareš is very intuitive. As an application of the main results, it is shown that if we identify every fuzzy number with the corresponding equivalence class, there wound be more differentiable fuzzy functions than what is found in the literature. After that, the fuzzy differential equations in the quotient space of fuzzy numbers were investigated [23, 22]. In this paper, we shall study the stability of the trivial solution of the fuzzy differential equations in the quotient space of fuzzy numbers by Lyapunov's second method.

2 Preliminaries

A fuzzy set $\widetilde{x}$ of $\mathbb{R}$ is characterized by a membership function $\mu_{\widetilde{x}} : \mathbb{R} \to [0,1]$. For each such fuzzy set $\widetilde{x}$, we denote by $[\widetilde{x}]^{\alpha} = \{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) \geq \alpha\}$ for any $\alpha \in (0,1]$, its α-level set. We define the set

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$[\widetilde{x}]^0$ by $[\widetilde{x}]^0 = \overline{\bigcup_{\alpha\in(0,1]} [\widetilde{x}]^\alpha}$, where $\overline{A}$ denotes the closure of a crisp set A. A fuzzy set $\widetilde{x}$ is said to be a fuzzy number if it satisfies the following conditions [3]:

(1) $\widetilde{x}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\mu_{\widetilde{x}}(x_0) = 1$;
(2) $\widetilde{x}$ is convex, i.e., $\mu_{\widetilde{x}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\widetilde{x}}(x_1), \mu_{\widetilde{x}}(x_2)\}$, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (0,1)$;
(3) $\widetilde{x}$ is upper semi-continuous;
(4) $[\widetilde{x}]^0$ is compact.

Equivalently, a fuzzy number $\widetilde{x}$ is a fuzzy set with non-empty bounded closed level sets $[\widetilde{x}]^\alpha = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ for all $\alpha \in [0,1]$, where $[\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ denotes a closed interval with the left end point $\widetilde{x}_L(\alpha)$ and the right end point $\widetilde{x}_R(\alpha)$. We denote the class of fuzzy numbers by $\mathscr{F}$. We say that a fuzzy number $\widetilde{s} \in \mathscr{F}$ is symmetric [15], if $\mu_{\widetilde{s}}(x) = \mu_{\widetilde{s}}(-x)$, for all $x \in \mathbb{R}$, i.e., $\widetilde{s} = -\widetilde{s}$. The set of all symmetric fuzzy numbers will be denoted by $\mathscr{S}$.

Definition 2.1 *[5] Let $\widetilde{x}, \widetilde{y} \in \mathscr{F}$. We say that $\widetilde{x}$ is equivalent to $\widetilde{y}$ and write $\widetilde{x} \sim \widetilde{y}$ if and only if there exist symmetric fuzzy numbers $\widetilde{s_1}, \widetilde{s_2} \in \mathscr{S}$ such that $\widetilde{x} + \widetilde{s_1} = \widetilde{y} + \widetilde{s_2}$.*

The equivalence relation defined above is reflexive, symmetric and transitive [15]. Let $\langle\widetilde{x}\rangle$ denote the equivalence class containing the element $\widetilde{x}$ and denote the set of equivalence classes by $\mathscr{F}/\mathscr{S}$.

Definition 2.2 *[10] Let $f : [a,b] \to \mathbb{R}$. f is said be of bounded variation if there exists a $C > 0$ such that*

$$\sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| \leq C$$

for every partition $a = x_0 < x_1 < \cdots < x_n = b$ on $[a,b]$. The total variation of f on $[a,b]$ is defined by

$$V_a^b(f) = \sup_p \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)|,$$

where p represents all partitions of $[a,b]$. The set of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

Definition 2.3 *[7] For a fuzzy number $\widetilde{x}$, we define a function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $\widetilde{x}_M(\alpha)$ for all $\alpha \in [0,1]$, i.e.,*

$$\widetilde{x}_M(\alpha) = \frac{\widetilde{x}_L(\alpha) + \widetilde{x}_R(\alpha)}{2}.$$

Then the function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ will be called the midpoint function of the fuzzy number $\widetilde{x}$.

Lemma 2.1 *[21] For any $\tilde{x} \in \mathscr{F}$, the midpoint function $\tilde{x}_M$ is continuous from the right at 0 and continuous from the left on $[0,1]$. Furthermore it is a function of bounded variation on $[0,1]$.*

Definition 2.4 *[16] Let $\widetilde{x} \in \mathscr{F}$ and let $\widehat{x}$ be a fuzzy number such that $\widetilde{x} = \widehat{x} + \widetilde{s}$ for some $\widetilde{s} \in \mathscr{S}$, if $\widehat{x} = \widetilde{y} + \widetilde{s}_1$ for some $\widetilde{y} \in \mathscr{F}$ and $\widetilde{s}_1 \in \mathscr{S}$, then $\widetilde{s}_1 = \widetilde{0}$. Then the fuzzy number $\widehat{x}$ will be called the Mareš core of the fuzzy number $\widetilde{x}$.*

Definition 2.5 *[22] Define $d_{\sup} : \mathscr{F}/\mathscr{S} \times \mathscr{F}/\mathscr{S} \to \mathbb{R}^+ \cup \{0\}$ by*

$$d_{\sup}\left(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle\right) = \sup_{\alpha\in[0,1]} \left|M_{\langle\widetilde{x}\rangle}(\alpha) - M_{\langle\widetilde{y}\rangle}(\alpha)\right|,$$

for any $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$.

We know that $(\mathscr{F}/\mathscr{S}, d_{\sup})$ is a metric space [21].

3 Main results

Definition 3.1 *[22] For each $m(t) \in C[J, \mathbb{R}]$, where J is a subinterval of $(0, +\infty)$, we will define $d^+ : C[J, \mathbb{R}] \to \mathbb{R}$ by*

$$d^+ m(t) = \overline{\lim_{h \to 0^+}} \frac{1}{h}(m(t+h) - m(t)).$$

Definition 3.2 *[23] A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is differentiable at $t_0 \in J$ if for small $|h| > 0$, there exists an $F'(t_0) \in \mathscr{F}/\mathscr{S}$ such that*

$$\lim_{h \to 0} d_{\sup}\left(\frac{F(t_0 + h) - F(t_0)}{h}, F'(t_0)\right) = 0.$$

Definition 3.3 *[23] A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is measurable if F is measurable with respect to $d_{\sup}$.*

A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is called integrably bounded if there exists an integrable function $h : J \to \mathbb{R}^+ \cup \{0\}$ such that $\left|M_{F(t)}(\alpha)\right| \leq h(t)$ for all $t \in J$ and $\alpha \in [0, 1]$; a mapping $F : J \to \mathscr{F}/\mathscr{S}$ is said to be of uniformly bounded variation with respect to $\alpha \in [0, 1]$ (for short, of uniformly bounded variation) if there exists a constant $K > 0$ such that $V_0^1\left(M_{F(t)}\right) \leq K$, for each $t \in J$ [23].

Definition 3.4 *[23] Let $F : J \to \mathscr{F}/\mathscr{S}$ be measurable. The integral of F over J, denoted $\int_J F(t)dt$, is a mapping $M_{\int_J F(t)dt} : [0, 1] \to \mathbb{R}$, which is defined by the equation*

$$M_{\int_J F(t)dt}(\alpha) = \int_J M_{F(t)}(\alpha)dt$$

for each $\alpha \in [0, 1]$. The mapping F is said to be integrable over J if there exists an $\langle \widetilde{x}_0 \rangle \in \mathscr{F}/\mathscr{S}$ such that $M_{\int_J F(t)dt} = M_{\langle \widetilde{x}_0 \rangle}$. In this case, we denote the integral by

$$\int_J F(t)dt = \langle \widetilde{x}_0 \rangle .$$

Assume that $f : \mathbb{R}_+ \times S(\rho) \to \mathscr{F}/\mathscr{S}$ is continuous and of uniformly bounded variation, where $S(\rho) = \{\langle \widetilde{x} \rangle \in \mathscr{F}/\mathscr{S} : d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{0} \rangle) < \rho\}$. We consider the initial value problem for the fuzzy differential equation

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1}$$

We assume that $f(t, \langle \widetilde{0} \rangle) = \langle \widetilde{0} \rangle$ so that we have the trivial solution $x(t) = \langle \widetilde{0} \rangle$ for (1).

We shall discuss some simple asymptotically stability results of solutions of (1) by Lyapunov's second method. First, we give some notions of concerning the stability of the trivial solution of (1). Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) existing on $[t_0, +\infty)$. Denote $\mathcal{K} = \{\omega \in C[\mathbb{R}_+, \mathbb{R}_+], \omega(0) = 0, \omega(\cdot)$ is increasing $\}$.

Definition 3.5 *The trivial solution $x(t) = \langle \widetilde{0} \rangle$ of (1) is said to be*

$(S1)$ *stable, if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that if $d_{\sup}(x_0, \langle \widetilde{0} \rangle) < \delta$ then*

$$d_{\sup}(x(t), \langle \widetilde{0} \rangle) < \varepsilon, \qquad t \geq t_0;$$

$(S2)$ *uniformly stable, if δ in $(S1)$ is independent of t_0;*

$(S3)$ *asymptotically stable, if it is stable and for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0) > 0$ and $T = T(t_0, x_0, \varepsilon) > 0$ such that if $d_{\sup}(x_0, \langle \widetilde{0} \rangle) < \delta$ then*

$$d_{\sup}(x(t), \langle \widetilde{0} \rangle) < \varepsilon, \qquad t \geq t_0 + T;$$

$(S4)$ *equi-asymptotically stable, if T in $(S3)$ is independent of x_0;*

$(S5)$ *uniformly asymptotically stable, if it is uniformly stable and δ and T in $(S4)$ are independent of t_0.*

Lemma 3.1 *[13] Suppose that $g(t,\varphi)$ be a continuous function on $\mathbb{R}_+^2$ and $r(t) = r(t, t_0, \varphi_0)$, $\varphi(t_0) = \varphi_0$ be the maximal solution of the scalar differential equation:*

$$\frac{d\varphi}{dt} = g(t,\varphi), \quad \varphi(t_0) = \varphi_0 \geq 0, \tag{2}$$

existing on $[t_0, +\infty)$. Let $m(t)$ be a continuous function on $\mathbb{R}_+$ satisfies

$$d^+m(t) = \overline{\lim_{h\to 0^+}} \frac{m(t+h) - m(t)}{h} \leq g(t, m(t)), \qquad t \geq t_0.$$

Then $m(t) \leq r(t)$, for each $t \geq t_0$ if $m(t_0) \leq \varphi_0$.

Let $V(t, \langle \widetilde{x} \rangle) : \mathbb{R}_+ \times S(\rho) \to \mathbb{R}$ be a given function. Then we define

$$D_f^+ V(t, \langle \widetilde{x} \rangle) = \overline{\lim_{h\to 0^+}} \frac{1}{h} \left(V(t+h, \langle \widetilde{x} \rangle + hf(t, \langle \widetilde{x} \rangle)) - V(t, \langle \widetilde{x} \rangle) \right),$$

where $f(\cdot)$ is the right-hand side of (1). Note that, if $V(t,x)$ is Lipchitzian in x, then we have

$$d^+V(t, x(t)) \leq D_f^+ V(t, x(t)).$$

Lemma 3.2 *[22] Suppose that*
(1) $|V(t, \langle \widetilde{x} \rangle) - V(t, \langle \widetilde{y} \rangle)| \leq L(t) d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $D_f^+ V(t, \langle \widetilde{x} \rangle) \leq g(t, V(t, \langle \widetilde{x} \rangle))$, $g(\cdot,\cdot) \in C[\mathbb{R}_+^2, \mathbb{R}]$.
If $x(t) = x(t, t_0, x_0)$ is any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$ such that $V(t_0, x_0) \leq \varphi_0$, then we have

$$V(t, x(t)) \leq r(t, t_0, \varphi_0), \qquad t \geq t_0,$$

where $r(t, t_0, \varphi_0)$ is the maximal solution of the scalar differential equation (2) existing on $[t_0, +\infty)$.

Lemma 3.3 *Suppose that*
(1) $|V(t, \langle \widetilde{x} \rangle) - V(t, \langle \widetilde{y} \rangle)| \leq L(t) d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $D_f^+ V(t, \langle \widetilde{x} \rangle) \leq -\omega(h(t, \langle \widetilde{x} \rangle)) + g(t, V(t, \langle \widetilde{x} \rangle))$, $h(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $\omega(\cdot) \in \mathcal{K}$ *and* $g(t,\varphi) \in C[\mathbb{R}_+^2, \mathbb{R}]$ *is nondecreasing with respect to φ for each $t \in \mathbb{R}_+$.*

If $x(t) = x(t, t_0, x_0)$ is any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$ such that $V(t_0, x_0) \leq \varphi_0$, then we have

$$V(t, x(t)) + \int_{t_0}^{t} \omega(h(s, x(s))) ds \leq r(t, t_0, \varphi_0), \qquad t \geq t_0,$$

where $r(t, t_0, \varphi_0)$ is the maximal solution of the scalar differential equation (2) existing on $[t_0, +\infty)$.

Proof. Let $m(t) = V(t, x(t)) + \int_{t_0}^{t} \omega(h(s, x(s))) ds \geq V(t, x(t))$ for each $t \geq t_0$. Then $m(t_0) = V(t_0, x_0) \leq \varphi_0$ and for small $h > 0$,

$$\begin{aligned}
m(t+h) - m(t) &= V(t+h, x(t+h)) + \int_{t_0}^{t+h} \omega(h(s, x(s))) ds \\
&- V(t, x(t)) - \int_{t_0}^{t} \omega(h(s, x(s))) ds \\
&= V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t))) \\
&+ V(t+h, x(t) + hf(t, x(t))) - V(t, x(t)) + \int_{t}^{t+h} \omega(h(s, x(s))) ds \\
&\leq L(t+h) d_{\sup}(x(t+h), x(t) + hf(t, x(t))) \\
&+ V(t+h, x(t) + hf(t, x(t))) - V(t, x(t)) + \int_{t}^{t+h} \omega(h(s, x(s))) ds.
\end{aligned}$$

Thus, we get

$$\begin{aligned} d^+m(t) &= \overline{\lim_{h\to 0^+}} \frac{m(t+h)-m(t)}{h} \\ &\leq D_f^+V(t,x(t)) + \overline{\lim_{h\to 0^+}} \frac{1}{h}\int_t^{t+h} \omega(h(s,x(s)))ds \\ &+ L(t)\overline{\lim_{h\to 0^+}} \frac{1}{h} d_{\sup}(x(t+h), x(t)+hf(t,x(t))) \\ &= D_f^+V(t,x(t)) + \omega(h(t,x(t))) \\ &+ L(t)\overline{\lim_{h\to 0^+}} d_{\sup}\left(\frac{x(t+h)-x(t)}{h}, f(t,x(t))\right) \\ &= D_f^+V(t,x(t)) + \omega(h(t,x(t))) + L(t)d_{\sup}\left(x'(t), f(t,x(t))\right) \\ &= D_f^+V(t,x(t)) + \omega(h(t,x(t))) \leq g(t,V(t,x(t))), \end{aligned}$$

for each $t \geq t_0$. By the monotonicity of $g(t,\varphi)$ with respect to φ for each $t \geq t_0$, we have

$$d^+m(t) \leq g(t,V(t,x(t))) \leq g(t,m(t)),$$

for each $t \geq t_0$. By Lemma 3.1, we obtain

$$V(t,x(t)) + \int_{t_0}^t \omega(h(s,x(s)))ds = m(t) \leq r(t,t_0,\varphi_0), \qquad t \geq t_0.$$

□

Theorem 3.1 *Suppose that there exists a function $V(t,\langle\widetilde{x}\rangle)$ satisfies the following conditions:*
(1) $|V(t,\langle\widetilde{x}\rangle) - V(t,\langle\widetilde{y}\rangle)| \leq L(t)d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$*;*
(2) $\omega(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t,\langle\widetilde{x}\rangle)$, $V(t,\langle\widetilde{0}\rangle) = 0$, $\omega(\cdot) \in \mathcal{K}$*;*
(3) $D_f^+V(t,\langle\widetilde{x}\rangle) \leq g(t,V(t,\langle\widetilde{x}\rangle))$, $g(\cdot,\cdot) \in C[\mathbb{R}_+^2, \mathbb{R}], g(t,0) = 0$.
If the solution $\varphi(t) = 0$ of (2) is asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is asymptotically stable.

Proof. If the solution $\varphi(t) = 0$ of (2) is asymptotically stable, then by (S3) of Definition 3.5, we have it is stable. Thus, by Theorem 3.1 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is stable.

Since for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, x_0, \varepsilon)$ such that if $0 \leq \varphi_0 < \delta_0$ then

$$|\varphi(t,t_0,\varphi_0)| < \omega(\varepsilon), \qquad\qquad t \geq t_0 + T.$$

Since $V(t,\langle\widetilde{0}\rangle) = 0$, we have

$$V(t_0,\langle\widetilde{x}\rangle) = |V(t_0,\langle\widetilde{x}\rangle) - V(t_0,\langle\widetilde{0}\rangle)| \leq L(t_0)d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle),$$

for each $\langle\widetilde{x}\rangle \in S(\rho)$. Thus, there exists $\delta = \delta(t_0)$ such that if $d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle) < \delta$, then $V(t_0,\langle\widetilde{x}\rangle) < \delta_0$.

Let $x(t) = x(t,t_0,x_0)$ be any solution of (1) through (t_0,x_0) existing on $[t_0,+\infty)$. Next, we shall show that if $d_{\sup}(x_0,\langle\widetilde{0}\rangle) < \delta$ then $d_{\sup}(x(t),\langle\widetilde{0}\rangle) < \varepsilon$ for each $t \geq t_0 + T$. By the conditions (1), (3) and Lemma 3.2, we get

$$V(t,x(t)) \leq r(t,t_0,V(t_0,x_0)), \qquad t \geq t_0 + T,$$

where $r(t,t_0,V(t_0,x_0))$ is the maximal solution of the scalar differential equation (2) existing on $[t_0,+\infty)$. Since $V(t_0,x_0) < \delta_0$, we have $r(t,t_0,V(t_0,x_0)) < \omega(\varepsilon)$ for each $t \geq t_0 + T$ and therefore

$$V(t,x(t)) \leq r(t,t_0,V(t_0,x_0)) < \omega(\varepsilon), \qquad t \geq t_0 + T.$$

By the condition (2), we get

$$\omega(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) \leq V(t, x(t)) < \omega(\varepsilon), \qquad t \geq t_0 + T.$$

By the monotonicity of ω, we have

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is asymptotically stable. □

Theorem 3.2 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the conditions (1), (2) and (3) of Theorem 3.1. If the solution $\varphi(t) = 0$ of (2) is equi-asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable.*

Proof. In fact, we can show Theorem 3.2 by a similar method of Theorem 3.1. □

Theorem 3.3 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*

(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot, \cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;

(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(t, d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(t, \cdot) \in \mathcal{K}$;

(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\beta V(t, \langle\widetilde{x}\rangle), \beta > 0$.

Then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. By Theorem 3.2 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is stable. Thus, taking $\varepsilon = \rho$, there exists a $\delta = \delta(t_0, \rho)$ such that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$, then

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \rho, \qquad t \geq t_0.$$

Let the function $g(t, \varphi) = -\beta\varphi$, $(t, \varphi) \in \mathbb{R}_+^2$ and $\varphi_0 = V(t_0, x_0)$ in Lemma 3.2. Then we know that

$$r(t, t_0, \varphi_0) = V(t_0, x_0) e^{-\beta(t-t_0)}, \qquad t \geq t_0,$$

is the unique solution of the scalar differential equation (2). Thus, by Lemma 3.2, we obtain

$$V(t, x(t)) \leq V(t_0, x_0) e^{-\beta(t-t_0)}, \qquad t \geq t_0.$$

For any given $\varepsilon > 0$, we take $T = T(t_0, \varepsilon) = \dfrac{1}{\beta} \ln \dfrac{\omega_2(t_0, \delta)}{\omega_1(\varepsilon)} + 1$. Then, by the condition (2), we get

$$\begin{aligned} \omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) &\leq V(t, x(t)) \leq V(t_0, x_0) e^{-\beta(t-t_0)} \\ &\leq e^{-\beta} \omega_2(t_0, d_{\sup}(x_0, \langle\widetilde{0}\rangle)) \frac{\omega_1(\varepsilon)}{\omega_2(t_0, \delta)} \\ &\leq e^{-\beta} \omega_2(t_0, \delta)) \frac{\omega_1(\varepsilon)}{\omega_2(t_0, \delta)} \\ &= e^{-\beta} \omega_1(\varepsilon) < \omega_1(\varepsilon), \end{aligned}$$

which implies that

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable. □

Theorem 3.4 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*

(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot, \cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;

(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$;

(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq g(t, V(t, \langle\widetilde{x}\rangle))$, $g(\cdot, \cdot) \in C[\mathbb{R}_+^2, \mathbb{R}], g(t, 0) = 0$.

If the solution $\varphi(t) = 0$ of (2) is uniformly asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable.

Proof. If the solution $\varphi(t) = 0$ of (2) is uniformly asymptotically stable, then by (S5) of Definition 3.5, we have it is uniformly stable. Thus, by Theorem 3.3 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly stable.

Since for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 > 0$ and $T = T(\varepsilon)$ such that if $0 \leq \varphi_0 < \delta_0$ then

$$|\varphi(t, t_0, \varphi_0)| < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

Since $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$, there exist a $\delta > 0$ such that $\omega_2(\delta) < \omega_1(\delta_0)$.

Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. Next, we shall show that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$ then $d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon$ for each $t \geq t_0 + T$. By the conditions (1), (3) and Lemma 3.2, we get

$$V(t, x(t)) \leq r(t, t_0, \omega_1^{-1}(V(t_0, x_0))), \qquad t \geq t_0 + T,$$

where $r(t, t_0, \omega_1^{-1}(V(t_0, x_0)))$ is the maximal solution of the scalar differential equation (2) existing on $[t_0, +\infty)$. By the condition (2), we have

$$V(t_0, x_0) \leq \omega_2(d_{\sup}(x_0, \langle\widetilde{0}\rangle)) \leq \omega_2(\delta) < \omega_1(\delta_0).$$

Thus, by the monotonicity of ω_1, we have $\omega_1^{-1}(V(t_0, x_0)) \leq \delta_0$, which implies that

$$r(t, t_0, \omega_1^{-1}(V(t_0, x_0))) < \omega_1(\varepsilon), \qquad t \geq t_0 + T$$

and therefore

$$V(t, x(t)) \leq r(t, t_0, \omega_1^{-1}(V(t_0, x_0))) < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

By the condition (2), we get

$$\omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) \leq V(t, x(t)) < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

By the monotonicity of ω_1, we have

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable. □

Theorem 3.5 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*

(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot, \cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;

(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$;

(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\omega_3(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_3(\cdot) \in \mathcal{K}$.

Then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) *is uniformly asymptotically stable.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. By Theorem 3.4 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly stable. Thus, taking $\varepsilon = \rho$, there exists a $\delta = \delta(\rho)$ such that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$, then

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \rho, \qquad t \geq t_0.$$

Let the function $g(t, \varphi) \equiv 0$, $(t, \varphi) \in \mathbb{R}_+^2$ and $\varphi_0 = V(t_0, x_0)$ in Lemma 3.3. Then we know that $r(t, t_0, \varphi_0) \equiv V(t_0, x_0)$ is the unique solution of the scalar differential equation (2). Thus, by Lemma 3.3, we obtain

$$V(t, x(t)) + \int_{t_0}^{t} \omega_3(d_{\sup}(x(s), \langle\widetilde{0}\rangle)) ds \leq V(t_0, x_0), \qquad t \geq t_0.$$

For any given $\varepsilon > 0$, we take $T = T(\varepsilon) = \dfrac{\omega_2(\delta)}{\omega_3\omega_2^{-1}\omega_1(\varepsilon)} + 1$. Suppose that $d_{\sup}(x(t), \langle\widetilde{0}\rangle) \geq \omega_2^{-1}\omega_1(\varepsilon)$ for each $t \in [t_0, t_0 + T]$. Then, by the condition (2), we get

$$\begin{aligned} V(t, x(t)) &= V(t_0, x_0) - \int_{t_0}^{t} \omega_3(d_{\sup}(x(s), \langle\widetilde{0}\rangle))ds \\ &\leq \omega_2(d_{\sup}(x_0, \langle\widetilde{0}\rangle)) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)(t - t_0) \\ &< \omega_2(\delta) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)(t - t_0), \end{aligned}$$

for each $t \in [t_0, t_0 + T]$. Thus, we obtain

$$0 \leq V(t_0 + T, x(t_0 + T)) < \omega_2(\delta) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)T = -\omega_3\omega_2^{-1}\omega_1(\varepsilon) < 0.$$

This is a contradiction, thus there exists a $t^* \in [t_0, t_0 + T]$ such that

$$d_{\sup}(x(t^*), \langle\widetilde{0}\rangle) < \omega_2^{-1}\omega_1(\varepsilon).$$

Since $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\omega_3(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq 0$, we have

$$V(t, x(t)) \leq V(t^*, x(t^*)), \qquad t \geq t^*.$$

Then, by the condition (2), we get

$$\begin{aligned} \omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) &\leq V(t, x(t)) \leq V(t^*, x(t^*)) \\ &\leq \omega_2(d_{\sup}(x(t^*), \langle\widetilde{0}\rangle)) \\ &< \omega_2\omega_2^{-1}\omega_1(\varepsilon) = \omega_1(\varepsilon), \end{aligned}$$

which implies that $d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon$ for each $t \geq t^*$. Hence, we obtain

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Consequently, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable. □

Example 3.1 *Define $F : \mathbb{R}_+ \to \mathscr{F}/\mathscr{S}$ by the α-level sets of the fuzzy mapping*

$$\left[\widehat{F(t)}\right]^\alpha = \left[-\frac{2e^{-\alpha}}{1+t}, 0\right], \qquad \alpha \in [0, 1],$$

where $\widehat{F(t)}$ is the Mareš core of $F(t)$, for each $t \in \mathbb{R}_+$. Thus, we have

$$M_{F(t)}(\alpha) = -\frac{e^{-\alpha}}{1+t}, \qquad \alpha \in [0, 1],$$

for each $t \in \mathbb{R}_+$. It is obvious that $M_{F(t)}(\alpha)$ is continuous from the right at 0 and continuous from the left on $[0, 1]$ with respect to α. Since $M_{F(t)}(\alpha)$ is increasing with respect to α, we get

$$V_0^1(M_{F(t)}) = \frac{1 - e^{-1}}{1+t} \leq 1 - e^{-1}, \qquad t \in \mathbb{R}_+.$$

Thus, we obtain that $F(t)$ is of uniformly bounded variation. Since $M_{F(t)}(\alpha)$ is uniformly continuous with respect to $t \in \mathbb{R}_+$, we get that $F(t)$ is continuous with respect to $d_{\sup}$. Define $f : \mathbb{R}_+ \times \mathscr{F}/\mathscr{S} \to \mathscr{F}/\mathscr{S}$ by

$$f(t, \langle\widetilde{x}\rangle) = F(t)\,\langle\widetilde{x}\rangle\,.$$

It is obvious that f is continuous with respect to $d_{\sup}$ and of uniformly bounded variation.

Consider a Lyapunov function $V(t, \langle \widetilde{x} \rangle) = d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{0} \rangle)$. *Then* $V(t, \langle \widetilde{0} \rangle) = d_{\sup}(\langle \widetilde{0} \rangle, \langle \widetilde{0} \rangle) = 0$ *and*

$$|V(t, \langle \widetilde{x} \rangle) - V(t, \langle \widetilde{y} \rangle)| = \left| d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{0} \rangle) - d_{\sup}(\langle \widetilde{y} \rangle, \langle \widetilde{0} \rangle) \right| \leq d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle),$$

for any $(t, \langle \widetilde{x} \rangle), (t, \langle \widetilde{y} \rangle) \in \mathbb{R}_+ \times \mathscr{F}/\mathscr{S}$. *By Definition 2.9, for a small* $h > 0$, *we have*

$$\begin{aligned} V(t+h, \langle \widetilde{x} \rangle + hf(t, \langle \widetilde{x} \rangle)) &= d_{\sup}(\langle \widetilde{x} \rangle + hf(t, \langle \widetilde{x} \rangle), \langle \widetilde{0} \rangle) = d_{\sup}(\langle \widetilde{x} \rangle + hF(t) \langle \widetilde{x} \rangle, \langle \widetilde{0} \rangle) \\ &= \sup_{\alpha \in [0,1]} \left| M_{\langle \widetilde{x} \rangle}(\alpha) + hM_{F(t)}(\alpha) M_{\langle \widetilde{x} \rangle}(\alpha) \right| \\ &\leq \sup_{\alpha \in [0,1]} \left| M_{\langle \widetilde{x} \rangle}(\alpha) \right| \left(1 + h \sup_{\alpha \in [0,1]} M_{F(t)}(\alpha) \right) \\ &= \left(1 - \frac{he^{-1}}{1+t} \right) d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{0} \rangle). \end{aligned}$$

Hence, we get

$$D_f^+ V(t, \langle \widetilde{x} \rangle) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \left(V(t+h, \langle \widetilde{x} \rangle + hf(t, \langle \widetilde{x} \rangle)) - V(t, \langle \widetilde{x} \rangle) \right) \leq -\frac{e^{-1}}{1+t} d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{0} \rangle).$$

Let $g(t, \varphi) = -\dfrac{e^{-1}}{1+t}\varphi$. *Then, we have*

$$D_f^+ V(t, \langle \widetilde{x} \rangle) \leq g(t, d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{0} \rangle)) = g(t, V(t, \langle \widetilde{x} \rangle)).$$

It's easy to show that the solution $\varphi = 0$ *of* (2) *is asymptotically stable. Hence, by Theorem 3.1, the trivial solution* $x(t) = \langle \widetilde{0} \rangle$ *of* (1) *is asymptotically stable.*

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ON DIFFERENTIAL EQUATIONS ASSOCIATED WITH SQUARED HERMITE POLYNOMIALS

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Abstract. In this paper, we investigate differential equations associated with squared Hermite polynomials and derive some new and explicit identities for these polynomials arising from the differential equations.

1. Introduction

As a method of obtaining new identities for special polynomials and numbers, in [8] T. Kim initiated a remarkable idea of using ordinary differential equations. Namely, he derived a family of nonlinear differential equations, indexed by positive integers, satisfied by the generating function of the Frobenius-Euler numbers and used them in order to get an interesting identity expressing higher-order Frobenius-Euler numbers in terms of (ordinary) Frobenius-Euler numbers. Here, more precisely, the differential equations are satisfied not by the generating function of the Frobenius-Euler numbers but by a constant multiple of that.

This method turned out to be very fruitful and can be applied to many interesting special polynomials and numbers (see [5, 8–11]). For example, linear differential equations are derived for Bessel polynomials, Changhee polynomials, actuarial polynomials, Meixner polynomials of the first kind, Poisson-Charlier polynomials, Laguerre polynomials, Hermite polynomials, and Stirling polynomials, while nonlinear ones are obtained for Bernoulli numbers of the

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second, Boole numbers, Chebyshev polynomials of the first, second, third, and fourth kind, degenerate Euler numbers, degenerate Eulerian polynomials, Korobov numbers, and Legendre polynomials.

To be specific, we will illustrate the results in the case of Bernoulli numbers of the second kind (see [5]). Firstly, it is shown that the function $F = F(t) = \frac{1}{log(1+t)}$ satisfies the family of nonlinear differential equations

$$F^{(N)}(t) = \frac{(-1)^N}{(1+t)^N} \sum_{j=2}^{N+1} (j-1)!(N-1)! H_{N-1,j-2} F^j \qquad (N = 1, 2, \cdots), \tag{1}$$

where H_N are the generalized harmonic numbers defined by

$$\begin{aligned} &H_{N,0} = 1, \quad \text{for all } N, \\ &H_{N,1} = \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}, \\ &H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-1,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j} \quad (N \geq j \geq 2). \end{aligned} \tag{2}$$

Recall that the Bernoulli numbers of the second b_n are given by the generating function

$$\frac{t}{log(1+t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \quad \text{(see [5])}. \tag{3}$$

More generally, the Bernoulli numbers of the second $b_n^{(r)}$ of order r are defined by the generating function

$$\left(\frac{t}{log(1+t)}\right)^r = \sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!} \quad \text{(see [5])}. \tag{4}$$

Then, secondly the family of differential equations in (1) are used to derive the following interesting identities: for $N = 1, 2, \cdots$ and $n = 0, 1, \cdots$, we have

$$\begin{aligned} &(-1)^n \sum_{j=0}^{\min\{n,N-1\}} (N-j)!(N-1)! H_{N-1,N-1-j} (n)_j b_{n-j}^{(N+1-j)} \\ &= \begin{cases} (-1)^N N!(N)_n & \text{if} \quad 0 \leq n \leq N, \\ \sum_{l=0}^{n-N-1} \binom{N}{l} \frac{b_{n-l}}{n-l} (n)_{l+N+1} & \text{if} \quad n \geq N+1. \end{cases} \end{aligned} \tag{5}$$

As a generalization of the usual factorial $n!$, the double factorial of a positive integer n is defined by

$$n!! = \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1 & \text{if} \quad n > 0 \text{ odd}, \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2 & \text{if} \quad n > 0, \text{ even}, \\ 1 & \text{if} \quad n = -1, \ 0. \end{cases} \tag{6}$$

(see [1]).

Throughout this paper, the double factorials will be used.

The Hermite polynomials are classical orthogonal polynomials used such diverse areas as combinatorics, numerical analysis, probability, finite element methods, systems theory and quantum mechanics (see [2–4, 6, 7, 12–14]).

With the Roman's definition of Hermite polynomials $H_n(x)$ as

$$H_n(x) = e^{xt-t^2/2}, \tag{7}$$

we see from ([3], p.250) that

$$(1-t^2)^{-1/2}e^{x[t/(1+t)]} = \sum_{n=0}^{\infty}[H_n(\sqrt{x})]^2\frac{t^n}{n!}. \tag{8}$$

For brevity, we denote $[H_n(\sqrt{x})]^2$ by $SH_n(x)$, and hence

$$(1-t^2)^{-1/2}e^{x[t/(1+t)]} = \sum_{n=0}^{\infty}SH_n(x)\frac{t^n}{n!}. \tag{9}$$

In this paper, we would like to derive a family of linear differential equations satisfied by the generating function of the squared Hermite polynomials in (9) and use them in order to get an interesting identity for those polynomials. As an easy consequence of this result, we will have an expression for the squared Hermite polynomials.

2. Differential equations for the squared Hermite polynomials

In this paper, all differentiations are taken with respect to t, while x being fixed. Let

$$\begin{aligned} F = F(t;x) &= \left(1-t^2\right)^{-\frac{1}{2}} e^{x\left(\frac{t}{t+1}\right)} \\ &= (1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}} e^{x\left(\frac{t}{t+1}\right)}. \end{aligned} \tag{10}$$

Then

$$\begin{aligned} F^{(1)} &= \frac{1}{2}(1-t)^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}}e^{x\left(\frac{t}{t+1}\right)} - \frac{1}{2}(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{3}{2}}e^{x\left(\frac{t}{t+1}\right)} \\ &\quad +(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}(1+t)^{-2}xe^{x\left(\frac{t}{t+1}\right)} \\ &= \left\{\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2}\right\}F. \end{aligned} \tag{11}$$

$$\begin{aligned} F^{(2)} &= \left\{\frac{1}{2}(1-t)^{-2} + \frac{1}{2}(1+t)^{-2} - 2x(1+t)^{-3}\right\}F \\ &+ \left\{\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2}\right\}^2 F \\ &= \left\{\frac{1}{2}(1-t)^{-2} + \frac{1}{2}(1+t)^{-2} - 2x(1+t)^{-3}\right\}F \\ &\quad + \left\{\frac{1}{4}(1-t)^{-2} + \frac{1}{4}(1+t)^{-2} + x^2(1+t)^{-4}\right. \\ &\quad \left. -\frac{1}{2}(1-t)^{-1}(1+t)^{-1} - x(1+t)^{-3} + x(1-t)^{-1}(1+t)^{-2}\right\}F \\ &= \left\{\frac{3}{4}(1-t)^{-2} - \frac{1}{2}(1-t)^{-1}(1+t)^{-1} + x(1-t)^{-1}(1+t)^{-2}\right. \\ &\quad \left. +\frac{3}{4}(1+t)^{-2} - 3x(1+t)^{-3} + x^2(1+t)^{-4}\right\}F. \end{aligned} \tag{12}$$

So, we are led to put

$$F^{(N)} = \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F. \tag{13}$$

Here $a_{i,j}(N,x)$ are polynomials in x.

$$\begin{aligned}
F^{(N+1)} &= \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} ia_{i,j}(N,x)(1-t)^{-(i+1)}(1+t)^{-j}\right)F \\
&\quad -\left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} ja_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+1)}\right)F \\
&\quad +\left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right. \\
&\quad \left.\times\left\{\frac{1}{2}(1-t)^{-1}-\frac{1}{2}(1+t)^{-1}+x(1+t)^{-2}\right\}\right)F \\
&= \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} (i+\frac{1}{2})a_{i,j}(N,x)(1-t)^{-(i+1)}(1+t)^{-j}\right)F \\
&\quad -\left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} (j+\frac{1}{2})a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+1)}\right)F \\
&\quad +\left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} xa_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+2)}\right)F \\
&= \left(\sum_{i=1}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} (i-\frac{1}{2})a_{i-1,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F \\
&\quad -\left(\sum_{i=0}^{N}\sum_{j=N-i+1}^{2(N-i)+1} (j-\frac{1}{2})a_{i,j-1}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F \\
&\quad +\left(\sum_{i=0}^{N}\sum_{j=N-i+2}^{2(N-i)+2} xa_{i,j-2}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F. \qquad (14)
\end{aligned}$$

On the other hand,

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} a_{i,j}(N+1,x)(1-t)^{-i}(1+t)^{-j}\right)F. \qquad (15)$$

In order to add the sums in (14), we decompose them as follows:

$$\begin{aligned}
\sum_{i=1}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} &= \sum_{i=1}^{N}\sum_{j=N+2-i}^{2(N-i)+1} + \sum_{i=1}^{N}\sum_{j=N+1-i} \\
&\quad + \sum_{i=1}^{N}\sum_{j=2(N+1-i)} + \sum_{i=N+1}\sum_{j=0}; \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{N}\sum_{j=N-i+1}^{2(N-i)+1} &= \sum_{i=1}^{N}\sum_{j=N-i+2}^{2(N-i)+1} + \sum_{i=1}^{N}\sum_{j=N-i+1} \\
&\quad + \sum_{i=0}\sum_{j=N+2}^{2N+1} + \sum_{i=0}\sum_{j=N+1}; \qquad (17)
\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{N}\sum_{j=N-i+2}^{2(N-i)+2} &= \sum_{i=1}^{N}\sum_{j=N-i+2}^{2(N-i)+1}+\sum_{i=1}^{N}\sum_{j=2(N-i)+2}\\ &+\sum_{i=0}\sum_{j=N+2}^{2N+1}+\sum_{i=0}\sum_{j=2N+2}. \end{aligned}\tag{18}$$

Now, the sum in (14) can be rewritten as

$$\begin{aligned}&F^{(N+1)}\\ =&\sum_{i=1}^{N}\sum_{j=N+2-i}^{2(N-i)+1}\left\{(i-\frac{1}{2})a_{i-1,j}(N,x)-(j-\frac{1}{2})a_{i,j-1}(N,x)+xa_{i,j-2}(N,x)\right\}\\ &\qquad\times(1-t)^{-i}(1+t)^{-j}F\\ &+\sum_{i=1}^{N}\left\{(i-\frac{1}{2})a_{i-1,N-i+1}(N,x)-(N-i+\frac{1}{2})a_{i,N-i}(N,x)\right\}\\ &\qquad\times(1-t)^{-i}(1+t)^{-(N-i+1)}F\\ &+\sum_{i=1}^{N}\left\{(i-\frac{1}{2})a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x)\right\}(1-t)^{-i}(1+t)^{-2(N+1-i)}F\\ &+\sum_{j=N+2}^{2N+1}\left\{-(j-\frac{1}{2})a_{0,j-1}(N,x)+xa_{0,j-2}(N,x)\right\}(1+t)^{-j}F\\ &-(N+\frac{1}{2})a_{0,N}(N,x)(1+t)^{-(N+1)}F+xa_{0,2N}(N,x)(1+t)^{-(2N+2)}F\\ &+(N+\frac{1}{2})a_{N,0}(N,x)(1-t)^{-(N+1)}F.\end{aligned}\tag{19}$$

Comparing (15) and (19), we obtain: for $1\le i\le N$, $N-i+2\le j\le 2(N-i)+1$,

$$a_{i,j}(N+1,x)=(i-\frac{1}{2})a_{i-1,j}(N,x)-(j-\frac{1}{2})a_{i,j-1}(N,x)+xa_{i,j-2}(N,x);\tag{20}$$

for $1\le i\le N$,

$$a_{i,N-i+1}(N+1,x)=(i-\frac{1}{2})a_{i-1,N-i+1}(N,x)-(N-i+\frac{1}{2})a_{i,N-i}(N,x);\tag{21}$$

for $1\le i\le N$,

$$a_{i,2(N+1-i)}(N+1,x)=(i-\frac{1}{2})a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x);\tag{22}$$

for $N+2\le j\le 2N+1$,

$$a_{0,j}(N+1,x)=-(j-\frac{1}{2})a_{0,j-1}(N,x)+xa_{0,j-2}(N,x);\tag{23}$$

$$a_{0,N+1}(N+1,x)=-(N+\frac{1}{2})a_{0,N}(N,x);\tag{24}$$

$$a_{0,2N+2}(N+1,x)=xa_{0,2N}(N,x);\tag{25}$$

$$a_{N+1,0}(N+1,x)=(N+\frac{1}{2})a_{N,0}(N,x).\tag{26}$$

Note here that all of these recurrence relations can be merged into one relation (20), for $0\le i\le N+1$, $N-i+1\le j\le 2(N-i+1)$, with the understanding that

$$a_{i,j}(N,x)=0,\tag{27}$$

unless $0 \leq i \leq N,\ N-i \leq j \leq 2(N-i)$. In addition to these, we have the following initial conditions:

$$F = F^{(0)} = a_{0,0}(0,x)F \longrightarrow a_{0,0}(0,x) = 1, \tag{28}$$

$$\begin{aligned}
F^{(1)} &= \left(\sum_{i=0}^{1} \sum_{j=1-i}^{2(1-i)} a_{i,j}(1,x)(1-t)^{-i}(1+t)^{-j} \right) F \\
&= \left(a_{0,1}(1,x)(1+t)^{-1} + a_{0,2}(1,x)(1+t)^{-2} + a_{1,0}(1,x)(1-t)^{-1} \right) F \\
&= \left(\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2} \right) F \\
&\longrightarrow \quad a_{1,0}(1,x) = \frac{1}{2},\ a_{0,1}(1,x) = -\frac{1}{2},\ a_{0,2}(1,x) = x.
\end{aligned} \tag{29}$$

As easy consequences, from (24)-(26) we get

$$\begin{aligned}
a_{N+1,0}(N+1,x) &= \left(N+\frac{1}{2}\right) a_{N,0}(N,x) \\
&= \left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) a_{N-1,0}(N-1,x) \\
&= \cdots \\
&= \left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) \cdots \frac{3}{2} a_{1,0}(1,x) \\
&= \left(\frac{1}{2}\right)^{N+1} (2N+1)!!
\end{aligned} \tag{30}$$

$$\begin{aligned}
a_{0,N+1}(N+1,x) &= -\left(N+\frac{1}{2}\right) a_{0,N}(N,x) \\
&= (-1)^2 \left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) a_{0,N-1}(N-1,x) \\
&= \cdots \\
&= (-1)^N \left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) \cdots \frac{3}{2} a_{0,1}(1,x) \\
&= \left(-\frac{1}{2}\right)^{N+1} (2N+1)!!
\end{aligned} \tag{31}$$

$$\begin{aligned}
a_{0,2N+2}(N+1,x) &= x a_{0,2N}(N,x) = x^2 a_{0,2(N-1)}(N-1,x) \\
&= x^N a_{0,2}(1,x) = x^{N+1} a_{0,0}(0,x) = x^{N+1}.
\end{aligned} \tag{32}$$

Let $N+2 \leq j \leq 2N+1$. Then, from (23), we have

$$a_{0,j}(N+1,x) = x a_{0,j-2}(N,x) - \left(j-\frac{1}{2}\right) a_{0,j-1}(N,x). \tag{33}$$

For $j = N+2$, we get the following:

$$\begin{aligned}
& a_{0,N+2}(N+1,x) \\
=\ & x a_{0,N}(N,x) - \left(N+\frac{3}{2}\right) a_{0,N+1}(N,x) \\
=\ & x a_{0,N}(N,x) - \left(N+\frac{3}{2}\right)\left(x a_{0,N-1}(N-1,x) - \left(N+\frac{1}{2}\right) a_{0,N}(N-1,x) \right) \\
=\ & x \left(a_{0,N}(N,x) - (N+\frac{3}{2}) a_{0,N-1}(N-1,x) \right) \\
& +(-1)^2 \left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\left(x a_{0,N-2}(N-2,x) - \left(N-\frac{1}{2}\right) a_{0,N-1}(N-2,x) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= x\sum_{k=0}^{N-1}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-k+\frac{5}{2}\right)a_{0,N-k}(N-k,x) \\
&\quad +(-1)^N\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\frac{5}{2}a_{0,2}(1,x) \\
&= x\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k(2N+3)(2N+1)\cdots(2N-2k+5)a_{0,N-k}(N-k,x) \\
&= x\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k\frac{(2N+3)!!}{(2N-2k+3)!!}a_{0,N-k}(N-k,x). \qquad (34)
\end{aligned}
$$

For $j=N+3$, we obtain the following:

$$
\begin{aligned}
&a_{0,N+3}(N+1,x) \\
&= xa_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)a_{0,N+2}(N,x) \\
&= xa_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)\left(xa_{0,N}(N-1,x)-\left(N+\frac{3}{2}\right)a_{0,N+1}(N-1,x)\right) \\
&= x\left(a_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)a_{0,N}(N-1,x)\right) \\
&\quad (-1)^2\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\left(xa_{0,N-1}(N-2,x)-\left(N+\frac{1}{2}\right)a_{0,N}(N-2,x)\right) \\
&= \cdots \\
&= x\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\left(N-k+\frac{7}{2}\right)a_{0,n-k+1}(N-k,x) \\
&\quad +(-1)^{N-1}\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\frac{9}{2}a_{0,4}(2,x) \\
&= x\sum_{k=0}^{N-1}(-1)^k\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\left(N-k+\frac{7}{2}\right)a_{0,n-k+1}(N-k,x) \\
&= x\sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k\frac{(2N+5)!!}{(2N-2k+5)!!}a_{0,N-k+1}(N-k,x). \qquad (35)
\end{aligned}
$$

Continuing this process, we can deduce that, for $N+2\le j\le 2N+1$,

$$
a_{0,j}(N+1,x)=x\sum_{k=0}^{2N+2-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}a_{0,j-k-2}(N-k,x). \qquad (36)
$$

Let $1\le i\le N$. Then, from (21), we have

$$
a_{i,N-i+1}(N+1,x)=\left(i-\frac{1}{2}\right)a_{i-1,N-i+1}(N,x)-\left(N-i+\frac{1}{2}\right)a_{i,N-i}(N,x). \qquad (37)
$$

For $i=1$, we obtain the following:

$$
\begin{aligned}
&a_{1,N}(N+1,x) \\
&= \frac{1}{2}a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)a_{1,N-1}(N,x) \\
&= \frac{1}{2}a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)\left(\frac{1}{2}a_{0,N-1}(N-1,x)-(N-\frac{3}{2})a_{1,N-2}(N-1,x)\right) \\
&= \frac{1}{2}\left(a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)a_{0,N-1}(N-1,x)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^2\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\left(\frac{1}{2}a_{0,N-2}(N-2,x)-\left(N-\frac{5}{2}\right)a_{1,N-3}(N-2,x)\right)\\
=\ &\cdots\\
=\ &\frac{1}{2}\sum_{k=0}^{N-1}(-1)^k\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\cdots\left(N-\frac{2k-1}{2}\right)a_{0,N-k}(N-k,x)\\
&+(-1)^N\left(N-\frac{1}{2}\right)\cdots\frac{1}{2}a_{1,0}(1,x)\\
=\ &\frac{1}{2}\sum_{k=0}^{N}(-1)^k\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\cdots\left(N-\frac{2k-1}{2}\right)a_{0,N-k}(N-k,x)\\
=\ &\frac{1}{2}\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k\frac{(2N-1)!!}{(2N-2k-1)!!}a_{0,N-k}(N-k,x). \qquad (38)
\end{aligned}
$$

For $i=2$, we get the following:

$$
\begin{aligned}
&a_{2,N-1}(N+1,x)\\
=\ &\frac{3}{2}a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)a_{2,N-2}(N,x)\\
=\ &\frac{3}{2}a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)\left(\frac{3}{2}a_{1,N-2}(N-1,x)-\left(N-\frac{5}{2}\right)a_{2,N-3}(N-1,x)\right)\\
=\ &\frac{3}{2}\left(a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)a_{1,N-2}(N-1,x)\right)\\
&+(-1)^2\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\left(\frac{3}{2}a_{1,N-3}(N-2,x)-\left(N-\frac{7}{2}\right)a_{2,N-4}(N-2,x)\right)\\
=\ &\cdots\\
=\ &\frac{3}{2}\sum_{k=0}^{N-2}(-1)^k\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\left(N-\frac{2k+1}{2}\right)a_{1,N-k-1}(N-k,x)\\
&+(-1)^{N-1}\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\frac{1}{2}a_{2,0}(2,x)\\
=\ &\frac{3}{2}\sum_{k=0}^{N-1}(-1)^k\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\left(N-\frac{2k+1}{2}\right)a_{1,N-k-1}(N-k,x)\\
=\ &\frac{3}{2}\sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k\frac{(2N-3)!!}{(2N-2k-3)!!}a_{1,N-k-1}(N-k,x). \qquad (39)
\end{aligned}
$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$
\begin{aligned}
&a_{i,N-i+1}(N+1,x)\\
=\ &\frac{2i-1}{2}\sum_{k=0}^{N-i+1}\left(-\frac{1}{2}\right)^k\frac{(2N-2i+1)!!}{(2N-2k-2i+1)!!}a_{i-1,N-k-i+1}(N-k,x). \qquad (40)
\end{aligned}
$$

Let $1\le i\le N$. Then, from (22), we have

$$
\begin{aligned}
&a_{i,2(N+1-i)}(N+1,x)\\
=\ &\left(i-\frac{1}{2}\right)a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x). \qquad (41)
\end{aligned}
$$

Then, proceeding analogously to the case of (37), we can deduce that, for $1\le i\le N$,

$$
a_{i,2(N+1-i)}(N+1)=\frac{2i-1}{2}\sum_{k=0}^{N-i+1}x^k a_{i-1,2(N-k-i+1)}(N-k,x), \qquad (42)
$$

For $1 \le i \le N$, $N-i+2 \le j \le 2(N-i)+1$, from (20) we have

$$\begin{aligned} & a_{i,j}(N+1,x) \\ = & \left(i-\frac{1}{2}\right) a_{i-1,j}(N,x) - \left(j-\frac{1}{2}\right) a_{i,j-1}(N,x) + x a_{i,j-2}(N,x). \end{aligned} \tag{43}$$

Let $i=1$, Then, with $N+1 \le j \le 2N-1$, (43) becomes

$$a_{1,j}(N+1,x) = \frac{1}{2} a_{0,j}(N,x) + x a_{1,j-2}(N,x) - \left(j-\frac{1}{2}\right) a_{1,j-1}(N,x). \tag{44}$$

For $j = N+1$, we get the following:

$$\begin{aligned} & a_{1,N+1}(N+1,x) \\ = & \frac{1}{2} a_{0,N+1}(N,x) + x a_{1,N-1}(N,x) - \left(N+\frac{1}{2}\right) a_{1,N}(N,x) \\ = & \frac{1}{2} a_{0,N+1}(N,x) + x a_{1,N-1}(N,x) \\ & - \left(N+\frac{1}{2}\right)\left(\frac{1}{2} a_{0,N}(N-1,x) + x a_{1,N-2}(N-1,x) - \left(N-\frac{1}{2}\right) a_{1,N-1}(N-1,x)\right) \\ = & \frac{1}{2}\left(a_{0,N+1}(N,x) - \left(N+\frac{1}{2}\right) a_{0,N}(N-1,x)\right) \\ & + x\left(a_{1,N-1}(N,x) - \left(N+\frac{1}{2}\right) a_{1,N-2}(N-1,x)\right) + (-1)^2\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) \\ & \times\left(\frac{1}{2} a_{0,N-1}(N-2,x) + x a_{1,N-3}(N-2,x) - \left(N-\frac{3}{2}\right) a_{1,N-2}(N-2,x)\right) \\ & = \cdots \\ & = \frac{1}{2}\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(N-\frac{2k-3}{2}\right) a_{0,N-k+1}(N-k,x) \\ & + x\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(N-\frac{2k-3}{2}\right) a_{1,N-k-1}(N-k,x) \\ & + (-1)^{N-1}\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(\frac{5}{2}\right) a_{1,2}(2,x) \\ & = \sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k \frac{(2N+1)!!}{(2N-2k+1)!!}\left(\frac{1}{2} a_{0,N-k+1}(N-k,x) + x a_{1,N-k-1}(N-k,x)\right). \end{aligned} \tag{45}$$

For $j = N+2$, we obtain the following:

$$\begin{aligned} & a_{1,N+2}(N+1,x) \\ = & \frac{1}{2} a_{0,N+2}(N,x) + x a_{1,N}(N,x) - \left(N+\frac{3}{2}\right) a_{1,N+1}(N,x) \\ & = \frac{1}{2} a_{0,N+2}(N,x) + x a_{1,N}(N,x) \\ & - \left(N+\frac{3}{2}\right)\left(\frac{1}{2} a_{0,N+1}(N-1,x) + x a_{1,N-1}(N-1,x) - \left(N+\frac{1}{2}\right) a_{1,N}(N-1,x)\right) \\ & = \frac{1}{2}\left(a_{0,N+2}(N,x) - \left(N+\frac{3}{2}\right) a_{0,N+1}(N-1,x)\right) \\ & + x\left(a_{1,N}(N,x) - \left(N+\frac{3}{2}\right) a_{1,N-1}(N-1,x)\right) \\ & + (-1)^2\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{2}a_{0,N}(N-2,x) + xa_{1,N-2}(N-2,x) - \left(N-\frac{1}{2}\right)a_{1,N-1}(N-2,x)\right)\\
&= \cdots\\
&= \frac{1}{2}\sum_{k=0}^{N-3}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-\frac{2k-5}{2}\right)a_{0,N-k+2}(N-k,x)\\
&\quad +x\sum_{k=0}^{N-3}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-\frac{2k-5}{2}\right)a_{1,N-k}(N-k,x)\\
&\quad +(-1)^{N-2}\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\frac{9}{2}a_{1,4}(3,x)\\
&= \sum_{k=0}^{N-2}\left(-\frac{1}{2}\right)^k\frac{(2N+3)!!}{(2N-2k+3)!!}\left(\frac{1}{2}a_{0,N-k+2}(N-k,x)+xa_{1,N-k}(N-k,x)\right). \qquad (46)
\end{aligned}
$$

Continuing this process, we can deduce that, for $N+1 \le j \le 2N-1$,

$$
\begin{aligned}
&a_{1,j}(N+1,x)\\
&= \sum_{k=0}^{2N-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{1}{2}a_{0,j-k}(N-k)+xa_{1,j-k-2}(N-k,x)\right). \qquad (47)
\end{aligned}
$$

Let $i=2$. Then, with $N \le j \le 2N-3$, (43) becomes

$$
\begin{aligned}
&a_{2,j}(N+1,x)\\
&= \frac{3}{2}a_{1,j}(N,x)+xa_{2,j-2}(N,x)-\left(j-\frac{1}{2}\right)a_{2,j-1}(N,x). \qquad (48)
\end{aligned}
$$

Then, proceeding analogously to the case of (44), we can deduce that, for $N \le j \le 2N-3$,

$$
\begin{aligned}
&a_{2,j}(N+1,x)\\
&= \sum_{k=0}^{2N-j-2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{3}{2}a_{1,j-k}(N-k,x)+xa_{2,j-k-2}(N-k,x)\right) \qquad (49)
\end{aligned}
$$

Thus we can deduce that, for $1 \le i \le N$, $N-i+2 \le j \le 2(N-i)+1$,

$$
\begin{aligned}
&a_{i,j}(N+1,x)\\
&= \sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\\
&\quad\times\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x)+xa_{i,j-k-2}(N-k,x)\right) \qquad (50)
\end{aligned}
$$

Our results can be summarized as:

$$
\begin{aligned}
&a_{0,0}(0,x)=1;\\
&a_{N+1,0}(N+1,x)=\left(-\frac{1}{2}\right)^{N+1}(2N+1)!!;\\
&a_{0,N+1}(N+1,x)=\left(-\frac{1}{2}\right)^{N+1}(2N+1)!!;\\
&a_{0,2N+2}(N+1,x)=x^{N+1};\\
&a_{0,j}(N+1,x)=x\sum_{k=0}^{2N+2-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}a_{0,j-k-2}(N-k,x)\\
&\quad \text{for } N+2 \le j \le 2N+1;
\end{aligned}
$$

$$a_{i,N-i+1}(N+1,x)=\frac{2i-1}{2}\sum_{k=0}^{N-i+1}\left(-\frac{1}{2}\right)^k\frac{(2N-2i+1)!!}{(2N-2k-2i+1)!!}a_{i-1,N-k-i+1}(N-k,x)$$

for $1 \le i \le N$;

$$a_{i,2(N+1-i)}(N+1,x)=\frac{2i-1}{2}\sum_{k=0}^{N-i+1}x^k a_{i-1,2(N-k-i+1)}(N-k,x),$$

for $1 \le i \le N$;

$$a_{i,j}(N+1,x) = \sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x)+xa_{i,j-k-2}(N-k,x)\right),$$

for $1 \le i \le N, N-i+2 \le j \le 2(N-i)+1$. (51)

From these, we can conclude that, for $0 \le i \le N+1$, $N+1-i \le j \le 2(N+1-i)$,

$$a_{i,j}(N+1,x) = \sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!} \times\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x)+xa_{i,j-k-2}(N-k,x)\right), \quad (52)$$

with $a_{0,0}(0,x)=1$, $a_{1,0}(1,x)=\frac{1}{2}$, $a_{0,1}(1,x)=-\frac{1}{2}$, $a_{0,2}(1,x)=x$, except for $i=0$ and $j=N+1$, in which case

$$a_{0,N+1}(N+1,x)=\left(-\frac{1}{2}\right)^{N+1}(2N+1)!!. \quad (53)$$

Our results can now be stated as the following theorem.

Theorem 1. The ordinary differential equations

$$F^{(N)}=\left(\frac{d}{dt}\right)^N F=\left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)}a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F, \quad (54)$$

$(N=0,1,2,\cdots,)$ have a solution $F=F(t,x)=(1-t^2)^{-\frac{1}{2}}e^{x\left(\frac{t}{1+t}\right)}$, where, for $0 \le i \le N$, $N-i \le j \le 2(N-i)$,

$$a_{i,j}(N,x) = \sum_{k=0}^{2N-j-2i}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!} \times\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k-1,x)+xa_{i,j-k-2}(N-k-1,x)\right), \quad (55)$$

with $a_{0,0}(0,x)=1$, $a_{1,0}(1,x)=\frac{1}{2}$, $a_{0,1}(1,x)=-\frac{1}{2}$, $a_{0,2}(1,x)=x$, except for $i=0$ and $j=N$, in which case

$$a_{0,N}(N,x)=\left(-\frac{1}{2}\right)^{N}(2N-1)!!. \quad (56)$$

3. Applications of differential equations

We recall from (9) that the squared Hermite polynomials $SH_k(x)$ are given by the generating function

$$F = F(t;x) = (1-t^2)^{-\frac{1}{2}} e^{\left(\frac{t}{1+t}\right)} = \sum_{k=0}^{\infty} SH_k(x)\frac{t^k}{k!}. \tag{57}$$

Here we derive some new and explicit identities for the squared Hermite polynomials from the differential equations in Theorem 1. Now, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} SH_{k+N}(x)\frac{t^k}{k!} &= \left(\sum_{k=0}^{\infty} SH_k(x)\frac{t^k}{k!}\right)^{(N)} \\
&= \left(\left(1-t^2\right)^{-\frac{1}{2}} e^{x\left(\frac{t}{1+t}\right)}\right)^{(N)} \\
&= \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right) F \\
&= \sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)\sum_{l=0}^{\infty}(i+l-1)_l\frac{t^l}{l!} \\
&\quad \times \sum_{m=0}^{\infty}(-1)^m(j+m-1)_m\frac{t^m}{m!}\sum_{n=0}^{\infty} SH_n(x)\frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)}\sum_{l+m+n=k}\binom{k}{l,m,n}\right. \\
&\quad \left. \times (-1)^m(i+l-1)_l(j+m-1)_m a_{i,j}(N,x)SH_n(x)\right)\frac{t^k}{k!}.
\end{aligned} \tag{58}$$

From this, we have, for $k, N = 0,1,2,\cdots$

$$\begin{aligned}
SH_{k+N}(x) &= \sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)}\sum_{l+m+n=k}\binom{k}{l,m,n} \\
&\quad \times(-1)^m(i+l-1)_l(j+m-1)_m a_{i,j}(N,x)SH_n(x).
\end{aligned} \tag{59}$$

Thus we obtain the following theorem.

Theorem 2. For $k, N = 0,1,2,\cdots$

$$\begin{aligned}
SH_{k+N}(x) &= \sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)}\sum_{l+m+n=k}\binom{k}{l,m,n} \\
&\quad \times(-1)^m(i+l-1)_l(j+m-1)_m a_{i,j}(N,x)SH_n(x),
\end{aligned}$$

where $a_{i,j}(N,x)$ are as in Theorem 1.

Letting $k = 0$ in (59), we obtain the following result giving expressions for the squared Hermite polynomials $SH_N(x)$.

Theorem 3. For $N = 0, 1, 2, \cdots$

$$SH_N(x) = \sum_{i=0}^{N} \sum_{j=N-i}^{2(N-i)} a_{i,j}(N, x),$$

where $a_{i,j}(N, x)$ are as in Theorem 1.

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Quenching for the discrete heat equation with a singular absorption term on finite graphs

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Abstract

We study the quenching for the discrete semi-linear heat equation with singular absorption $u_t = \Delta_\omega u - \lambda u^{-p}$ on finite graph with Dirichlet boundary condition and the positive initial condition $u_0(x)$. When $\lambda^{-p} \geq \max_{x\in S} u_0(x)$, we prove that the solution will quench in finite time by comparison principal. Meanwhile, we study the quenching rate. Moreover, we also prove that there exists a critical exponent λ^* such that the problem admits a global solution for all $\lambda \leq \lambda^*$. Finally, a numerical experiment on two finite graphs is given to illustrate our results.

Keywords: Discrete heat equation; singular absorption; quenching; graphs.
MSC: 35B05, 35B33, 45G05

1 Introduction

Let G be a graph with vertex set V and edge set E, where the vertex set is divided into the boundary vertices ∂S and the interior vertices S which is connected, and we always assume G is a finite, connected, simple (without multiple edges and loops) graph in the following context. In this paper, we mainly study the quenching phenomena for the following semi-linear discrete heat equation with singular absorption on finite graph

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$$\begin{cases} u_t = \Delta_\omega u - \lambda u^{-p}, & x \in S \text{ and } t \in (0,T), \\ u(x,t) = 1, & x \in \partial S \text{ and } t \in (0,T), \\ u(x,0) = u_0(x), & x \in S, \end{cases} \tag{1}$$

here p, λ and T are positive constants, the initial value $u_0(x) \in C(V)$ and satisfies $0 < u_0(x) \leq 1$ for any $x \in S$. The function space $C(V)$ denotes the set of all functions which are definite on the vertices V of the graph G, and Δ_ω denotes the discrete Laplacian operator on finite graph, which is defined as follows (see [1]),

$$\Delta_\omega u(x) = \sum_{y \in V} [u(y) - u(x)] \cdot \omega(x,y),$$

where the function $\omega(x,y)$ is called the weighted function, and satisfies

$$\begin{aligned} &(i)\ \omega(x,x) = 0, \text{ for any } x \in V, \\ &(ii)\ \omega(x,y) = \omega(y,x) \geq 0, \text{ for any } x, y \in V, \\ &(iii)\ \omega(x,y) = 0, \text{ if and only if } (x,y) \notin E. \end{aligned}$$

Moreover, $d_\omega(x) = \sum_{x \in V} \omega(x,y)$ denotes the degree of the node $x \in V$ of the weighted graph G, and we assume that $d_\omega(x) \leq 1$ for any $x \in S$.

By introducing $v(x,t) = 1 - u(x,t)$, it is not difficult to verify that the function $v(x,t)$ satisfies the following initial boundary value problem

$$\begin{cases} v_t = \Delta_\omega v + \lambda(1-v)^{-p}, & x \in S \text{ and } t \in (0,T), \\ v(x,t) = 0, & x \in \partial S \text{ and } t \in (0,T), \\ v(x,0) = 1 - u_0(x), & x \in S. \end{cases} \tag{2}$$

In the continuous case including the local and nonlocal diffusion equation likes (1) or (2), its quenching phenomena has attracted much attention from the work of H. Kawarada [2] in 1975. This type of the diffusion equation with a singular absorption term (or a reaction term) comes form the polarization phenomena in ionic conductors [2], and can be considered as a limiting case of models in chemical catalyst kinetics or models of in enzyme kinetics [4, 5, 3, 6]. The detailed researches on the quenching phenomena can be found in [9, 6, 7, 8] and the references therein. Especially, for the nonlinear diffusion equation

$$u_t - u_{xx} = -u^{-p}, -l < x < l$$

with non-homogeneous Dirichlet boundary condition and the positive initial value, its quenching occurs in finite time for sufficiently large l in[2, 7]. Moreover, the quenching of the semilinear parabolic equation

$$u_t - \Delta u = g(u)$$

with homogeneous Dirichlet boundary condition and the positive initial value was also studied, the readers can refer to [10, 11]. On the other hand, the authors of [9] considered the quenching behaviour of the following nonlocal diffusion equation

$$u_t = J * u - u - \lambda u^{-p},$$

the critical parameter λ^* and the quenching rate and the quenching set were also given.

Recently, the ω-harmonic function and the $\omega-$heat equation were considered by many authors since the discrete heat equation has been widely applied to the fields of heat and energy transfer, electrical networks, image processing and so on [1, 12, 13]. In [14], Y.S. Chung, Y.S. Lee et.al considered the extinction and positivity of the discrete heat equation with absorption on network

$$u_t = \Delta_\omega u - u^p,$$

where $p > 0$. Furthermore, the extinction and positivity for the p, ω-heat equation with absorption was also studied in [16, 15]. Blow-up for the $\omega-$heat equation with a reaction term on graphs

$$u_t = \Delta_\omega u + \lambda u^p,$$

where $p > 0$ was researched in [17, 18]. The asymptotic behavior of solutions for the ω-heat equation with reaction and absorption term was considered in [19].

Motivated by the above works, the purpose of this paper is to discuss the quenching phenomenons for the discrete heat equation with singular absorption term and the non-homonomous Dirichlet boundary conditions. The local existence and uniqueness of solutions are obtained in the next section. In the third section, we will show the comparison principal for the discrete heat equation (1). The sufficient conditions on quenching and quenching rate are proved in the section 4. In the section 5, we mainly discuss the existence of the global solution. In the last section, we give some numerical experiments to illustrate our results.

2 Local existence and uniqueness of solutions

Lemma 2.1 *Suppose $0 < u_0(x) \leq 1$, then, there exists a unique solution $u \in C[0,T) \times C(V)$ for the problem (1). Moreover, if T is finite, then*

$$\lim_{t \to T^-} u(x,t) = 0 \tag{3}$$

for some $x \in S$.

Proof. Since $0 < u_0(x) \leq 1$, there exists a positive constant ε, such that $2\varepsilon < u_0(x) \leq 1$. Set

$$X_0 = \{u \in C[0, t_0] \times C(V), \varepsilon \leq u \leq K \text{ and } u(x) \equiv 1 \text{ for any } x \in \partial S\},$$

where $K > 1$ and

$$t_0 < \min\left\{\frac{K-1}{K}, \frac{\varepsilon}{K + \lambda\varepsilon^{-p}}, \frac{1}{2 + \lambda p \varepsilon^{-p-1}}\right\}. \tag{4}$$

Now, we define the operator as follows:

$$T_{u_0}[u](x,t) = \begin{cases} u_0(x) + \int_0^t \Delta_\omega u(x,s)ds - \lambda \int_0^t u^{-p}(x,s)ds, & x \in S, 0 \leq t \leq t_0, \\ 1, & x \in \partial S, 0 \leq t \leq t_0, \end{cases}$$

and the norm of the Banach space X_0

$$\|u(x,t)\|_{X_0} = \max_{x \in V} \max_{t \in [0,t_0]} |u(x,t)|$$

for any $u(x,t) \in X_0$.

First, we prove that the operator T_{u_0} maps X_0 into X_0. It is easy to verify that $T_{u_0}[u](x,t)$ is continuous about the time t for any fixed node $x \in V$. On the other hand, for any $u(x,t) \in X_0$, we have

$$T_{u_0}[u](x,t) \geq 2\varepsilon - (K + \lambda\varepsilon^{-p})t_0 \geq \varepsilon, \tag{5}$$

moreover, we also have

$$T_{u_0}[u](x,t) \leq 1 + Kt_0 = K(\frac{1}{K} + t_0) \leq K. \tag{6}$$

Next, we show that T_{u_0} is a strict contraction in X_0. That is to say, for any $u, v \in X_0$, we get

$$\begin{aligned} \|u - v\|_{X_0} &\leq \left\| \int_0^t \sum_{y \in V} [u(y,s) - v(y,s)]\, \omega(x,y) ds \right\|_{X_0} \\ &+ \left\| \int_0^t [u(x,s) - v(x,s)] ds \right\|_{X_0} + \lambda \left\| \int_0^t [v^{-p}(x,s) - u^{-p}(x,s)] ds \right\|_{X_0} \\ &\leq 2t_0 \|u - v\|_{X_0} + \lambda p \left\| \int_0^t |\xi|^{-p-1} |u(x,s) - v(x,s)| ds \right\|_{X_0} \\ &\leq t_0(2 + \lambda p \varepsilon^{-p-1}) \|u - v\|_{X_0} < \|u - v\|_{X_0}. \end{aligned}$$

Hence, by Banach fixed point theorem, there exists a unique $u \in X_0$ such that $u = T_{u_0(x)}[u]$, so, for any $x \in S$, we have

$$u(x,t) = \begin{cases} u_0(x) + \int_0^t \Delta_\omega u(x,s)ds - \lambda \int_0^t u^{-p}(x,s)ds, & x \in S \\ 1, & x \in \partial S, \end{cases} \tag{7}$$

thus, we can get $u(x,t)$ is the unique solution to the problem (1) in $t \in [0, t_0]$. Now, if $u(x, t_0) > 0$, we can continue the above procedure, and then, the solution can be extend to the time interval $[t_0, t_1]$. This procedure can be continued again and again until $\lim_{t \to T^-} u(x,t) \to 0$ for some time T which may be infinite.

3 Comparison principle

In this section, we mainly show a comparison principal. To do this, we begin with the definition of the super-solution and sub-solution to the problem (1).

Definition 3.1 *A function $\overline{u} \in C(V) \times C[0,T)$ is a super-solution to the problem* (1) *if $\overline{u}$ is a positive function and satisfies*

$$\begin{cases} \overline{u}_t \geq \Delta_\omega \overline{u} - \lambda \overline{u}^{-p}, & x \in S \text{ and } t \in (0,T), \\ \overline{u}(x,t) \geq 0, & x \in \partial S \text{ and } t \in (0,T), \\ \overline{u}(x,0) \geq u_0(x), & x \in S, \end{cases} \tag{8}$$

Analogously, we say that $\underline{u} \in C(V) \times C[0,T)$ is a sub-solution if it satisfies the reverses above inequalities.

Now, we have the following comparison principle.

Theorem 3.1 ***(Comparison principle)*** *Suppose $\overline{u}$ and $\underline{u}$ be a super-solution and a sub-solution to the problem* (1.1)*, respectively, then $\overline{u} \geq \underline{u}$ in $(x,t) \in V \times [0,T)$.*

Proof. For any $0 < t_0 < T$, set $m = \min\limits_{S\times[0,t_0]} \{\overline{u}, \underline{u}\}$ and $M = \max\limits_{S\times[0,t_0]} \{\overline{u}, \underline{u}\}$, thus, we know that m, M are the positive constants. And then, suppose $v(x,t) = \underline{u} - \overline{u}$. Notice that $v(x,0) > 0$ for any $x \in S$. By the definitions of the super-solution and the sub-solution, we can get

$$v_t \geq \Delta_\omega v - \lambda(\underline{u}^{-p} - \overline{u}^{-p}), \tag{9}$$

let $v^+(x,t) = \max\{v(x,t), 0\} \geq 0$. Thus, multiplying v^+ both sides of the above inequality, and integrating on S, we obtain

$$\begin{aligned} &\frac{1}{2}\left(\int_{x\in S} (v^+(x,t))^2\right)_t \\ &\leq \int_{x\in S} \Delta_\omega v(x,t) v^+(x,t) + \int_{x\in S} (\underline{u}^{p(x)} - \overline{u}^{p(x)}) v^+(x,t), \end{aligned} \tag{10}$$

For the first term of the right part of the above inequality, we have

$$\int_{x\in S} \Delta_\omega v(x,t) v^+(x,t) \leq 0. \tag{11}$$

In fact, let $J(t) = \{x \in V : v(x,t) > 0\}$, if $J(t)$ is empty set, we have the desired results. Now, assume $J(t)$ is not an empty set. Due to $\underline{u}(x,t) \leq 0$, $\overline{u}(x,t) \geq 0$ for any $x \in \partial S$ and $0 \leq t \leq t_0$, so $v(x,t) = \underline{u}(x,t) - \overline{u}(x,t) \leq 0$ for any $x \in \partial S$ and $0 \leq t \leq t_0$.

Now, we get $J(t) \subset S$. Thus, if $x \in J(t)$ and $y \in V \setminus J(t)$, we have $v(x,t) > 0$ and $v(y,t) - v(x,t) < 0$, hence, we have

$$\sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) < 0.$$

Furthermore, we get

$$\begin{aligned}
&\sum_{x\in S}\sum_{y\in V} v^+(x,t)[v(y,t) - v(x,t)]\omega(x,y) \\
&= \sum_{x\in J(t)} \sum_{y\in J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) \\
&+ \sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) \\
&= -\frac{1}{2} \sum_{x\in J(t)} \sum_{y\in J(t)} [v(y,t) - v(x,t)]^2\omega(x,y) \\
&+ \sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) < 0.
\end{aligned} \tag{12}$$

On the other hand, for any fixed $x \in S$, by mean value theorem, we have

$$\underline{u}^{-p}(x,t) - \overline{u}^{-p}(x,t) = -p\xi^{-p-1}(x,t)v(x,t),$$

where $\xi(x,t) = \theta(x)\underline{u}(x,t) + (1 - \theta(x))\overline{u}(x,t)$ and $0 \le \theta(x) \le 1$. And then, we have $m \le \xi(x,t) \le M$. Thus, for the second term of the right part of the inequality (10), we also have

$$\int_{x\in S} (\underline{u}^{p(x)} - \bar{u}^{p(x)})v^+(x,t) \le -m^{-p-1} \int_{x\in S} (v^+(x,t))^2. \tag{13}$$

Combine the inequalities (10), (12) and (13), we obtain

$$\left(\int_{x\in V} (v^+(x,t))^2\right)_t < 0. \tag{14}$$

There exists a contradiction. Hence $J(t) = \emptyset$. By the arbitrariness of t_0, we obtain $\overline{u}(x,t) \ge \underline{u}(x,t)$, for $(x,t) \in V \times [0,T)$.

4 Quenching phenomena and quenching rate

In this section, similar to the method used in [9], we mainly propose the quenching condition and quenching rate. Before the discussions and proofs, we firstly give some notes about the initial value condition and also the boundary condition. Since the absorption term is singular at points which satisfy $u(x) = 0$, we need the initial value $u_0(x) > 0$. Moreover, if $\max_{x\in S} u_0(x) > 1$, we can set

$$U(t) = (\lambda p)^{\frac{1}{p+1}}(A - t)^{\frac{1}{p+1}},$$

where $A = \max_{x \in S} u_0(x)$, and then, it is easy to verify that $U(t)$ is a super-solution to the discrete diffusion equation (1) when $U(t) \geq 1$. Thus, by the comparison principle, there exists t_0 such that $1 \geq U(t_0) \geq u(x, t_0)$. Hence, we can discuss the quenching pheromone to the problem (1) with the large initial value beginning with the initial time time $t = t_0$. The following proof can be similarly done. Finally, if we choose the homogenous Dirichlet boundary condition, i.e. set $u(x,t) = 0$ for any $x \in \partial S$, and then, we can also get $U(t)$ is also a super-solution to the problem (1) for any $t < A$, and then, we have $u(x,t)$ always quenches in finite time, i.e. the solution to the problem (1) is not global.

Next, we give the proof of the quenching phenomena about the problem (1), we mainly have the following two results.

Theorem 4.1 *If the initial value $u_0(x)$ satisfies that*

$$\max_{x \in S} u_0(x) \leq \lambda^{\frac{1}{p}} < 1, \tag{15}$$

and then, the solution to the problem (1) *quenches in finite time T.*

Proof. It is easy to verify that

$$v(x,t) = \begin{cases} \lambda^{\frac{1}{p}}, & x \in S, \\ 1, & x \in \partial S, \end{cases} \tag{16}$$

is the super-solution to the problem (1), thus, by the comparison principle, we have $u(x,t) \leq \lambda^{\frac{1}{p}}$ for any $x \in S$ and $t \in [0, T)$.

Now, assume $u(x,t)$ attains its minimum value at the nodes x^* for any fix time t. At this point, we have

$$\begin{aligned} u_t(x^*, t) &= \sum_{y \in V} u(y,t)\omega(x^*, y) - d^* u(x^*, t) - \lambda u^{-p}(x^*, t) \\ &\leq d^* - d^* u(x^*, t) - \lambda u^{-p}(x^*, t) \\ &\leq -d^* u(x^*, t), \end{aligned} \tag{17}$$

where $d^* = d_\omega(x^*)$. Integrating both sides of the above inequality in $[0, t]$, we can get

$$u(x^*, t) \leq u_0(x^*) e^{-d^* t} \leq \lambda^{\frac{1}{p}} e^{-d^* t}. \tag{18}$$

Thus, for the equality in (17), note that the function $-s^{-p}$ is increasing, hence, choose $t_0 \geq \frac{ln(2d^*)}{pd^*}$, and then, for any $t \geq t_0$, we can also get

$$\begin{aligned} u_t(x^*, t) &\leq d^* - d^* u(x^*, t) - \lambda u^{-p}(x^*, t) \\ &\leq d^* - \frac{\lambda}{2} u^{-p}(x^*, t) - \frac{\lambda}{2} u^{-p}(x^*, t) \\ &\leq d^* - \frac{1}{2} e^{pd^* t} - \frac{\lambda}{2} u^{-p}(x^*, t) \\ &\leq -\frac{\lambda}{2} u^{-p}(x^*, t), \end{aligned} \tag{19}$$

Integrating both sides of the above inequality in $[t_0, t]$, we can obtain

$$\begin{aligned}
&u^{p+1}(u(x^*,t))\\
&\leq u^{p+1}(u(x^*,t_0)) - \frac{(p+1)\lambda}{2}(t-t_0)\\
&\leq u_0^{p+1}(x^*)e^{-d^*(p+1)t_0} - \frac{(p+1)\lambda}{2}(t-t_0),
\end{aligned}$$

from this inequality, we have $u(x,t)$ quenches at finite time T, moreover, we have

$$T \leq t_0 + \frac{(p+1)\lambda}{2}u_0^{p+1}(x^*)e^{-d^*(p+1)t_0}. \tag{20}$$

Theorem 4.2 *If $\lambda \geq 1$, then the solution to the problem* (1) *also quenches in finite time.*

Proof. Since $\lambda \geq 1$ and $0 < u_0(x) \leq 1$, we have $\max\limits_{x\in V} u_0(x) \leq \lambda^{\frac{1}{p+1}}$. Now, it is easy to verify that $v(x,t) \equiv \lambda^{\frac{1}{p+1}}, x \in V$ is a super-solution to the problem (1). Thus, by the comparison principle, we also have $u(x,t) \leq \lambda^{\frac{1}{p+1}}$ for any $x \in V$.

Now, also assume $u(x,t)$ attains its minimum value at the nodes x^* for any fix time t. At this point, we have

$$\begin{aligned}
u_t(x^*,t) &= \sum_{y\in V} u(y,t)\omega(x^*,y) - d^*u(x^*,t) - \lambda u^{-p}(x^*,t)\\
&\leq \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \lambda u^{-p}(x^*,t)\\
&\leq -d^*u(x^*,t),
\end{aligned} \tag{21}$$

Integrating both sides of the above inequality on $[0, t]$, we can get

$$u(x^*,t) \leq u_0(x^*)e^{-d^*t}. \tag{22}$$

Thus, for any $t \geq t_0$, from the inequality in (21) and by choosing $t_0 \geq \frac{\ln 2 - \frac{p\ln\lambda}{p+1}}{pd^*}$, it follows that

$$\begin{aligned}
u_t(x^*,t) &\leq \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \lambda u^{-p}(x^*,t)\\
&= \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \frac{\lambda}{2}u^{-p}(x^*,t) - \frac{\lambda}{2}u^{-p}(x^*,t)\\
&\leq \lambda^{\frac{1}{p+1}} - \frac{\lambda}{2}e^{pd^*t} - \frac{\lambda}{2}u^{-p}(x^*,t)\\
&\leq -\frac{\lambda}{2}u^{-p}(x^*,t),
\end{aligned} \tag{23}$$

Integrating both sides of the above inequality on $[t_0, t]$, we can obtain

$$\begin{aligned}
&u^{p+1}(u(x^*,t))\\
&\leq u^{p+1}(u(x^*,t_0)) - \frac{(p+1)\lambda}{2}(t-t_0)\\
&\leq u_0^{p+1}(x^*)e^{-(p+1)d^*t_0} - \frac{(p+1)\lambda}{2}(t-t_0),
\end{aligned}$$

by this inequality, we get $u(x,t)$ quenches at finite time T, moreover, we also have

$$T \leq t_0 + \frac{(p+1)\lambda}{2} u_0^{p+1}(x^*) e^{-(p+1)d^* t_0}. \tag{24}$$

Theorem 4.3 ***(The quenching rate)*** *If the solution $u(x,t)$ to the problem (1) quenches in finite time T at the node x^*, and then, we have*

$$\lim_{t \to T^-} (T-t)^{\frac{-1}{p+1}} u(x^*,t) = [(p+1)\lambda]^{\frac{1}{p+1}}.$$

Proof. Since $0 < u_0(x) \leq 1$, and then, it is easy to verify that $v(x,t) \equiv 1$ is a super-solution to the problem (1), by the comparison principle, we know that $0 < u(x,t) \leq 1$ for any $x \in V$ and $t \in [0,T)$.

Now, multiply u^p on the both sides of the discrete heat equation in the problem (1), and then, we get

$$u^p u_t = u^p \Delta_\omega u - \lambda, x \in S, t \in [0,T). \tag{25}$$

Next, we establish the upper bound of the quenching rate. Due to $0 < u(x,t) \leq 1$, we have

$$\begin{aligned} u^p u_t &= u^p \Delta_\omega u - \lambda \\ &= u^p \sum_{y \in V} u(y,t)\omega(x,y) - d_\omega(x) u^{p+1} - \lambda \\ &\geq -u^{p+1} - \lambda \geq -1 - \lambda \end{aligned} \tag{26}$$

for any $x \in S, t \in [0,T)$. Assume that $u(x,t)$ quenches in finite time T at the node x^*, and then, integrating the inequality $u^p u_t \geq -1-\lambda$ on the time t on $[t,T]$ on the node x^*, due to $u(x,t) \to 0$ when $t \to T^-$, we can get

$$u^{p+1}(x^*,t) \leq (p+1)(\lambda+1)(T-t).$$

Moreover, due to the inequality $u^p u_t \geq -u^{p+1} - \lambda$, thus, at the quenching node x^*, we also have

$$u^p u_t(x^*,t) \geq -(p+1)(\lambda+1)(T-t) - \lambda, \tag{27}$$

Integrating again in the time interval $[t,T]$, we have

$$-\frac{1}{p+1} u^{p+1}(x^*,t) \geq \frac{1}{2}(p+1)(\lambda+1)(T-t)^2 - \lambda(T-t), \tag{28}$$

thus, we get

$$\frac{u^{p+1}(x^*,t)}{T-t} \leq (p+1)\lambda \left(-(p+1)\frac{2(\lambda+1)}{\lambda}(T-t) + 1 \right). \tag{29}$$

Now, we establish the lower bound of the quenching rate. By the equation (31) and $0 < u(x,t) \le 1$, we also have

$$u^p u_t = u^p \sum_{y \in V} u(y,t)\omega(x,y) - d_\omega(x)u^{p+1} - \lambda \le u^p - \lambda.$$

Thus, by the inequality (26), at the quenching node x^*, we can obtain the following inequality

$$u^p u_t \le [(p+1)(\lambda+1)(T-t)]^{\frac{p}{p+1}} - \lambda.$$

Integrating in the time interval $[t,T]$, we have

$$\frac{u^{p+1}(x^*,t)}{T-t} \ge (p+1)\lambda \left(-\frac{(p+1)^{\frac{2p+1}{p+1}}(\lambda+1)^{\frac{p}{p+1}}}{(2p+1)\lambda}(T-t) + 1 \right). \tag{30}$$

Combine the inequalities (29) and (30), and let $t \to T^-$, we can obtain the need results.

5 The existence of a global solution

In this section, we investigate the existence of a global solution to the problem (1) with the initial value $u_0(x) \equiv 1$ for any $x \in S$. To do this, we begin with the following lemma.

Lemma 5.1 *There exists a small nonnegative constant λ^*, such that if $\lambda \le \lambda^*$, then the eigenvalue problem*

$$\begin{cases} \Delta_\omega u(x) = \lambda u^{-p}(x), & x \in S, \\ u(x) = 1, & x \in \partial S, \end{cases} \tag{31}$$

exists at least one solution.

Proof. Let $C(V)$ denotes the set of all the functions which are defined on the finite graph G with its nodes V, and then, the norm on $C(V)$ is as follows:

$$\|v\|_{C(V)} = \max_{x \in V} v(x). \tag{32}$$

Furthermore, set $C_0(V) = \{v(x) \in C(V)$ and $v(x) \equiv 0$ for any $x \in \partial S\}$ and assume that $A = \{v \in C_0(V) : -\varepsilon < v(x) < 1\}$ is a open subset of $C_0(V)$, the nonlinear function $F(\lambda, v) : (-\varepsilon, \varepsilon) \times A \to C(S)$ is defined as

$$F(\lambda, v) = \Delta_\omega v + \lambda(1-v)^{-p}, \tag{33}$$

where ε is a small enough constant.

It is obviously that $F(\lambda, v)$ is differentiable function and $F(0,0) = 0$. Moreover, the Fréchet derivative of $F(\lambda, v)$ at $(0,0)$ is

$$F_v(0,0)[z(x)] = \Delta_\omega z(x) \tag{34}$$

is a continuous linear operator for any $z(x) \in A$. In fact, for any sequence $z_m(x) \to z(x)$, we have $\|\Delta_\omega[z_m(x) - z(x)]\|_{C(V)} \leq |V|\|z_m - z\|_{C(V)}$, so $F_v(0,0)$ is a continuous operator. Moreover, its kernel is the function $z = 0$ (see [1]), and then, it is injective. On the other hand, $F_v(0,0)$ is a linear transformation on finite dimensional space, and then, it is also a compact linear operator, hence, it is also bijective. By the Open-Mapping Theorem we deduce that $F_v(0,0)$ is a linear homeomorphism of $C_0(V)$ into $C_0(V)$. By the Implicit Function Theorem in the appendix A of [20], there exists a neighborhoods $U \in (-\varepsilon, \varepsilon)$ of $\lambda = 0$ and $W \in A$ of $v(x) \equiv 0$ such that $F(\lambda, v_\lambda) = 0$ for any $\lambda \in U$, and $v_\lambda \in W$ is unique. Thus, for any $\lambda < \lambda^* \in U$, suppose $u_\lambda(x) = 1 - v_\lambda(x)$, it is easy to verify that $u_\lambda(x)$ is a solution to the equation (31).

Based on the above lemma, we have the following theorem on the existence of the global solution to the problem (1) with $u_0(x) = 1$.

Theorem 5.1 *There exists a constat λ^*, such that $\lambda \leq \lambda*$, the problem* (1) *with the initial value $u_0(x) = 1$ has a global solution, while for $\lambda > \lambda^*$, then no global solution exists.*

Proof. Firstly, from the proofs of Theorem 4.1 and 4.2, we have the solution to the problem (1) with the initial value $u_0(x) = 1$ quenching in infinite time is impossible. Moreover, set $w(x,t) = u_t(x,t)$, and then, we get w satisfies

$$\begin{cases} w_t = \Delta_\omega w + p\lambda u^{-p-1} w, & (x,t) \in S \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial S \times (0,T), \\ w(x,0) = -\lambda, & x \in S. \end{cases} \tag{35}$$

Then, by comparison principle, we obtain that $w = u_t \leq 0$. On the other hand, by the Lemma 5.1, we have λ is small enough, the equation (31) exists a positive solution $v_\lambda(x)$, in fact, it is also a sub-solution to the problem (1) with the initial value $u_0(x) = 1$. Hence, the solution of (1) with the initial value $u_0(x) = 1$ satisfies that, either it quenches in finite time, or it converges to a stationary solution

Next, we discuss the critical exponent of the quenching and the global existence. In fact, If $u(x,t)$ is a global solution to the problem (1), then, we know that $u(x,t) \to u_\infty$ as $t \to \infty$ and u_∞ is a solution the the problem (31), is a stationary solution to the equation (31). Moreover, for any fix constant λ_1, if there exists a solution $v_{\lambda_1}(x)$ to the problem (31), i.e. $v_{\lambda_1}(x)$ satisfies

$$\Delta_\omega v_{\lambda_1}(x) = \lambda v_{\lambda_1}^{-p}(x), \tag{36}$$

furthermore, it is easy to verify that $v_{\lambda_1}(x)$ is a sub-solution to the problem (1) with the initial value $u_0(x) = 1$ and $\lambda \leq \lambda_1$. Thus, the solution to the problem (1) with the initial value $u_0(x) = 1$ is global when $\lambda \leq \lambda_1$. By this monotonicity property given

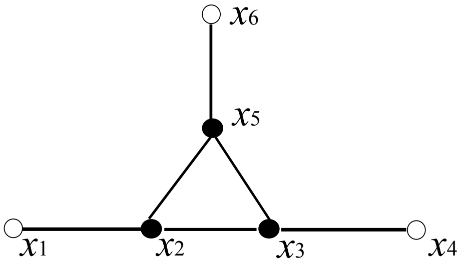

Figure 1: The graph G_1

above discussion, set $\lambda^* = \sup_{\lambda \in B} \lambda$, where the set $B = \{\lambda : v_\lambda(x)$ exists to (36)$\}$. This completes the proof.

6 Numerical experiments

In this section, we consider a graph G_1 (as shown in Figure 1), which has six nodes $x_1, x_2, \cdots, x_6$, where x_2, x_3, x_5 are interior and x_1, x_4, x_6 are the boundary. Moreover, we only consider the weight function $\omega \equiv \frac{1}{3}$. Thus, the discrete heat equation in (1) is

$$\begin{cases} u_t(x_2,t) = \frac{1}{3} + \frac{1}{3}u(x_3,t) + \frac{1}{3}u(x_5,t) - u(x_2,t) - \lambda u^{-p}(x_2,t) \\ u_t(x_3,t) = \frac{1}{3} + \frac{1}{3}u(x_2,t) + \frac{1}{3}u(x_5,t) - u(x_3,t) - \lambda u^{-p}(x_3,t) \\ u_t(x_5,t) = \frac{1}{3} + \frac{1}{3}u(x_2,t) + \frac{1}{3}u(x_3,t) - u(x_5,t) - \lambda u^{-p}(x_5,t) \end{cases} \tag{37}$$

Now, we also suppose that the exponent $p = 1.2, \lambda = 0.8$. Moreover, the discrete Laplacian operator Δ_ω on the graph G_1 is as follows:

$$\Delta_\omega = -\frac{1}{3}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \tag{38}$$

Thus, set $U(t) = (u(x_2,t), u(x_3,t), u(x_5,t))^T$, and then, we have the equation (37) can be rewrote as follows:

$$U_t = \frac{\mathbf{1}}{3} + \Delta_\omega * U(t) - 0.8U^{-2}(t), \text{ with } U(0) = (0.3, 0.35, 0.4)^T, \tag{39}$$

where $\mathbf{1} = (1,1,1)^T$.

By Theorem 4.1, we get $U(t)$ quenches in finite time, moreover, U_t blows up in finite time. Since the system (39) is nonlinear, it is difficult to compute its analytic solutions. Hence, we consider its numerical solutions. The explicit difference scheme to the system (39) is as follows:

$$U_{n+1} = U_n + \Delta t(\frac{\mathbf{1}}{3} + \Delta_\omega * U_n - 0.8U_n^{-2}), \text{ with } U_0 = (0.3, 0.35, 0.4)^T, \tag{40}$$

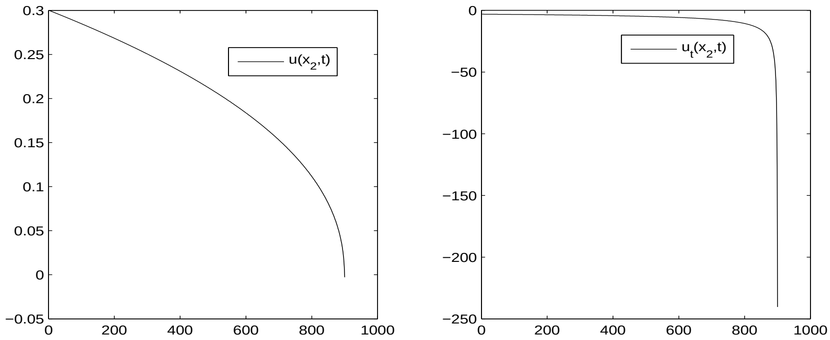

Figure 2: Quenching of $u(x_2, t)$ and Blow-up of $u_t(x_2, t)$ in finite time

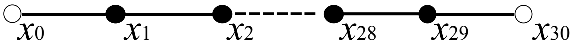

Figure 3: The graph G_2

where U_n denotes $U(n\Delta t)$ for $n = 1, 2, 3, \cdots$ and Δt is the time step which taking as $0.043/n$ in the numerical experiment. The numerical experiment result is shown in Figure 2. From this numerical experiment, we know that the solution $U(t)$ quenches and U_t blows up in finite time.

At the end of this section, we give another example. Now, we consider the discrete heat equation (1) on the following finite graph G_2 (as shown in Figure 3), which has six nodes $x_0, x_2, \cdots, x_{30}$, where $x_1, x_2, \cdots, x_{29}$ are interior and x_0, x_{30} are the boundary. Moreover, we only consider the weight function $\omega(x_i, x_j) \equiv \frac{1}{4}$. Thus, the discrete heat equation in (1) is

$$\begin{cases} u_t(x_1, t) = \frac{1}{4}(1 + u(x_2, t) - 2u(x_1, t)) - \lambda u^{-p}(x_1, t), \\ u_t(x_i, t) = \frac{1}{4}(u(x_{i-1}) + u(x_{i+1}, t) - 2u(x_i, t)) - \lambda u^{-p}(x_i, t), 1 \leq i \leq 28, \\ u_t(x_{29}, t) = \frac{1}{4}(1 + u(x_{28}, t) - 2u(x_{29}, t)) - \lambda u^{-p}(x_{29}, t), \end{cases} \tag{41}$$

where $\lambda = 1$, $p = 1.2$, and then, let the initial value $u_0(x_i) = 1 - 0.9 \sin\left(\frac{i}{30}\pi\right)$, where $1 \leq i \leq 29$ and $u(x_0, t) = u(x_{30}, t) = 1$. Thus, by the theorem 4.2, we have the solution $u(x_i, t)$ will quench in finite time. Also since the nonlinear of the system (41), we consider the following difference scheme:

$$V_{n+1} = V_n + \Delta t \left(B + \Delta_\omega V_n - \lambda V_n^{-p}\right), n = 0, 1, 2, \cdots, \tag{42}$$

where $V_n = (u(x_1, n\Delta t), u(x_2, n\Delta t), \cdots, u(x_{29}, n\Delta t))^T$, $B = (1/4, 0, \cdots, 0, 1/4)^T$ is a 29− dimensions vector, $\Delta t = 0.0001/n$ is the time step, and the discrete Laplacian

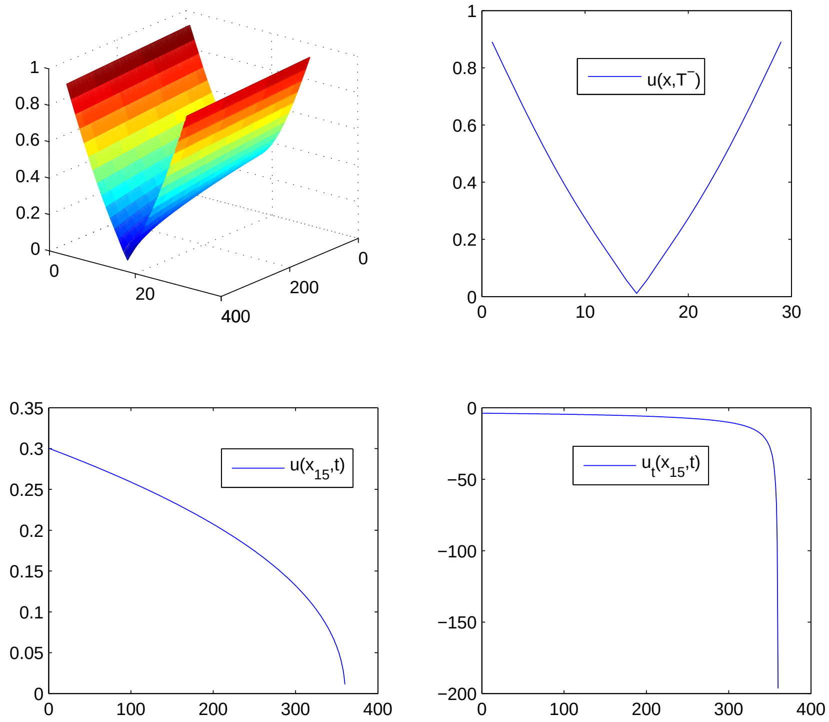

Figure 4: Quenching of $u(x,t)$ and Blow-up of $u_t(x_{15},t)$ in finite time

operator on the graph G_2 is as follows:

$$\Delta_\omega = \frac{1}{4}\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}_{29\times 29}. \tag{43}$$

Moreover, the initial value $V_0 = (u_0(x_1), u_0(x_2)), \cdots, u_0(x_{29}))$. The numerical experiment results can be found in Figure 4.

7 Conclusion

In this paper, we mainly consider the quenching problem and the global solution of the discrete heat equation with a singular absorption, the quenching time, quenching rate and the critical exponent were also given. We only prove the existence of the critical exponent, its upper and lower bounds may be established by the Kaplan's method in the further work.

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Nonlocal fractional-order boundary value problems with generalized Riemann-Liouville integral boundary conditions

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Abstract

In this paper, we study existence and uniqueness of solutions for nonlocal boundary value problems of Caputo fractional differential equations equipped with generalized Riemann-Liouville integral boundary conditions. A variety of fixed point theorems such as Banach's fixed point theorem, nonlinear contractions, Krasnoselskii's fixed point theorem, Schaefer's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder degree theory are applied to obtain the desired results. Several examples are discussed for illustration of the obtained results.

Key words and phrases: Caputo fractional derivative; generalized Riemann-Liouville integral; nonlocal boundary conditions; fixed point theorems.
AMS (MOS) Subject Classifications: 26A33; 34A08

1 Introduction

We investigate the sufficient criteria for existence of solutions for the following Caputo fractional differential equation

$$D^q x(t) = f(t, x(t)), \qquad 0 < t < T, \tag{1}$$

subject to nonlocal generalized Riemann-Liouville fractional integral boundary conditions of the form

$$\begin{aligned} x(0) &= \gamma \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\zeta \frac{s^{\rho-1} x(s)}{(\zeta^\rho - s^\rho)^{1-\alpha}} ds := \gamma \ {}^\rho I^\alpha x(\zeta), \\ x(T) &= \delta \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^\xi \frac{s^{\rho-1} x(s)}{(\xi^\rho - s^\rho)^{1-\beta}} ds := \delta \ {}^\rho I^\beta x(\xi), \quad 0 < \zeta, \xi < T, \end{aligned} \tag{2}$$

where D^q denote the Caputo fractional derivative of order q, ${}^\rho I^z$, $z \in \{\alpha, \beta\}$, is the generalized Riemman-Liouville fractional integral of order $z > 0$, $\rho > 0$, ζ, ξ arbitrary, with $\zeta, \xi \in (0, T)$, $\gamma, \delta \in \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

As a second problem, we study Caputo fractional differential equation (1) supplemented with a combination of Riemman-Liouville and generalized Riemman-Liouville integral boundary conditions:

$$\begin{aligned} x(0) &= \gamma \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} (\zeta - s)^{\alpha-1} x(s) ds := \gamma \ J^{\alpha} x(\zeta), \\ x(T) &= \delta \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^{\xi} \frac{s^{\rho-1} x(s)}{(t^{\rho} - s^{\rho})^{1-\beta}} ds := \delta \ {}^{\rho}I^{\beta} x(\xi), \quad 0 < \zeta, \xi < T, \end{aligned} \tag{3}$$

where J^q is the Riemman-Liouville fractional integral of order $q > 0$ while ${}^{\rho}I^{\beta}$ denote generalized Riemman-Liouville fractional integral of order $\beta > 0$, $\rho > 0$.

The subject of fractional differential equations has evolved into an interesting and popular field of research during the last few decades. The surge in developing several aspects of fractional calculus owes to its extensive applications in several branches of engineering and technical sciences such as physics, chemical technology, population dynamics, biotechnology, biosciences, control theory and economics. The nonlocal nature of fractional derivatives, which takes into account memory and hereditary properties of various materials and processes, has played a key role in improving the mathematical modeling based on integer-order derivatives, for instance, see [1, 2, 3, 4].

Fractional-order boundary value problems supplemented with different kinds of boundary conditions have been studied by several researchers. In particular, integral boundary conditions involving classical, Riemann-Liouville or Hadamard or Erdélyi-Kober type integral operators have received significant attention. In [5], Riemann-Liouville and Hadamard fractional integrals are jointly represented by a single integral, which is called generalized Riemann-Liouville fractional integral (see Definition 2.2). For some recent works on the topic we refer the reader to a series of papers [6]-[20] and the references cited therein.

The purpose of the present study is to develop the existence theory for problems (1)-(2) and (1)-(3) by means of standard tools of fixed point theory. In Section 2 we recall some preliminary facts that we need in the sequel. In Section 3 we present our main results, while Section 4 contains examples illustrating the results obtained in Section 3.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

Definition 2.1 *The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.2 *[5] The generalized Riemann-Liouville fractional integral of order $q > 0$ and $\rho > 0$ of a function $f(t)$ for all $0 < t < \infty$, is defined as*

$${}^{\rho}I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^{\rho} - s^{\rho})^{1-q}} ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Remark 2.3 *From the above definition it follows that when $\rho = 1$ we arrive at the standard Riemann-Liouville fractional integral, which is used to define both the Riemann-Liouville and Caputo fractional derivatives, while when $\rho \to 0$ we have*

$$\lim_{\rho \to 0} {}^{\rho}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s}\right)^{q-1} \frac{f(s)}{s} ds,$$

which is the famous Hadamard fractional integral. See [5].

Definition 2.4 *The Riemann-Liouville fractional derivative of order $q>0,\ n-1<q<n,\ n\in\mathbb{N}$, is defined as*

$$D^q_{0+}f(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-q-1}f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.5 *The Caputo derivative of order q for a function $f:[0,\infty)\to\mathbb{R}$ can be written as*

$${}^cD^qf(t)=D^q\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right),\quad t>0,\quad n-1<q<n.$$

Remark 2.6 *If $f(t)\in C^n[0,\infty)$, then*

$${}^cD^qf(t)=\frac{1}{\Gamma(n-q)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{q+1-n}}ds=I^{n-q}f^{(n)}(t),\ t>0,\ n-1<q<n.$$

Lemma 2.7 *Let constants $q>0$ and $p>0$. Then:*

$${}^\rho I^qt^p=\frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)}\frac{t^{p+\rho q}}{\rho^q}. \tag{4}$$

Proof. By Definition 2.2, we have

$$\begin{aligned}{}^\rho I^qt^p &= \frac{\rho^{1-q}}{\Gamma(q)}\int_0^t\frac{s^{\rho-1}s^p}{(t^\rho-s^\rho)^{1-q}}\,ds=\frac{\rho^{1-q}}{\Gamma(q)}\frac{t^{p+\rho q}}{\rho}\int_0^1\frac{u^{\frac{p}{\rho}}}{(1-u)^{1-q}}\,du\\ &= \frac{\rho^{1-q}}{\Gamma(q)}\frac{t^{p+\rho q}}{\rho}B\left(\frac{p+\rho}{\rho},q\right)=\frac{t^{p+\rho q}}{\rho^q}\frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)}.\end{aligned}$$

This completes the proof. □

Lemma 2.8 *For any $y\in AC([0,T],\mathbb{R})$, x is a solution of the linear fractional boundary value problem*

$$\begin{cases}{}^cD^qx(t)=y(t), & 1<q\le 2,\\ x(0)=\gamma\ {}^\rho I^\alpha x(\zeta), \quad x(T)=\delta\ {}^\rho I^\beta x(\xi), & 0<\zeta,\xi<T,\end{cases} \tag{5}$$

if and only if

$$x(t)=J^qy(t)+\frac{\gamma}{\Lambda}(v_4-tv_3)\ {}^\rho I^\alpha J^qy(\zeta)+\frac{1}{\Lambda}(v_2+tv_1)\Big(\delta\ {}^\rho I^\beta J^qy(\xi)-J^qy(T)\Big), \tag{6}$$

where

$$\begin{aligned} v_1 &= 1-\gamma\frac{\zeta^{\rho\alpha}}{\rho^\alpha}\frac{1}{\Gamma(\alpha+1)}, & v_2=\gamma\frac{\zeta^{\rho\alpha+1}}{\rho^\alpha}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\alpha+\rho}{\rho})},\\ v_3 &= 1-\delta\frac{\xi^{\rho\beta}}{\rho^\beta}\frac{1}{\Gamma(\beta+1)}, & v_4=T-\delta\frac{\xi^{\rho\beta+1}}{\rho^\beta}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})},\end{aligned} \tag{7}$$

and

$$\Lambda=v_1v_4+v_2v_3\neq 0. \tag{8}$$

Proof. For arbitrary constants $c_0, c_1 \in \mathbb{R}$, the general solution of the fractional differential equation in (5) can be written as [2]

$$x(t) = c_0 + c_1 t + J^q y(t). \tag{9}$$

Applying the generalized fractional integral operator on (9) and using Lemma 2.7, we get

$${}^\rho I^z x(t) = {}^\rho I^z J^q y(t) + c_0 \frac{t^{\rho z}}{\rho^z} \frac{1}{\Gamma(z+1)} + c_1 \frac{t^{\rho z+1}}{\rho^z} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho z+\rho}{\rho})}. \tag{10}$$

Using (9) and (10) in boundary conditions of (5), we get the system

$$\begin{aligned} \left(1 - \gamma \frac{\zeta^{\rho\alpha}}{\rho^\alpha} \frac{1}{\Gamma(\alpha+1)}\right) c_0 - \gamma \frac{\zeta^{\rho\alpha+1}}{\rho^\alpha} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\alpha+\rho}{\rho})} c_1 &= \gamma \ {}^\rho I^\alpha J^q y(\zeta), \\ \left(1 - \delta \frac{\xi^{\rho\beta}}{\rho^\beta} \frac{1}{\Gamma(\beta+1)}\right) c_0 + \left(T - \delta \frac{\xi^{\rho\beta+1}}{\rho^\beta} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})}\right) c_1 &= \delta \ {}^\rho I^\beta J^q y(\xi) - J^q y(T). \end{aligned} \tag{11}$$

Solving (11) together with the notations (7) and (8), we find that

$$c_0 = \frac{1}{\Lambda} \Big\{ \gamma v_4 \ {}^\rho I^\alpha J^q y(\zeta) + v_2 \Big(\delta \ {}^\rho I^\beta J^q y(\xi) - J^q y(T) \Big) \Big\},$$

$$c_1 = \frac{1}{\Lambda} \Big\{ v_1 \Big(\delta \ {}^\rho I^\beta J^q y(\xi) - J^q y(T) \Big) - \gamma v_2 \ {}^\rho I^\alpha J^q y(\zeta) \Big\}.$$

Substituting the values of c_0 and c_1 in (9) yields the solution (6). Conversely, it can easily be shown by direct computation that the integral equation (6) satisfies the problem (5). This completes the proof.□

Our next lemma deals with the linear variant of (1)-(3). We do not provide the proof of this result as it is similar to the preceding one.

Lemma 2.9 *For any $y \in AC([0,T],\mathbb{R})$, x is a solution of the linear fractional boundary value problem*

$$\begin{cases} {}^c D^q x(t) = y(t), \quad 1 < q \le 2, \\ x(0) = \gamma \ J^\alpha x(\zeta), \quad x(T) = \delta \ {}^\rho I^\beta x(\xi), \quad 0 < \zeta, \xi < T, \end{cases} \tag{12}$$

if and only if

$$x(t) = J^q y(t) + \frac{\gamma}{\Lambda_1}(u_4 - t u_3) \ J^{q+\alpha} y(\zeta) + \frac{1}{\Lambda_1}(u_2 + t u_1) \Big(\delta \ {}^\rho I^\beta J^q y(\xi) - J^q y(T) \Big), \tag{13}$$

where

$$u_1 = 1 - \gamma \frac{\zeta^\alpha}{\Gamma(\alpha+1)}, \quad u_2 = \gamma \frac{\zeta^{\alpha+1}}{\Gamma(\alpha+2)}, \quad u_3 = 1 - \delta \frac{\xi^{\rho\beta}}{\rho^\beta} \frac{1}{\Gamma(\beta+1)}, \quad u_4 = T - \delta \frac{\xi^{\rho\beta+1}}{\rho^\beta} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})}, \tag{14}$$

and

$$\Lambda_1 = u_1 u_4 + u_2 u_3 \neq 0. \tag{15}$$

3 Existence results

Let us denote by $\mathcal{C} = C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. By $L^1([0,T],\mathbb{R})$ we mean the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \displaystyle\int_0^T |x(t)| dt$.

In view of Lemma 2.8, we introduce operators $\mathcal{Q}, \widehat{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}$ associated with problems (1)-(2) and (1)-(3) respectively by

$$\begin{aligned}(\mathcal{Q}x)(t) &= J^q f(s,x(s))(t) + \frac{\gamma}{\Lambda}(v_4 - tv_2)\ {}^\rho I^\alpha J^q f(s,x(s))(\zeta) \\ &+\frac{1}{\Lambda}(v_2+tv_1)\Big(\delta\ {}^\rho I^\beta J^q f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big),\ t\in[0,T],\end{aligned} \tag{16}$$

$$\begin{aligned}(\widehat{\mathcal{Q}}x)(t) &= J^q f(s,x(s))(t) + \frac{\gamma}{\Lambda_1}(u_4 - tu_3)\ J^{q+\alpha} f(s,x(s))(\zeta) \\ &+\tfrac{1}{\Lambda_1}(u_2+tu_1)\Big(\delta\ {}^\rho I^\beta J^q (s,x(s))(\xi) - J^q f(s,x(s))(T)\Big),\quad t\in[0,T].\end{aligned} \tag{17}$$

In the sequel, we use the following expression:

$${}^\rho I^h f(s,x(s))(y) = \frac{\rho^{1-h}}{\Gamma(h)}\int_0^y \frac{s^{\rho-1}f(s,x(s))}{(y^\rho - s^\rho)^{1-h}}ds, \quad h\in\{\alpha,\beta\}.$$

Further, we set the constants

$$\begin{aligned}\Omega: &= \frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\alpha+\rho}{\rho}\Big)} \\ &+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)} + \frac{T^q}{\Gamma(q+1)}\right).\end{aligned} \tag{18}$$

$$\begin{aligned}\Omega_1: &= \frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|u_4|+T|u_2|)\zeta^{\alpha+q}}{|\Lambda_1|\Gamma(\alpha+q+1)} \\ &+\frac{(|u_2|+T|u_1|)}{|\Lambda_1|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)} + \frac{T^q}{\Gamma(q+1)}\right).\end{aligned} \tag{19}$$

In the following subsections, we establish several existence and uniqueness results for problems (1)-(2) and (1)-(3) by applying a variety of fixed point theorems. We present in details the proofs for problem (1)-(2), while the proofs for problem (1)-(3) are omitted as they are similar to the ones obtained for problem (1)-(2).

3.1 Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.1 *Assume that:*

(H_1) *there exists a positive constant L such that $|f(t,x)-f(t,y)|\le L|x-y|$, for each $t\in[0,T]$ and $x,y\in\mathbb{R}$.*

If

$$L\Omega < 1, \tag{20}$$

where Ω is defined by (18), then the boundary value problem (1)-(2) has a unique solution on $[0,T]$.

Proof. Observe that a fixed point problem equivalent to problem (1)-(2) is $x=\mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined by (16), and that the existence of a fixed point of the operator $\mathcal{Q}$ implies the existence of a solution for problem (1)-(2). Applying the Banach contraction mapping principle, we shall show that $\mathcal{Q}$ has a unique fixed point. For that we let $\sup_{t\in[0,T]}|f(t,0)| = M < \infty$ and choose $r \ge \frac{M\Omega}{1-L\Omega}$. To show that $\mathcal{Q}B_r\subset B_r$, where $B_r=\{x\in\mathcal{C}: \|x\|\le r\}$, we have for any $x\in B_r$ that

$$|(\mathcal{Q}x)(t)| \le \sup_{t\in[0,T]}\left\{J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))|(\zeta)\right.$$

$$
\begin{aligned}
& +\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(\delta\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi)+J^q|f(s,x(s))|(T)\Big)\\
\leq\ & J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(T)\\
& +\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(\zeta)\\
& +\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(\xi)\\
& +J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(T)\Big)\\
\leq\ & (L\|x\|+M)J^q(1)(T)+(L\|x\|+M)\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta)\\
& +(L\|x\|+M)\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi)+J^q(1)(T)\Big)\\
\leq\ & (Lr+M)\left\{\frac{T^q}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)}\right.\\
& \left.+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)}+\frac{T^q}{\Gamma(q+1)}\right)\right\}\\
\leq\ & (Lr+M)\Omega\leq r,
\end{aligned}
$$

which implies that $\mathcal{Q}B_r\subset B_r$.

Next, we let $x,y\in\mathcal{C}$. Then for $t\in[0,T]$, we have

$$
\begin{aligned}
|\mathcal{Q}x(t)-\mathcal{Q}y(t)| \leq\ & \sup_{t\in[0,T]}\Big\{J^q|f(s,x(s))-f(s,y(s))|(t)\\
& +\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))-f(s,y(s))|(\zeta)\\
& +\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(\delta\ {}^\rho I^\beta J^q|f(s,x(s))-f(s,y(s))|(\xi)\\
& +J^q|f(s,x(s))-f(s,y(s))|(T)\Big)\\
\leq\ & L\|x-y\|J^q(1)(T)+L\|x-y\|\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta)\\
& +L\|x-y\|\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi)+J^q(1)(T)\Big)\\
=\ & L\Omega\|x-y\|,
\end{aligned}
$$

which leads to $\|\mathcal{Q}x-\mathcal{Q}y\|\leq L\Omega\|x-y\|$. As $L\Omega<1$, $\mathcal{Q}$ is a contraction. Therefore, it follows by the Banach's contraction mapping principle that $\mathcal{Q}$ has a fixed point which in fact is the unique solution of problem (1)-(2). The proof is completed. □

Theorem 3.2 *Assume that* (H_1) *holds. If*

$$L\Omega_1<1, \tag{21}$$

where Ω_1 *is defined by (19), then the boundary value problem (1)-(3) has a unique solution on* $[0,T]$.

3.2 Existence result via Krasnoselskii's fixed point theorem

Lemma 3.3 *(Krasnoselskii's fixed point theorem) [21]. Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) *$Ax+Bx\in M$ whenever*

$x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3.4 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition we assume that*

(H_2) $|f(t,x)| \le \varphi(t), \quad \forall (t,x) \in [0,T] \times \mathbb{R}$, *and* $\varphi \in C([0,T], \mathbb{R}^+)$.

Then the problem (1)-(2) has at least one solution on $[0,T]$ provided

$$L\left\{ \frac{|\gamma|(|v_4| + T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha \Gamma(q+1)} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)} + \frac{(|v_2| + T|v_1|)}{|\Lambda|} \left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta \Gamma(q+1)} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)} \right) \right\} < 1. \tag{22}$$

Proof. Define the operators $\mathcal{Q}_1, \mathcal{Q}_2 : \mathcal{C} \to \mathcal{C}$ as follows

$$\begin{aligned} \mathcal{Q}_1 x(t) &= J^q f(s, x(s))(t), \quad t \in [0,T], \\ \mathcal{Q}_2 x(t) &= \frac{\gamma}{\Lambda}(v_4 - tv_2)\, {}^\rho I^\alpha J^q f(s,x(s))(\zeta) \\ &\quad + \frac{1}{\Lambda}(v_2 + tv_1)\Big(\delta\, {}^\rho I^\beta J^q f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big), \quad t \in [0,T]. \end{aligned}$$

Setting $\sup_{t\in[0,T]} \varphi(t) = \|\varphi\|$ and choosing $\rho \ge \|\varphi\|\Omega$, where Ω is defined by (18), we consider $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$. For any $x, y \in B_\rho$, we have

$$\begin{aligned} |\mathcal{Q}_1 x(t) + \mathcal{Q}_2 y(t)| &\le \sup_{t\in[0,T]} \Bigg\{ J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\, {}^\rho I^\alpha J^q |f(s,x(s))|(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\, {}^\rho I^\beta J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\Bigg\} \\ &\le \|\varphi\| \Bigg\{ \frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|v_4| + T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)} \\ &\quad + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right)\Bigg\} \\ &= \|\varphi\|\Omega \le \rho. \end{aligned}$$

This shows that $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_\rho$. Using (22), it ca easily be established that $\mathcal{Q}_2$ is a contraction.

Continuity of f implies that the operator $\mathcal{Q}_1$ is continuous. Also, $\mathcal{Q}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{Q}_1 x\| \le \frac{T^q}{\Gamma(q+1)}\|\varphi\|.$$

Now we prove the compactness of the operator $\mathcal{Q}_1$.

We define $\sup_{(t,x)\in[0,T]\times B_\rho} |f(t,x)| = \bar{f} < \infty$, and consequently, for $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\begin{aligned} |\mathcal{Q}_1 x(t_2) - \mathcal{Q}_1 x(t_1)| &= \Big| J^q f(s,x(s))(t_2) - J^q f(s,x(s))(t_1) \Big| \\ &\le \frac{\bar{f}}{\Gamma(q)} \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]ds + \int_{t_1}^{t_2} (\tau_2 - s)^{q-1} ds \right| \end{aligned}$$

$$\leq \frac{\bar{f}}{\Gamma(q+1)}[|t_2^q - t_1^q| + |t_2 - t_1|^q],$$

which tends to zero as $t_2 - t_1 \to 0$ is independent of x. Thus, $\mathcal{Q}_1$ is equicontinuous. So $\mathcal{Q}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{Q}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 3.3 are satisfied. So the conclusion of Lemma 3.3 implies that problem (1)-(2) has at least one solution on $[0,T]$ □

Theorem 3.5 *Assume that* (H_1) *and* (H_2) *hold. Then the problem (1)-(3) has at least one solution on* $[0,T]$ *provided*

$$L\left\{\frac{|\gamma|(|u_4|+T|u_2|)\zeta^{\alpha+q}}{|\Lambda_1|\Gamma(\alpha+q+1)} + \frac{(|v_2|+T|v_1|)}{|\Lambda_1|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right)\right\} < 1. \tag{23}$$

3.3 Existence and uniqueness result via nonlinear contractions

Definition 3.6 *Let E be a Banach space and let $\mathcal{F}: E \to E$ be a mapping. $\mathcal{F}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Theta: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:*

$$\|\mathcal{F}x - \mathcal{F}y\| \leq \Theta(\|x-y\|), \qquad \forall x, y \in E.$$

Lemma 3.7 *(Boyd and Wong)[22]. Let E be a Banach space and let $\mathcal{F}: E \to E$ be a nonlinear contraction. Then $\mathcal{F}$ has a unique fixed point in E.*

Theorem 3.8 *Let $f: [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_3) $|f(t,x) - f(t,y)| \leq z(t)\dfrac{|x-y|}{A^* + |x-y|}$, *for $t \in [0,T]$, $x, y \geq 0$, where $z: [0,T] \to \mathbb{R}^+$ is continuous and A^* is the constant given by*

$$A^* := J^q z(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q z(\zeta) + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big\{|\delta|\ {}^\rho I^\beta J^q z(\xi) + J^q z(T)\Big\}.$$

Then the problem (1)-(2) has a unique solution on $[0,T]$.

Proof. Consider the operator $\mathcal{Q}: \mathcal{C} \to \mathcal{C}$ defined by (16) and a continuous nondecreasing function $\Theta: \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Theta(\varepsilon) = \frac{A^*\varepsilon}{A^* + \varepsilon}, \qquad \forall \varepsilon \geq 0.$$

Note that the function Θ satisfies $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&|\mathcal{Q}x(t) - \mathcal{Q}y(t)| \\
\leq\ & \sup_{t\in[0,T]}\Bigg\{J^q|f(s,x(s)) - f(s,y(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s)) - f(s,y(s))|(\zeta) \\
&+\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s,x(s)) - f(s,y(s))|(\xi) + J^q|f(s,x(s)) - f(s,y(s))|(T)\Big)\Bigg\} \\
\leq\ & J^q\left(z(s)\frac{|x-y|}{A^* + |x-y|}\right)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q\left(z(s)\frac{|x-y|}{A^* + |x-y|}\right)(\zeta)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\left\{|\delta|\ {}^{\rho}I^{\beta}J^{q}\left(z(s)\frac{|x-y|}{A^*+|x-y|}\right)(\xi)+J^{q}\left(z(s)\frac{|x-y|}{A^*+|x-y|}\right)(T)\right\}\\
\leq\ & \frac{\Theta(\|x-y\|)}{A^*}\Big[J^{q}z(T)+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^{\rho}I^{\alpha}J^{q}z(\zeta)+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big\{|\delta|\ {}^{\rho}I^{\beta}J^{q}z(\xi)+J^{q}z(T)\Big\}\Big]\\
=\ & \Theta(\|x-y\|).
\end{aligned}
$$

This implies that $\|\mathcal{Q}x-\mathcal{Q}y\|\leq\Theta(\|x-y\|)$. Therefore $\mathcal{Q}$ is a nonlinear contraction. Hence, by Lemma 3.7 the operator $\mathcal{Q}$ has a unique fixed point which is the unique solution of the problem (1)-(2). This completes the proof. □

Theorem 3.9 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption:*

$(H_3)'$ $|f(t,x)-f(t,y)|\leq z(t)\dfrac{|x-y|}{A_1^*+|x-y|}$, *for $t\in[0,T]$, $x,y\geq 0$, where $z:[0,T]\to\mathbb{R}^+$ is continuous and A_1^* is the constant given by*

$$
A_1^*:=J^{q}z(T)+\frac{|\gamma|}{|\Lambda|}(|u_4|+T|u_2|)\ J^{\alpha+q}z(\zeta)+\frac{1}{|\Lambda|}(|u_2|+T|u_1|)\Big\{|\delta|\ {}^{\rho}I^{\beta}J^{q}z(\xi)+J^{q}z(T)\Big\}.
$$

Then the problem (1)-(3) has a unique solution on $[0,T]$.

3.4 Existence result via Schaefer fixed point theorem

Lemma 3.10 *[23] Let X be a Banach space. Assume that $T:X\to X$ is a completely continuous operator and the set $V=\{u\in X\mid u=\mu Tu, 0<\mu<1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.11 *Assume that there exists a positive constant L_1 such that $|f(t,x)|\leq L_1$ for $t\in[0,1]$, $x\in\mathbb{R}$. Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.*

Proof. As a first step, it will be shown that the operator $\mathcal{Q}$ defined by (16) is completely continuous. Observe that continuity of $\mathcal{Q}$ follows from the continuity of f. For a positive constant r, let $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$ be a bounded ball in $\mathcal{C}$. Then for $t\in[0,T]$ we have

$$
\begin{aligned}
|\mathcal{Q}x(t)| \leq\ & J^{q}|f(s,x(s))|(t)+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^{\rho}I^{\alpha}J^{q}|f(s,x(s))|(\zeta)\\
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^{\rho}I^{\beta}J^{q}|f(s,x(s))|(\xi)+J^{q}|f(s,x(s))|(T)\Big)\\
\leq\ & L_1J^{q}(1)(T)+L_1\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^{\rho}I^{\alpha}J^{q}(1)(\zeta)\\
&+L_1\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^{\rho}I^{\beta}J^{q}(1)(\xi)+J^{q}(1)(T)\Big),\\
\leq\ & L_1\left\{\frac{T^{q}}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)}\right.\\
&\left.+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^{\beta}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)}-\frac{T^{q}}{\Gamma(q+1)}\right)\right\}\\
=\ & L_1\Omega.
\end{aligned}
$$

Now, for $\tau_1,\tau_2\in[0,1]$ with $\tau_1<\tau_2$, we get

$$
|\mathcal{Q}x(\tau_2)-\mathcal{Q}x(\tau_1)| \leq |J^{q}f(s,x(s))(\tau_2)-J^{q}f(s,x(s))(\tau_1)|+\frac{|\gamma||v_2||\tau_2-\tau_1|}{|\Lambda|}\ {}^{\rho}I^{\alpha}J^{q}|f(s,x(s))|(\zeta)
$$

$$
\begin{aligned}
& +\frac{|v_1||\tau_2-\tau_1|}{|\Lambda|}\Big(|\delta|\ {}^{\rho}I^{\beta}J^{q}|f(s,x(s))|(\xi)+J^{q}|f(s,x(s))|(T)\Big)\\
\le\ & \frac{L_1}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}]ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}ds\right|\\
& +\frac{L_1|\gamma||v_2||\tau_2-\tau_1|}{|\Lambda|}\ {}^{\rho}I^{\alpha}J^{q}(\zeta)+\frac{L_1|v_1||\tau_2-\tau_1|}{|\Lambda|}\Big(|\delta|\ {}^{\rho}I^{\beta}J^{q}(\xi)+J^{q}(T)\Big).
\end{aligned}
$$

As $\tau_2-\tau_1\to 0$, the right-hand side of the above inequality tends to zero independently of $x\in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{Q}:\mathcal{C}\to\mathcal{C}$ is completely continuous.

Next, we consider the set $V=\{x\in\mathcal{C}: x=\mu\mathcal{Q}x,\ 0<\mu<1\}$. In order to show that V is bounded, let $x\in V$ and $t\in[0,T]$. Then

$$
\begin{aligned}
\|x\|\ \le\ & L_1\left\{\frac{T^q}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)}\right.\\
& \left.+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^{\beta}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)}-\frac{T^q}{\Gamma(q+1)}\right)\right\}\\
=\ & L_1\Omega.
\end{aligned}
$$

Therefore, V is bounded. Hence, by Lemma 3.10, the boundary value problem (1)-(2) has at least one solution. □

Theorem 3.12 *Assume that there exists a positive constant L_1 such that $|f(t,x)|\le L_1$ for $t\in[0,1]$, $x\in\mathbb{R}$. Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.*

3.5 Existence result via Leray-Schauder's Degree Theory

Theorem 3.13 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that*

(H_4) *there exist constants $0\le\nu<\Omega^{-1}$, and $M>0$ such that*

$$|f(t,x)|\le\nu|x|+M\quad\text{for all}\quad (t,x)\in[0,T]\times\mathbb{R},$$

where Ω is defined by (18).

Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. In view of the fixed point problem

$$x=\mathcal{Q}x, \tag{24}$$

where the operator $\mathcal{Q}:\mathcal{C}\to\mathcal{C}$ is given by (16), we have to establish that there exists at least one solution $x\in C[0,T]$ satisfying (24). Set a ball $B_R\subset C[0,T]$ with a constant radius $R>0$ as

$$B_R=\{x\in\mathcal{C}:\max_{t\in[0,T]}|x(t)|<R\}.$$

Then we have to show that the operator $\mathcal{Q}:\overline{B}_R\to C[0,T]$ satisfies the condition

$$x\neq\theta\mathcal{Q}x,\quad\forall x\in\partial B_R,\quad\forall\theta\in[0,1]. \tag{25}$$

Next, we introduce

$$H(\theta,x)=\theta\mathcal{Q}x,\quad x\in\mathcal{C},\quad\theta\in[0,1].$$

As shown in Theorem 3.16 we have that the operator $\mathcal{Q}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_θ defined by $h_\theta(x)=x-H(\theta,x)=$

$x - \theta\mathcal{Q}x$ is completely continuous. If (25) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\theta, B_R, 0) &= \deg(I - \theta\mathcal{Q}, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(x) = x - \mathcal{Q}x = 0$ for at least one $x \in B_R$. Let us assume that $x = \theta\mathcal{Q}x$ for some $\theta \in [0,1]$ and for all $t \in [0,T]$. Then

$$\begin{aligned}|x(t)| &= |\theta\mathcal{Q}x(t)|\\ &\leq J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))|(\zeta)\\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\\ &\leq (\nu|x| + M)J^q p(s)(T) + (\nu|x| + M)\frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta)\\ &\quad + (\nu|x| + M)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi) + J^q(1)(T)\Big)\\ &= (\nu|x| + M)\Omega,\end{aligned}$$

which, on taking the norm $\sup_{t\in[0,T]}|x(t)| = \|x\|$ and solving for $\|x\|$, yields

$$\|x\| \leq \frac{M\Omega}{1 - \nu\Omega}.$$

If $R = \dfrac{M\Omega}{1-\nu\Omega} + 1$, (25) holds. This completes the proof. □

Theorem 3.14 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that*

$(H_4)'$ *there exist constants $0 \leq \nu < \Omega_1^{-1}$, and $M > 0$ such that*

$$|f(t,x)| \leq \nu|x| + M \quad \textit{for all} \quad (t,x) \in [0,T] \times \mathbb{R},$$

where Ω_1 is defined by (19).

Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.

3.6 Existence result via Leray-Schauder's nonlinear alternative

Lemma 3.15 *(Nonlinear alternative for single valued maps [24]). Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) $\mathcal{A}$ has a fixed point in $\bar{U}$, or

(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}(x)$.

Theorem 3.16 *Assume that*

(H_5) *there exists a continuous nondecreasing function $\Phi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T], \mathbb{R}^+)$ such that*

$$|f(t,x)| \leq p(t)\Phi(\|x\|) \quad \textit{for each} \quad (t,x) \in [0,T] \times \mathbb{R};$$

(H_6) *there exists a constant $N > 0$ such that*

$$\frac{N}{\Phi(N)\Big\{J^q p(s)(T) + A_1 + A_2\Big\}} > 1,$$

where

$$\begin{aligned} A_1 &= \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta), \\ A_2 &= \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big). \end{aligned}$$

Then the boundary value problem (1)-(2) has at least one solution on $[0, T]$.

Proof. Let the operator $\mathcal{Q}$ be defined by (16). We first show that $\mathcal{Q}$ *maps bounded sets (balls) into bounded sets in* $C([0, T], \mathbb{R})$. For a positive constant r, let $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ be a bounded ball in $\mathcal{C}$. Then for $t \in [0, T]$ we have

$$\begin{aligned} |\mathcal{Q}x(t)| &\le J^q|f(s, x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s, x(s))|(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \\ &\le \Phi(\|x\|)J^q p(s)(T) + \Phi(\|x\|)\frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta) \\ &\quad + \Phi(\|x\|)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big), \end{aligned}$$

and consequently,

$$\begin{aligned} \|\mathcal{Q}x\| &\le \Phi(r)\Big\{J^q p(s)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big)\Big\}. \end{aligned}$$

Next we will show that *the operator $\mathcal{Q}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$*. Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned} |\mathcal{Q}x(\tau_2) - \mathcal{Q}x(\tau_1)| &\le |J^q f(s, x(s))(\tau_2) - J^q f(s, x(s))(\tau_1)| + \frac{|\alpha||v_2||\tau_2 - \tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q|f(s, x(s))|(\zeta) \\ &\quad + \frac{|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \\ &\le \frac{\Phi(r)}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]p(s)ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}p(s)ds\right| \\ &\quad + \frac{\Phi(r)|\gamma||v_2||\tau_2 - \tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q p(s)(T) \\ &\quad + \frac{\Phi(r)|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big). \end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Finally, we show that there exists an open set $U \subset \mathcal{C}$ with $x \ne \theta\mathcal{P}x$ for $\theta \in (0, 1)$ and $x \in \partial U$.

Let x be a solution. Then, for $t \in [0, T]$, and following the similar computations as in the first step, we have

$$|x(t)| \le \Phi(\|x\|)\Big\{J^q p(s)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta)$$

$$+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi)+J^q p(s)(T)\Big)\Big\}$$

which leads to

$$\frac{\|x\|}{\Phi(\|x\|)\Big\{J^q p(s)(T)+A_1+A_2\Big\}}\le 1.$$

In view of (H_6), there exists N such that $\|x\|\neq N$. Let us set

$$\mathcal{U}=\{x\in C([0,T],\mathbb{R})\,:\,\|x\|<N\}.$$

We see that the operator $\mathcal{Q}:\overline{\mathcal{U}}\to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x\in\partial\mathcal{U}$ such that $x=\theta\mathcal{Q}x$ for some $\theta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{Q}$ has a fixed point $x\in\overline{\mathcal{U}}$ which is a solution of the boundary value problem (1)-(2). This completes the proof. □

Theorem 3.17 *Assume that* (H_5) *holds. In addition we suppose that:*

$(H_6)'$ *there exists a constant* $N'>0$ *such that*

$$\frac{N'}{\Phi_1(N')\Big\{J^q p(s)(T)+A_1'+A_2'\Big\}}>1, \tag{26}$$

where

$$\begin{aligned}A_1' &= \frac{|\gamma|}{|\Lambda_1|}(|u_4|+T|u_2|)J^{\alpha+q}p(s)(\zeta),\\ A_2' &= \frac{1}{|\Lambda_1|}(|u_2|+T|u_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi)+J^q p(s)(T)\Big).\end{aligned}$$

Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.

4 Examples

In this section, we present some examples to illustrate our results.

Example 4.1 *Consider the following nonlocal boundary value problem involving generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{3}{2}}x(t)=\dfrac{3}{25}\left(\dfrac{4x^2(t)+5|x(t)|}{3+4|x(t)|}\right)e^{-2t}+\dfrac{1}{2}\cos^2 t+1, & t\in\left[0,\dfrac{5}{3}\right],\\ x(0)=\dfrac{1}{2}\,{}^{\frac{\sqrt{3}}{2}}I^{\frac{4}{\sqrt{3}}}x\left(\dfrac{2}{3}\right),\quad x\left(\dfrac{5}{3}\right)=\dfrac{3}{4}\,{}^{\frac{\sqrt{3}}{2}}I^{\frac{\pi}{2}}x\left(\dfrac{4}{3}\right), \end{cases} \tag{27}$$

where $q=3/2$, $T=5/3$, $\gamma=1/2$, $\rho=\sqrt{3}/2$, $\alpha=4/\sqrt{3}$, $\zeta=2/3$, $\delta=3/4$, $\beta=\pi/2$, $\xi=4/3$ and $f(t,x)=(3/25)((4x^2+5|x|)/(3+4|x|))e^{-2t}+(1/2)\cos^2 t+1$. Using given information, we find that $v_1=0.8856776719$, $v_2=0.02007036728$, $v_3=0.0060494642$, $v_4=1.202612652$, $\Lambda=1.065248589$ and $\Omega=4.304419870$. Also $|f(t,x)-f(t,y)|\le(1/5)|x-y|$. Thus the condition (H_1) is satisfied with $L=1/5$ and $L\Omega=0.8608839740<1$. Therefore, by Theorem 3.1, problem (27) has a unique solution on $[0,5/3]$.

Example 4.2 *Consider the following nonlocal boundary value problem*

$$\begin{cases} D^{\frac{5}{3}}x(t)=\dfrac{5}{48}(1+\sin^2 t)\dfrac{|x(t)|}{1+|x(t)|}+3t^2+\dfrac{2}{3}, & t\in\left[0,\dfrac{7}{4}\right],\\ x(0)=\dfrac{3}{2}\,{}^{\frac{5}{6}}I^{\frac{e}{\sqrt{2}}}x\left(\dfrac{5}{4}\right),\quad x\left(\dfrac{7}{4}\right)=\dfrac{4}{5}\,{}^{\frac{5}{6}}I^{\frac{11}{13}}x\left(\dfrac{3}{4}\right). \end{cases} \tag{28}$$

Here $q = 5/3$, $T = 7/4$, $\gamma = 3/2$, $\rho = 5/6$, $\alpha = e/\sqrt{2}$, $\zeta = 5/4$, $\delta = 4/5$, $\beta = 11/13$, $\xi = 3/4$ and $f(t,x) = (5(1+\sin^2 t)/48)(|x|/(1+|x|)) + 3t^2 + (2/3)$. Using the given data, we obtain $v_1 = -0.633695322$, $v_2 = 0.5982054854$, $v_3 = 0.1931118977$, $v_4 = 1.448388097$, $\Lambda = -0.8023161650 \neq 0$. As $|f(t,x) - f(t,y)| \leq (5/24)|x-y|$, we have that (H_1) is satisfied with $L = 5/24$. Further, we have $\Omega_2 = 0.9828570350 < 1$. Also

$$|f(t,x)| \leq \frac{5}{48}(1+\sin^2 t) + 3t^2 + \frac{2}{3} := \varphi(t),$$

which implies that the condition (H_2) holds true. In consequence, the conclusion of Theorem 3.4 applies and problem (28) has at least one solution on $[0, 7/4]$.

Example 4.3 *Consider the following nonlocal boundary value problem*

$$\begin{cases} D^{\frac{4}{3}}x(t) = \frac{1}{4}(t^{\frac{1}{3}}+1)\left(\frac{|x(t)|}{1+|x(t)|}\right) + \frac{3}{2}t + \frac{1}{3}, \qquad t \in \left[0, \frac{1}{2}\right], \\ x(0) = \frac{2}{\sqrt{\pi}}\, {}^{\frac{1}{\sqrt{3}}}I^{\frac{7}{4}}x\left(\frac{1}{4}\right), \quad x\left(\frac{1}{2}\right) = \frac{3}{e^2}\, {}^{\frac{1}{\sqrt{3}}}I^{\frac{8}{13}}x\left(\frac{1}{8}\right). \end{cases} \tag{29}$$

Here $q = 4/3$, $T = 1/2$, $\gamma = 2/\sqrt{\pi}$, $\rho = 1/\sqrt{3}$, $\alpha = 7/4$, $\zeta = 1/4$, $\delta = 3/e^2$, $\beta = 8/13$, $\xi = 1/8$ and $f(t,x) = ((t^{1/3}+1)/4)(|x|/(1+|x|)) + (3/2)t + (1/3)$. Using the previous information, we have $v_1 = 0.5478797820$, $v_2 = 0.02539640314$, $v_3 = 0.6962686485$, $v_4 = 0.4808910650$ and $\Lambda = 0.2811532112$. Choosing $z(t) = (t^{1/3}+1)/4$, find that $A^* = 0.2768779852$ and also

$$|f(t,x) - f(t,y)| \leq \frac{1}{4}(t^{\frac{1}{3}}+1)\frac{|x-y|}{0.2768779852 + |x-y|}.$$

Therefore, all assumptions of Theorem 3.8 are satisfied. Hence the problem (29) has at least one solution on $[0, 1/2]$.

Example 4.4 *Consider the following nonlocal boundary value problem with both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{5}{4}}x(t) = \tan^{-1}\left(\frac{x^4(t)+3x^2(t)}{1+|x(t)|}\right)\left(e^{\frac{3}{2}-t}+1\right) + 3\pi, \qquad t \in \left[0, \frac{3}{2}\right], \\ x(0) = \frac{4}{\sqrt{7}}J^{\frac{5}{\sqrt{3}}}x\left(\frac{1}{2}\right), \quad x\left(\frac{3}{2}\right) = \frac{\pi}{2}\, {}^{\frac{2}{7}}I^{\frac{3}{8}}x\left(\frac{5}{4}\right). \end{cases} \tag{30}$$

Here $q = 5/4$, $T = 3/2$, $\gamma = 4/\sqrt{7}$, $\alpha = 5/\sqrt{3}$, $\zeta = 1/2$, $\delta = \pi/2$, $\rho = 2/7$, $\beta = 3/8$, $\xi = 5/4$ and $f(t,x) = \tan^{-1}((x^4+3x^2)/(1+|x|))(e^{(3/2)-t}+1) + 3\pi$. From the given constants, we have $u_1 = 0.9607949552$, $u_2 = 0.005043420754$, $u_3 = -1.895136694$, $u_4 = -0.378780447$ and $\Lambda_1 = -0.3734883143 \neq 0$. As $f(t,x) \leq 4\pi := L_1$ for all $x \in \mathbb{R}$, therefore from Theorem 3.11, the problem 30 has at least one solution on $[0, 3/2]$.

Example 4.5 *Consider the following nonlocal boundary value problem subjected to both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{8}{5}}x(t) = \frac{1}{(t^{\frac{1}{2}}+10)^2}\left(\frac{10x^2(t)+1}{3+|x(t)|}\right) + e^{-|x(t)|} + \frac{1}{3}, \qquad t \in [0, \pi], \\ x(0) = \frac{\log 2}{\sqrt{3}}J^{\frac{3}{4}}x\left(\frac{\pi}{2}\right), \quad x(\pi) = \frac{\log 3}{\sqrt{8}}\, {}^{\frac{5}{\sqrt{7}}}I^{\frac{3}{\sqrt{e}}}x\left(\frac{\pi}{3}\right). \end{cases} \tag{31}$$

Here $q = 8/5$, $T = \pi$, $\gamma = \log 2/\sqrt{3}$, $\alpha = 3/4$, $\zeta = \pi/2$, $\delta = \log 3/\sqrt{8}$, $\rho = 5/\sqrt{7}$, $\beta = 3/\sqrt{e}$, $\xi = \pi/3$ and $f(t,x) = (1/(t^{1/2}+10)^2)((10x^2+1)/(3+|x|)) + e^{-|x|} + (1/3)$. By direct computation of given constants, we obtain $u_1 = 0.9607949552$, $u_2 = 0.2381638392$, $u_3 = 0.9635754531$, $u_4 = 3.121155944$ and $\Lambda_1 = 3.228279714 \neq 0$. In addition, we can find that $\Omega_1 = 8.997039531$. It is easy to see that

$$|f(t,x)| \leq \frac{1}{10}|x| + \frac{4}{3},$$

which leads to $\nu := 1/10 < \Omega_1^{-1} = 0.1111476721$ and $M := 4/3 > 0$. Applying the conclusion of Theorem 3.13, we get that the problem (31) has at least one solution on $[0, \pi]$.

Example 4.6 *Consider the following nonlocal boundary value problem supplemented with both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{7}{4}}x(t) = \dfrac{(\sqrt{t}+1)}{12}\left(\dfrac{x^2(t)\sin^2 x(t)}{3(1+|x(t)|)} + e^{-t}\cos^2 t\right), & t \in \left[0, \dfrac{12}{5}\right], \\ x(0) = \dfrac{1}{\sqrt{3}}J^{\frac{7}{9}}x\left(\dfrac{8}{5}\right), \quad x\left(\dfrac{12}{5}\right) = \dfrac{3}{16}\,{}^{\frac{1}{\sqrt{\pi}}}I^{\frac{1}{\sqrt{e}}}x\left(\dfrac{11}{5}\right), \end{cases} \tag{32}$$

where $q = 7/4$, $T = 12/5$, $\gamma = 1/\sqrt{3}$, $\alpha = 7/9$, $\zeta = 8/5$, $\delta = 3/16$, $\rho = 1/\sqrt{\pi}$, $\beta = 1/\sqrt{e}$, $\xi = 11/15$ and $f(t,x) = ((\sqrt{t}+1)/12)((x^2\sin^2 x)/(3(1+|x|)) + e^{-t}\cos^2 t)$. By the given values, we get $u_1 = 0.1010372543$, $u_2 = 0.8090664711$, $u_3 = 0.6114216572$, $u_4 = 1.970342759$, $\Lambda_1 = 0.6937587849 \neq 0$. Since

$$|f(t,x)| \leq \frac{(\sqrt{t}+1)}{12}\left(\frac{1}{3}|x| + 1\right) := p(t)\Phi_1(|x|),$$

the condition (H_4) is satisfied. Also $A_1' = 0.4202876316$, $A_2' = 0.7604168186$. Clearly condition (26) is satisfied for $N' > 3.560603169$. Therefore, by Theorem 3.17, problem (32) has at least one solution on $[0, 12/5]$.

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On entire function sharing a small function CM with its high order forward difference operator

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Abstract: In this paper, we investigate the uniqueness of an entire function of finite order sharing a small entire function with its high order forward difference operator. The results obtained extend some known theorems and also show the exact solutions of some certain difference equations.

Key words and phrases: uniqueness; entire function; difference equation; differential equation; small function.

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1 Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane C. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the standard notations such as $T(r, f), m(r, f), N(r, f), \overline{N}(r, f)$ in value distribution theory (see [11, 18, 19]). And we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to \infty$, possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. A meromorphic function a is said to be a small function with respect to f if and only if $T(r, a) = S(r, f)$. We use $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of zeros of f and the order of f respectively. We say that two meromorphic functions f and g share a value a IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that they share the value a CM (counting multiplicities). We define the forward difference operator $\Delta f = f(z+1) - f(z)$ and the high order forward difference operator $\Delta^n f = \Delta^{n-1}(\Delta f)$ by recurrence. Moreover, $\Delta^n f = \sum_{j=0}^{n} C_n^j (-1)^{n-j} f(z+j)$.

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In 1976, L. Rubel and C.C. Yang [7] studied the uniqueness of an entire function sharing two values with its derivative and they proved the following classical result.

Theorem 1 *Let f be a nonconstant entire function. If f and f' share two distinct finite values CM, then $f \equiv f'$.*

In 1996, R. Brück [2] studied the uniqueness theory about an entire function sharing one value with its first derivative and posed the following interesting conjecture.

Conjecture 1 *Let f be nonconstant entire function satisfying that the super order $\sigma_2(f) < \infty$ is not a positive integer. If f and f' share one finite value a CM, then $f' - a = c(f - a)$ holds for some nonzero constant c.*

It is well known that Δf can be considered as the difference counterpart of f'. So regarding Theorem A and Conjecture, it is natural to ask that what can be said about the relationship between Δf and f if they share one or two values CM. The difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been founded recently (see [3, 8, 9]), which brings about a number of papers focusing on such uniqueness problems. The authors in [17, 16, 20], for example, obtained the following results by considering the special case of entire functions of order less than 1 or 2 respectively.

Theorem 2 *[17] Let f be a transcendental entire function such that $\sigma(f) < 1$, n be a positive integer and η be a nonzero complex number. If f and $\Delta_\eta^n f$ share a finite value a CM, then $\Delta_\eta^n f - a = c(f - a)$ holds for some nonzero complex number c.*

Theorem 3 *[16] Let f be a transcendental entire function of order $\sigma(f) < 2$ and $\eta \neq 0$ be a complex number that is not a period of f. If f and $\Delta_\eta^n f$ share the value 0 CM, then $\Delta_\eta^n f/f$ reduces to a nonzero constant.*

Theorem 4 *[20] Let f be a transcendental entire function such that $\sigma(f) < 2$ and $\lambda(f) < \sigma(f)$. If f and $\Delta^n f$ share the value 0 CM, then f must be form of $f(z) = Ae^{\alpha z}$, where A and α are two nonzero constants.*

In this paper, we deal with the general case of entire function of finite order and obtain the following results which extend Theorem 2 and Theorem 4.

Theorem 5 *Let f be a transcendental entire function such that $\sigma(f) < \infty$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < 1$ and $\lambda(f - a) < \sigma(f)$. If f and $\Delta^n f$ share a CM, then a must reduce to a polynomial with degree at most $n - 1$ and f must be form of*

$$f(z) = a + bae^{\beta z},$$

where b and β are two nonzero constants such that $e^\beta = 1$.

Theorem 6 *Let f be a transcendental entire function such that $\lambda(f) < \sigma(f) < \infty$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If f and $\Delta^n f$ share a CM, then f must be form of $f(z) = be^{\beta z}$, where b and β are two nonzero constants such that $(e^{\beta} - 1)^n = 1$.*

Theorem 7 *Let f be a transcendental entire function such that $\lambda(f) < \max\{\sigma(f) - 1, 1\} < \infty$. If $f(z)$ and $\Delta^n f$ share the value 0 CM, then f must be form of $f(z) = he^{\beta z}$, where h and β are two nonzero constants.*

2 Some lemmas

Lemma 1 *(see[3]) Let f be a transcendental meromorphic function with finite order σ and η be a nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r),$$

$$i.e.,\ T(r, f(z+\eta)) = T(r, f) + S(r, f).$$

Lemma 2 *(see[3]) Let f be a transcendental meromorphic function with finite order σ. Then for each $\varepsilon > 0$, we have*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 3 *(see[3]) Let η be a nonzero complex number and f be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$, we have*

$$e^{-r^{\sigma-1+\varepsilon}} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq e^{r^{\sigma-1+\varepsilon}}.$$

Lemma 4 *(see [4]) Let f be a nonconstant meromorphic function of order $\sigma < \infty$, and let λ' and λ'' be, respectively, the exponent of convergence of the zeros and poles of f. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, +\infty)$ of $|z| = r$ of finite logarithmic measure, so that*

$$2\pi i n_{z,\eta} + \log\frac{f(z+\eta)}{f(z)} = \eta\frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon}), \tag{1}$$

or equivalently,

$$\frac{f(z+\eta)}{f(z)} = e^{\eta\frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon})},$$

holds for $r \notin E \cup [0, 1]$, where $n_{z,\eta}$ in (1) is an integer depending on both z and η, $\beta = \max\{\sigma - 2, 2\lambda - 2\}$ if $\lambda < 1$ and $\beta = \max\{\sigma - 2, \lambda - 1\}$ if $\lambda \geq 1$ and $\lambda = \max\{\lambda', \lambda''\}$.

Lemma 5 *(see [5]) Suppose that $P(z) = (\alpha+i\beta)z^n+\cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z) \not\equiv 0$ is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}, z = re^{i\theta}, \delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, there is $R > 0$ such that for $|z| = r > R$, we have*
(i) if $\delta(P, \theta) > 0$, then

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1+\varepsilon)\delta(P,\theta)r^n\};$$

(ii) if $\delta(P, \theta) < 0$, then

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1-\varepsilon)\delta(P,\theta)r^n\},$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ is a finite set.

Lemma 6 *(see [1]) Let g be a transcendental function of order less than 1, and h be a positive constant. Then there exists an ε set E such that*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \ \frac{g(z+\eta)}{g(z)} \to 1, as\ z \to \infty\ in\ C\backslash E$$

uniformly in η for $|\eta| \leq h$. Further, the set E may be chosen so that if $z \notin E$ and $|z|$ is sufficiently large, the function g has no zeroes or poles in $|\zeta - z| \leq h$.

Remark 1 *According to Hayman [12], an ε set is defined to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. Suppose E is an ε set, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.*

Lemma 7 *(see [18]) Suppose that $f_1, f_2, \cdots, f_n$ ($n \geq 2$) are meromorphic functions and $g_1, g_2, \cdots, g_n$ are entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j e^{g_j} \equiv 0$;
(ii) $g_j - g_k$ *are not constants for* $1 \leq j < k \leq n$;
(iii) *For* $1 \leq j \leq n, 1 \leq h < k \leq n,\ T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E)$.

Then $f_j \equiv 0$ ($j = 1, 2, \cdots, n$).

Lemma 8 *(see [6]) Let w be a transcendental meromorphic function with $\sigma < \infty$. Let $\Gamma = \{(k_1, j_1), \cdots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ for $i = 1, 2, \cdots, m$. Also let $\varepsilon > 0$ be a given constant, then there exists a set $E \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in \Gamma$, one has*

$$\left|\frac{w^{(k)}(z)}{w^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

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Lemma 9 *(see[18]) Let f be a nonconstant meromorphic function in the complex plane and $R(f) = p(f)/q(f)$, where $p(f) = \sum_{k=0}^{p} a_k f^k$ and $q(f) = \sum_{j=0}^{q} b_j f^j$ are two mutually prime polynomials in f. If the coefficients a_k, b_j are small functions of f and $a_k \not\equiv 0, b_j \not\equiv 0$, then*

$$T(r, R(f)) = \max\{p, q\}T(r, f) + S(r, f).$$

Lemma 10 *Let g be polynomial of degree at lest two. Then*

$$m\big(r, \sum_{j=0}^{n} a_j e^{g(z+j)-g(z)}\big) = m(r, e^{g(z+n)-g(z)}) + S(r, e^{g(z+n)-g(z)}),$$

where the coefficients a_j are small meromorphic functions of $e^{g(z+n)-g(z)}$.
Proof. Set $g(z) = a_l z^l + a_{l-1}z^{l-1} + \ldots + a_0$, $a_l \neq 0$, $l \geq 2$ and $H(z) = e^{la_l z^{l-1}}$. Then we get $g(z+j) - g(z) = jla_l z^{l-1} + \cdots$, and then $e^{g(z+j)-g(z)} = b_j e^{jla_l z^{l-1}}$, where $\sigma(b_j) \leq l-2$. So we have

$$\sum_{j=0}^{n} a_j e^{g(z+j)-g(z)} = \sum_{j=0}^{n} \tilde{a}_j e^{jla_l z^{l-1}} = \sum_{j=0}^{n} \tilde{a}_j H^j,$$

where $\tilde{a}_j = a_j b_j$ are small function of H. Application Lemma 9 to the equation above gives our conclusion immediately.

Lemma 11 *Let f be a transcendental entire function such that $2 \leq \sigma(f) < \infty$, also let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$ and $\lambda(f-a) < \sigma(f)$. If the difference equation*

$$\Delta^n f - a = (f-a)e^Q \tag{2}$$

holds, where Q is a nonconstant entire function, then Q is a polynomial such that $\deg Q = \sigma(f) - 1$.

Proof. From our assumption and Lemma 1, it is obvious for us to get that Q is a polynomial and

$$F := f - a = he^g \tag{3}$$

holds, where g is a polynomial with degree l satisfying $l = \sigma(f) \geq 2$, and h is an entire function originated from the canonical product of $f-a$ satisfying $\lambda(h) = \sigma(h) < \sigma(f)$. Set $g(z) = a_l z^l + a_{l-1}z^{l-1} + \ldots + a_0$ and $Q(z) = b_s z^s + a_{s-1}z^{s-1} + \ldots + b_0$ respectively. Substitution (3) into (2) yields

$$e^Q = \frac{\Delta^n f - a}{f-a} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{F(z+j)}{F(z)} + \frac{\Delta^n a - a}{F(z)}. \tag{4}$$

First of all, we estimate the first term $\sum_{j=0}^{n} C_n^j (-1)^{n-j} F(z+j)/F(z)$ on the right side of (4). Employing the definition of F, it turns out that $\sigma(F) = \sigma(f) =$

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$l \geq 2$ and $\lambda(F) = \sigma(h) < \sigma(f)$. By applying Lemma 4 to F, for any given $\varepsilon > 0$ small enough, there exists a set E with finite logarithmic measure such that

$$\frac{F(z+j)}{F(z)} = e^{j\frac{F'(z)}{F(z)} + O(r^{\beta+\varepsilon})}, \text{ as } r \to \infty, \text{ not in } E \cup [0,1], \tag{5}$$

where $\beta = \sigma(f) - 2$ if $\sigma(h) < 1$ or $\beta = \max\{\sigma(f) - 2, \sigma(h) - 1\}$ if $\sigma(h) \geq 1$. Combining the fact $\sigma(h) < \sigma(f) = l$, we get $\beta < \sigma(f) - 1 = l - 1$. By Lemma 8, we see, for any given $\varepsilon > 0$ small enough, that

$$\left|\frac{h'(z)}{h(z)}\right| \leq r^{\sigma(h)-1+\varepsilon} = o(r^{l-1}) \tag{6}$$

holds for $|z| = r \notin E$. Thus from (3) and (6), we obtain

$$\frac{F'(z)}{F(z)} = g'(z) + \frac{h'(z)}{h(z)} = la_l z^{l-1}(1 + o(1)) \tag{7}$$

as $|z| = r \to \infty$ not in E. So from (5) and (7), we obtain

$$\frac{F(z+j)}{F(z)} = e^{jla_l z^{l-1}(1+o(1))}, \quad r \notin E. \tag{8}$$

Secondly, we estimate the second term $(\Delta^n a - a)/F$ on the right side of (4). It is easy to see $N := \sigma(\Delta^n a - a) \leq \sigma(a) < \sigma(f) = l$ in a similar way by Lemma 1, which gives, for any given $\varepsilon > 0$, that

$$M(r, \Delta^n a - a) < e^{r^{N+\varepsilon}} \tag{9}$$

holds for all r large sufficiently. Let $\delta(\theta) = \cos((l-1)\theta + \arg a_l)$, $\delta(g,\theta) = \cos(l\theta + \arg a_l)$ and $z = re^{i\theta}$. It follows Lemma 5 that for any given $\varepsilon > 0$, there exists a set $H \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus H$, there is $R > 0$ such that for $|z| = r > R$, we have

$$\exp\{(1-\varepsilon)|a_l|\delta(g,\theta)r^l\} < |F(re^{i\theta})| \tag{10}$$

if $\delta(g,\theta) > 0$. So by (10) and (9), we see $(\Delta^n a - a)/F \to 0$, as $z = re^{i\theta} \to \infty$ such that $\delta(g,\theta) > 0$. By Lemma 3, for any for any given $\varepsilon > 0$ small enough, we have

$$e^{-r^{\sigma(h)-1+\varepsilon}} \leq \left|\frac{h(z+c)}{h(z)}\right| \leq e^{r^{\sigma(h)-1+\varepsilon}} \tag{11}$$

holds for all sufficient large $r \notin E$.

Lastly, we take such $z = re^{i\theta}$ that $\theta \in [0, 2\pi) \setminus H; \delta(g,\theta) > 0$ and consider three cases separately in the next section.

Case 1 If $\delta(\theta) < 0$, then

$$|e^{jla_l z^{l-1}(1+o(1))}| = e^{jl|a_l|r^{l-1}\delta(\theta)(1+o(1))} \to 0, \text{ as } r \to \infty.$$

By (4), (9), (11) and the equation above, we obtain $e^{Q(z)} = (-1)^n + o(1)$. It means Q is bounded on such θ and $r \notin E$, which implies Q is a constant. And then by (3) and (4), we obtain

$$k := e^Q = (-1)^n + \sum_{j=1}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} + \frac{\Delta^n a - a}{h(z)e^{g(z)}}. \qquad (12)$$

If $\Delta^n a - a \not\equiv 0$, then by (11), (12), and the fact $\sigma((\Delta^n a - a)/h) < \sigma(e^g)$, we see

$$\begin{aligned} &\tfrac{|a_l|}{\pi} r^l (1+o(1)) + S(r,e^g) = m(r,e^{-g}) + S(r,e^g) = m(r, \tfrac{\Delta^n a - a}{he^g}) \\ &\leq \sum_{j=1}^{n} m(r, \tfrac{h(z+j)}{h(z)}) + \sum_{j=1}^{n} m(r, e^{g(z+j)-g(z)}) \\ &\leq r^{\sigma(h)-1+\varepsilon} + \tfrac{n(n+1)}{2} \tfrac{|a_l l|}{\pi} r^{l-1}(1+o(1)), r \notin E, \end{aligned}$$

which is impossible. If $\Delta^n a - a \equiv 0$, then by (12), we see

$$k = (-1)^n + \sum_{j=1}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)}. \qquad (13)$$

Employing representation $\sigma(h) < \deg g(z) = l$ and (11), we see

$$\left| \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} \right| = e^{jl|a_l|r^{l-1}\delta(\theta)(1+o(1))}.$$

holds for $r \notin E$. And then in this situation, $(h(z+n)/h(z))e^{g(z+n)-g(z)}$ is the only maximal magnitude of module term in (13) by taking such z that $\delta(\theta) > 0$, which is also impossible.

Case 2 If $\delta(\theta) > 0$, then by (4), (8),(9) and (10), we obtain

$$e^{|b_s|r^s \cos(\arg b_s + s\theta)(1+o(1))} = |e^Q| = (1+o(1))e^{nl|a_l|r^{l-1}\delta(\theta)(1+o(1))} \to \infty.$$

It means $s = l-1$ on such θ and $r \notin E$, which yields $s = l-1$.

Case 3 $\delta(\theta) = 0$. Since the set $\{\theta : \delta(\theta) = 0\}$ is just a finite set and $\delta(g,\theta)$ is a continuous function of θ, so we can chose another $\tilde{\theta}$ near θ, possibly outside of a set with the linear measure zero, such that $\delta(g,\tilde{\theta}) > 0$ and $\delta(\tilde{\theta}) \neq 0$, and then this case can be transformed into case 1 or case 2.

Using the similar method in Lemma 11, we can prove the following lemma.

Lemma 12 *Let f be a transcendental entire function such that $2 \leq \sigma(f) < \infty$ and $\lambda(f) < \sigma(f)$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If the difference equation $\Delta^n f - a = (f-a)e^Q$ holds, where Q is a nonconstant entire function, then Q is a polynomial such that $\deg Q = \sigma(f) - 1$.*

Lemma 13 *Let a be an entire function of order less than 1. If a satisfies the difference equation $\Delta^n a - a = 0$, then $a \equiv 0$.*

Proof. Suppose on the contrary $a \not\equiv 0$. Then by Lemma 6, we see

$$1 = \frac{\Delta^n a}{a} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{a(z+j)}{a} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0$$

as $r \to +\infty, r \notin E_\varepsilon$, where E_ε is an ε set. It is impossible.

Lemma 14 *Let a be an entire function of order less than 1. Then a satisfies the difference equation $\Delta^n a = 0$ implies a is a polynomial of degree at most $n-1$.*

Proof. Set $H_i := \Delta^{n-i} a, \quad j = 0, 1, \ldots, n$. Then $H_1(z+1) - H_1(z) = \Delta H_1 = H_0 = \Delta^n a = 0$. If H_1 is a nonconstant entire function, then it is easy to see that $z_k = k \in Z$ are some different zeros of $H_1(z) - H_1(0)$, which implies

$$\overline{N}(r, \frac{1}{H_1(z) - H_1(0)}) \geq r(1 + o(1)).$$

So $\sigma(H_1) \geq 1$, which is a contradiction. Thus H_1 is a constant, and then $0 = {H_1}' = (\Delta H_2)' = \Delta H_2'$. By a similar discussion, we see H_2' is a constant and then $H_2'' = 0$. Repeating this process, we can obtain $a^{(n)} = H_n^{(n)} = 0$. Thus a is a polynomial whose degree is at most $n-1$.

3 The proofs of main theorems

1. Proof of theorem 5.
Since $\Delta^n f$ and f share the function a CM, so there exists a polynomial Q by Lemma 1 such that

$$\Delta^n f - a = (f - a)e^Q. \tag{14}$$

It follows $\lambda(f-a) < \sigma(f)$ that

$$f - a = he^g, \tag{15}$$

where g is a polynomial whose degree l satisfying $l = \sigma(f) \geq 1$, and h is an entire function originated from the canonical product of $f - a$ satisfying $\lambda(h) = \sigma(h) < \sigma(f) = l$. By substituting (15) into (14), we can obtain

$$[\Delta^n a - a] + \sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) e^{g(z+j)} = h(z) e^{g(z)+Q(z)}. \tag{16}$$

In what follows, we shall consider two cases separately to our discussion.

Case 1 $\sigma(f) \geq 2$. We rewrite (16) as the following form

$$[\Delta^n a - a] + [\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) e^{g(z+j)-g(z)} - h(z)e^{Q(z)}] e^{g(z)} = 0. \tag{17}$$

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By applying Lemma 11 to (14), we see $\deg Q = l-1$. Applying Lemma 7 to (17) and invoking the relation $\deg Q = l-1$, it turns out that $\Delta^n a - a = 0$, which mans $a \equiv 0$ by Lemma 13. Thus we get a contradiction with our assumption.

Case 2 $l = \deg g = \sigma(f) < 2$, in other worlds, $\sigma(f) = 1$. Thus without loss of generality, we can rewrite (15) as the form of $f - a = he^{\beta z}$, where β is a nonzero constant. By (14), we see $\deg(Q) \leq \sigma(f) = 1$, and then we shall consider two subcases in this case respectively as follows.

Case 2.1 Q is a constant. Then we can rewrite (17) as the following form

$$[\Delta^n a - a] + [H_n - he^Q]e^{\beta z} = 0, \tag{18}$$

where $H_n = \sum\limits_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j)k^j, \quad k = e^\beta$. It follows (18) and Lemma 7 that $\Delta^n a - a = 0$, which leads to a contradiction with our assumption similarly.

Case 2.2 $\deg(Q) = 1$. Set $Q(z) = \gamma z + d$, where γ is a nonzero constant. By substituting $Q(z) = \gamma z + d$ into (16), we see

$$[\Delta^n a - a] + H_n e^{\beta z} = e^d h e^{(\beta+\gamma)z}. \tag{19}$$

If $\beta + \gamma \neq 0$, then by (19) and Lemma 7, we get $h \equiv 0$, which is a contradiction. If $\beta + \gamma = 0$, then (19) reduces to

$$[\Delta^n a - a] + H_n e^{\beta z} = e^d h. \tag{20}$$

Then by (20) and Lemma 7, we see

$$H_n = \sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j)k^j = 0 \tag{21}$$

and

$$[\Delta^n a - a] = e^d h. \tag{22}$$

Employing representation (21) and Lemma 6, it turns out that

$$0 = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h} k^j \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j = (k-1)^n$$

as $z \to \infty$ not in an ε set. Thus we obtain $k = e^\beta = 1$ from the equation above. Substituting $k = 1$ into (21), we see $H_n = \sum\limits_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) = \Delta^n h = 0$. By Lemma 14 and the equation above, we see that h is a polynomial whose degree is at most $n-1$. If a is a transcendental function, and we take z such that $|z| = r$ and $|a(z)| = M(r, a)$, then we have

$$\lim_{z \to \infty} e^d \frac{h(z)}{a(z)} = 0.$$

However, we have by (22) that

$$e^d\frac{h}{a}=\frac{\Delta^n a}{a}-1=\sum_{j=0}^{n}C_n^j(-1)^{n-j}\frac{a(z+j)}{a}-1\to\sum_{j=0}^{n}C_n^j(-1)^{n-j}-1=-1$$

as $z\to\infty$ in $z\in\{z:|a(z)|=M(r,a)\}\backslash E_\varepsilon$, where E_ε is an ε set, which is impossible. Thus a is a polynomial and then $\deg(a)=\deg(\Delta^n a-a)=\deg e^d h=\deg h$, which leads to that a is a polynomial with degree at most $n-1$. Furthermore we get $\Delta^n a=0$ and $-a=e^d h$ from (22) and then f must be form of

$$f(z)=a(z)+ba(z)e^{\beta z},$$

where $b:=-e^{-d}$ and β are two nonzero constants such that $e^\beta=1$.

2. Proof of Theorem 6.

Using the same method as in Theorem 1, we see

$$\Delta^n f-a=(f-a)e^Q \tag{23}$$

and

$$f=he^g, \tag{24}$$

where g is a polynomial of degree l satisfying $l=\sigma(f)\geq 1$, h is an entire function originated from the canonical product of f satisfying $\lambda(h)=\sigma(h)<\sigma(f)=l$, and Q is a polynomial of degree at most l. From (23)-(24), we obtain

$$\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)e^{g(z+j)}=h(z)e^{g(z)+Q(z)}+a(z)-a(z)e^{Q(z)}. \tag{25}$$

In the next section, we shall consider two cases separately.

Case 1 $\sigma(f)\geq 2$. We rewrite (25) as the following form

$$[\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)e^{g(z+j)-g(z)}-h(z)e^{Q(z)}]e^{g(z)}=a(z)-a(z)e^{Q(z)}. \tag{26}$$

From Lemma 12, we see $\deg Q=l-1\geq 1$. Then by (26) and Lemma 7, we obtain $a-ae^Q=0$. Thus $e^Q\equiv 1$ or $a\equiv 0$, which is impossible.

Case 2 $l=\deg g=\sigma(f)<2$, in other words, $\sigma(f)=1$. Thus without loss of generality, we can rewrite (24) as the form of $f=he^{\beta z}$, where β is a nonzero constant. It is easy to see $\deg(Q)\leq 1$.We shall consider two subcases.

Case 2.1 Q is a constant. Then by (26), we see $e^Q=1$ and

$$\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)k^j-h(z)=0, \tag{27}$$

where $k=e^\beta$. From (27), we see

$$1=\sum_{j=0}^{n}C_n^j(-1)^{n-j}k^j\frac{h(z+j)}{h}\to\sum_{j=0}^{n}C_n^j(-1)^{n-j}k^j=(k-1)^n \tag{28}$$

as $z \to \infty$ not in an ε set. It means $(k-1)^n = 1$ and then

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j = 1. \tag{29}$$

By (27) and (29), we see

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j [h(z+j) - h(z)] = 0. \tag{30}$$

Set $B(z) = \Delta h = h(z+1) - h(z)$, then from Lemma 1, it is easy for us to see $\sigma(B) \leq \sigma(h) < 1$. From the definition of $B(z)$. Using the same method in Theorem 4 [20], we can proof $B(z) \equiv 0$. That is $h(z+1) = h(z)$. So we get h is a nonzero constant using the same method as in Lemma 14, and then f must be form of $f(z) = be^{\beta z}$, where $b := h$ and β are two nonzero constants such that $(e^{\beta} - 1)^n = 1$.

Case 2.2 $\deg(Q) = 1$. Set $Q(z) = \gamma z + d$, where γ is a nonzero constant. Then (25) becomes

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j h(z+j) e^{\beta z} - a = e^d h(z) e^{(\beta+\gamma)z} - e^d a e^{\gamma z}. \tag{31}$$

If $\beta + \gamma \neq 0$ and $\beta - \gamma \neq 0$, then by (31) and Lemma 7, we get $a \equiv 0$ and $h \equiv 0$, which is a contradiction. If $\beta - \gamma = 0$, then (31) becomes

$$\{[\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) k^j] + ae^d\} e^{\beta z} - a = e^d e^{2\beta z},$$

and we also get a contradiction by applying Lemma 7 to the equation above. If $\beta + \gamma = 0$, then (31) becomes

$$\{\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) k^j\} e^{2\beta z} = (e^d h(z) + a) e^{\beta z} - ae^d,$$

we can get a contradiction in a same way.

3. Proof of theorem 7.

We shall consider the following three cases separately to our discussion.

Case 1 $\sigma(f) < 1$. By Theorem 2, we get $\Delta^n f = cf$ holds for some nonzero complex number c. Then by Lemma 6, we get

$$c = \frac{\Delta^n f}{f} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{f(z+j)}{f(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0$$

as $z \to \infty$, possibly outside of a ε set. Therefore $c = 0$, which is a contradiction.

Case 2 $1 \leq \sigma(f) < 2$ and $\lambda(f) < 1$. Then we can get our conclusion immediately by Theorem 4.

Case 3 $\sigma(f) \geq 2$ and $\lambda(f) < \sigma(f) - 1$. Using the same method as in Theorem 5, we see

$$\Delta^n f = fe^Q \tag{32}$$

and

$$f = he^g, \tag{33}$$

where $g(z) = a_l z^l + a_{l-1} z^{l-1} + \ldots + a_0$, $Q(z) = b_s z^s + a_{s-1} z^{s-1} + \ldots + b_0$, $l \geq 2$, $s \leq k$, are polynomials, h is an entire function originated from the canonical product of f satisfying $\lambda(h) = \sigma(h) < \sigma(f) - 1 = l - 1$. From (32)-(33), we obtain

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} = e^{Q(z)}. \tag{34}$$

Recall $g(z+j) - g(z) = ja_l l z^{l-1}(1 + o(1))$. By (34), Lemma 1 and 10, we see

$$\begin{aligned} \tfrac{|b_s|}{\pi} r^s \sim m(r, e^Q) \qquad &= m\big(r, \textstyle\sum\limits_{j=0}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)}\big) \\ &= m\big(r, e^{g(z+n)-g(z)}\big) + S(r, e^{g(z+n)-g(z)}) \sim \tfrac{nl|a_l|}{\pi} r^{l-1}. \end{aligned}$$

It means $s = l - 1$ and $|b_s| = nl|a_l|$. We can rewrite (34) as the following form

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{ja_l l z^{l-1}(1+o(1))} + \frac{h(z+n)}{h(z)} e^{A_n} e^{na_l l z^{l-1}} = e^B e^{b_{l-1} z^{l-1}}, \tag{35}$$

where A_n, B are two polynomials with degree at most $l - 2$. Recalling (11) and taking any θ such that $\delta(\theta) = \cos((l-1)\theta + \arg a_l) > 0$, then we get $\tilde{\delta}(\theta) = \cos((l-1)\theta + \arg b_{l-1}) > 0$ by (35), and then

$$e^{nl|a_l| r^{l-1} \delta(\theta)(1+o(1))} = e^{|b_{l-1}| r^{l-1} \tilde{\delta}(\theta)(1+o(1))}.$$

That means $\delta(\theta) = \tilde{\delta}(\theta)$. By the arbitrariness of θ, we see $\arg a_l = \arg b_{l-1}$. Thus we obtain $b_s = nla_l$, and then (35) becomes

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{ja_l l z^{l-1}(1+o(1))} = e^B (1 - \frac{h(z+n)}{h(z)} e^{A_n - B}) e^{nla_l z^{l-1}}. \tag{36}$$

It is obvious $\sigma(e^B(1 - (h(z+n)/h) e^{A_n - B})) \leq \max\{\sigma(h), l-2\} < l - 1$. If $e^B - (h(z+n)/h) e^{A_n} \not\equiv 0$, then from (36) and Lemma 10, we see

$$\frac{nl|a_l|}{\pi} r^{l-1} \sim T(r, e^B (1 - \frac{h(z+n)}{h(z)} e^{A_n - B}) e^{nla_l z^{l-1}}) \sim \frac{(n-1)l|a_l|}{\pi} r^{l-1},$$

which is impossible. If $e^B - (h(z+n)/h(z)) e^{A_n} \equiv 0$, then (36) becomes

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{ja_l l z^{l-1}(1+o(1))} = 0, \tag{37}$$

however $(h(z+n-1)/h(z))e^{(n-1)a_l l z^{l-1}}$ is the only maximal magnitude of module term in (37) when taking $\delta(\theta) > 0$, which is impossible.

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Global Attractivity for Nonautonomous Difference Equation via Linearization

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Abstract. Consider the difference equation

$$\vec{x}_{n+1} = f(n, \vec{x}_n, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots,$$

where $k \in \{0, 1, \ldots\}$ and the initial conditions are real vectors. We investigate the asymptotic behavior of the solutions of the considered equation. We give some effective conditions for the global stability and global asymptotic stability of the zero or positive equilibrium of this equation. Our results are based on application of the linearizations technique. We illustrate our results with many examples that include some equations from mathematical biology.

Keywords: attractivity, difference equations, discrete dynamical system, global, linear fractional, rational, stability

AMS 2000 Mathematics Subject Classification: 39A10, 39A20, 37B25, 37D10, 37M99.

1 Introduction and preliminaries

Consider the difference equation

$$\vec{x}_{n+1} = f(n, \vec{x}_n, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots \tag{1}$$

where $k \in \{0, 1, \ldots\}$ and the initial conditions are real vectors in $\mathbb{R}^p, p \geq 2$. In many cases we investigate equation(1) by embedding equation(1) into a higher iteration of the form

$$\vec{x}_{n+l} = F(n, \vec{x}_{n+l-1}, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots \tag{2}$$

where $l \in \{1, 2, \ldots\}$, see [4, 5, 8]. By linearizing equation (2) and bringing it to the form

$$\vec{x}_{n+1} = \sum_{i=1-l}^{k} g_i \vec{x}_{n-i}, \tag{3}$$

where g_i in general, depend on n and the state variables $\vec{x}_k$ we can prove very general attractivity and asymptotic stability results for both autonomous and nonautonomous difference equations. The functions g_i are in general matrices but they can also be the scalars as well, see Section 3. This approach was used to get effective and applicable global asymptotic and global attractivity results for linear fractional difference equation, see [2] and quadratic fractional difference equation, see [3] with both constant and nonconstant coefficients. Furthermore, this approach produced global asymptotic and global attractivity results for nonautonomous difference equations with very general coefficients which can be discontinuous functions of n or state variables, see [4, 5, 8]. See [1, 7, 10, 11] for the case of monotone systems, where more precise results were obtained.

In this paper we use method of linearization to extend some of the results about the global attractivity and asymptotic stability of scalar equation from [4] to the case of vector equation (2). We illustrate our results with many examples that include some transition functions from mathematical biology such as linear, Beverton-Holt, sigmoid Beverton-Holt, etc., see [6, 7, 9, 11, 12] for related results. The rest of this section contains some definitions and preliminary results. Second section contains our main results on global attractivity in the case when the sum of the norms of g_i is less than 1. The third section

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gives some results on global attractivity in the delicate case when the sum of the scalar functions g_i is 1. The fourth section provides several examples which illustrate our results.

Denote by $\|\vec{x}\|$ any norm in $\mathbb{R}^p$.

Definition 1 *The zero equilibrium of equation* (3) *is stable if for* $(\forall\epsilon > 0)(\exists\delta > 0, N)$*:*

$$\|\vec{x}_i\| < \delta, i = -k, \dots, 0 \Longrightarrow \|\vec{x}_n\| < \epsilon, \text{ for all } n \geq N.$$

The zero equilibrium is asymptotically stable if it is stable and

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

Lemma 1 *Let* $\mathbf{I} - \sum_{i=0}^{k} g_i$ *be invertible for* $n = 1, 2, \dots,$ *where* $\mathbf{I}$ *is identity matrix.Then equation* (3) *has no nonzero equilibrium.*

Proof. Otherwise, equation (3) has the equilibrium $\bar{\mathbf{x}} \neq \vec{0}$. By pluging $\vec{x}_n = \bar{\mathbf{x}}$ in equation (3) we get

$$(\mathbf{I} - \sum_{i=0}^{k} g_i)\bar{\mathbf{x}} = \vec{0},$$

which implies $\bar{\mathbf{x}} = \vec{0}$, which is a contradiction. □

Remark 1 The matrix $\mathbf{I} - \sum_{i=0}^{k} g_i$ is invertible if the condition

$$\|\sum_{i=0}^{k} g_i\| < 1 \tag{4}$$

is satisfied in which case we have

$$(\mathbf{I} - \sum_{i=0}^{k} g_i)^{-1} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} g_i. \tag{5}$$

The condition (4) is implied by more applicable condition

$$\sum_{i=0}^{k} \|g_i\| < 1. \tag{6}$$

Remark 2 Equation (1) admits the following generalized identity

$$\vec{x}_{n+1} - \sum_{i=0}^{k} g_i\vec{K} = \sum_{i=0}^{k} g_i(\vec{x}_{n-i} - \vec{K}), \tag{7}$$

where $\vec{K}$ is an arbitrary vector. Generalized identity (7) implies

$$\|\vec{x}_{n+1} - \sum_{i=0}^{k} g_i\vec{K}\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{n-i} - \vec{K}\|. \tag{8}$$

Furthermore by taking $\vec{K} = \vec{0}$ in equation (8), we obtain another useful inequality

$$\|\vec{x}_{n+1}\| - L\sum_{i=0}^{k} \|g_i\| \leq \sum_{i=0}^{k} \|g_i\|(\|\vec{x}_{n-i}\| - L), \tag{9}$$

where L is an arbitrary constant.

Lemma 2 *Suppose that equation* (1) *has the linearization* (3) *and the functions* $g_i : R^{p+1} \to M_{p\times p}$, *where* $M_{p\times p}, p \geq 1$ *is the set of all real* $p \times p$ *matrices, are such that*

$$\sum_{i=0}^{k} \|g_i\| \leq 1, \quad n = 0, 1, \ldots$$

Then if equation (1) *has the zero equilibrium it is a stable fixed point.*

Proof. Assume that equation (1) has the zero equilibrium and the linearization (3). By taking $\vec{K} = \vec{0}$ in equation (8) we have

$$\|\vec{x}_{n+1}\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{n-i}\|.$$

Assume that $\sum_{i=0}^{k} \|\vec{x}_{-i}\| < \delta$. Take $\delta = \epsilon$. Then $\|\vec{x}_{-i}\| < \delta$ for $i = 0, 1, \ldots$. Hence

$$\|\vec{x}_1\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{-i}\| < \delta \sum_{i=0}^{k} \|g_i\| \leq \delta = \epsilon,$$

$$\|\vec{x}_2\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{1-i}\| < \delta \sum_{i=0}^{k} \|g_i\| \leq \delta = \epsilon$$

and so by induction $\|\vec{x}_n\| < \epsilon$ for $n \geq -k$. □

2 Main results

In this section we present our main results on global attractivity and global asymptotic stability of the equilibrium solutions of equation (1) which has the linearization (3).

Theorem 1 *Let* $l \in \{1, 2, \ldots\}$. *Suppose that equation* (1) *has the linearization* (3) *subject to the condition*

$$\sum_{i=1-l}^{k} \|g_i\| \leq 1, n = 0, 1, \ldots \tag{10}$$

Let $M_0 = \max\{\|\vec{x}_{l-1}\|, \ldots, \|\vec{x}_{-k}\|\}$. *Then every solution of equation* (1) *is bounded. In particular* $\|\vec{x}_n\| \leq M_0$ *for* $n \geq -k$.

Proof. Let $L \in R$. Then equation (9) implies

$$\|\vec{x}_{n+l}\| - L \sum_{i=1-l}^{k} \|g_i\| \leq \sum_{i=1-l}^{k} \|g_i\|(\|\vec{x}_{n-i}\| - L), \quad n = 0, 1, \ldots \tag{11}$$

By taking $L = M_0$ and $n = 0$ in equation (11), we obtain

$$\|\vec{x}_l\| - M_0 \sum_{i=1-l}^{k} \|g_i\| \leq \|g_{1-l}\|(\|\vec{x}_{l-1}\| - M_0) + \ldots + \|g_k\|(\|\vec{x}_{-k}\| - M_0) \leq 0,$$

which in view of equation (10) implies $\|x_l\| \leq M_0$. By using induction, we obtain

$$\|\vec{x}_{n+l}\| - M_0 \sum_{i=1-l}^{k} \|g_i\| \leq \|g_{1-l}\|(\|\vec{x}_{n+l-1}\| - M_0) + \ldots + \|g_k\|(\|\vec{x}_{n-k}\| - M_0) \leq 0, \quad n = 0, 1, \ldots$$

and so

$$\|\vec{x}_{n+l}\| \leq M_0 \sum_{i=1-l}^{k} \|g_i\| \leq M_0, \quad n = 0, 1, \ldots$$

Thus $\|\vec{x}_{n+l}\| \leq M_0$ for $n \geq -k$. □

Theorem 2 *Let $l \in \{1, 2, \ldots\}$. Suppose that equation (1) has the linearization (3) where the functions $g_i : R^{k+1} \to M_{p\times p}$ are such that*

$$\sum_{i=1-l}^{k} \|g_i\| \leq a < 1, \quad n = 0, 1, \ldots \tag{12}$$

Then

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

Proof. Let $L \in R$. Then every solution of equation (3) satisfies the inequality (11). Let $\gamma = l + k$. Define $M_N = \max\{\|\vec{x}_{\gamma N+l-1}\|, \ldots, \|\vec{x}_{\gamma N-k}\|\}$ for $N = 0, 1, \ldots$. Observe that if $\|\vec{x}_{\gamma N+l-1}\| = \ldots = \|\vec{x}_{\gamma N-k}\| = \vec{0}$ for some $N \geq 0$, then by (11) with $L = 0$ we get that

$$\|\vec{x}_{\gamma N+l+j}\| = \vec{0}, \quad j = 0, 1, \ldots$$

and so $\lim_{n\to\infty} \vec{x}_n = \vec{0}$.
Assume that $M_N > 0$ for all $N \geq 0$. By using (11) with $L = M_N$ and $n = \gamma N$ we obtain

$$\|\vec{x}_{\gamma N+l}\| - \sum_{i=1-l}^{k} \|g_i\| M_N \leq \|g_{1-l}\|(\|\vec{x}_{\gamma N+l-1}\| - M_N) + \ldots + \|g_k\|(\|\vec{x}_{\gamma N-k}\| - M_N) \leq 0$$

and so

$$\|\vec{x}_{\gamma N+l}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Similarly, by taking $n = \gamma N + 1$ in (11) we obtain

$$\|\vec{x}_{\gamma N+l+1}\| - \sum_{i=1-l}^{k} \|g_i\| M_N \leq \|g_{1-l}\|(\|\vec{x}_{\gamma N+l}\| - M_N) + \ldots + \|g_k\|(\|\vec{x}_{\gamma N-k+1}\| - M_N) \leq 0$$

and so

$$\|\vec{x}_{\gamma N+l+1}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Hence by induction we have that

$$\|\vec{x}_{\gamma N+l+j}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Thus

$$M_{N+1} \leq a M_N < M_N, \tag{13}$$

and so the sequence $\{M_N\}_{N=0}^{\infty}$ is decreasing sequence bounded below by zero. Furthermore (13) implies

$$M_N \leq a^{N+1} M_0 \to 0 \qquad as \qquad N \to \infty.$$

Hence

$$0 \leq \lim_{N\to\infty} \vec{x}_{\gamma N-j} \leq \lim_{N\to\infty} M_N = 0, \quad j = 1-l, \ldots, k.$$

Therefore

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

□

Corollary 1 *Suppose that equation* (1) *has the linearization* (3), *where* $l = 1$ *and the functions* $g_i : R^{k+1} \to M_{p\times p}$ *are such that*

$$\sum_{i=0}^{k} \|g_i\| \leq a < 1, \quad n = 0, 1, \ldots.$$

Then if equation (1) *has a zero equilibrium it is globally asymptotically stable.*

Assuming that f is differentiable in some neighborhood of the equilibrium solution $\bar{x}$, by applying Theorem 2 and Lemma 2 to the standard linearization of equation (1) about the equilibrium solution $\bar{x}$

$$\vec{x}_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})\vec{x}_{n-i}, \quad n = 0, 1, \ldots, \tag{14}$$

where $\frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})$ is the Jacobian matrix evaluated at the equilibrium point, we obtain the following result, which is local in the nature because of the fact that the standard linearization is local.

Corollary 2 *Assume that* f *is differentiable in some neighborhood of the equilibrium solution* $\bar{x}$. *The equilibrium* $\bar{x}$ *of equation* (1) *is locally asymptotically stable if*

$$\sum_{i=0}^{k} \|\frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})\| \leq a < 1.$$

3 The case when g_i are scalar functions

In this section we consider the case when all g_i are scalar functions. In this case the linearization (3) is equivalent to p scalar equations of the form

$$x^m_{n+1} = \sum_{i=1-l}^{k} g_i x^m_{n-i}, \quad n = 0, 1, \ldots; m = 1, \ldots, p. \tag{15}$$

For instance, in the case of second order difference equation in $\mathbb{R}^2$, we have that vector equation

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = g_0 \begin{bmatrix} x_n \\ y_n \end{bmatrix} + g_1 \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} \quad n = 0, 1, \ldots \qquad g_0, g_1 \geq 0 \tag{16}$$

is equivalent to the system

$$\begin{aligned} x_{n+1} &= g_0 x_n + g_1 x_{n-1} \\ y_{n+1} &= g_0 y_n + g_1 y_{n-1}. \end{aligned} \tag{17}$$

The next results apply to a special linearization (3) of equation (1), where all g_i are scalar functions.

Theorem 3 *Let $l \in \{1, 2, \ldots\}$. Suppose that equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that*

$$\sum_{i=1-l}^{k} g_i \geq a > 1, \quad n \geq 0.$$

Then if for some $n \geq 0$

(a) $\vec{x}_{n+l-1}, \ldots, \vec{x}_{n-k} > 0$, then $\lim_{n\to\infty} \vec{x}_n = \infty$, componentwise;

(b) $\vec{x}_{n+l-1}, \ldots, \vec{x}_{n-k} < 0$, then $\lim_{n\to\infty} \vec{x}_n = -\infty$, componentwise.

Proof. Proof follows from Theorem 2 in [4] applied to equation(15). □

A delicate case when

$$\sum_{i=1-l}^{k} g_i = 1, \quad n = 0, 1, \ldots. \tag{18}$$

is treated in the following three results.

Theorem 4 *Suppose that on some interval I equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Then there exists $A > 0$ such that for $n \geq 0$ every positive g_i satisfies*

$$A \leq g_i \leq 1, \quad n = 0, 1, \ldots. \tag{19}$$

Proof. Proof follows from Proposition 3 in [4] applied to equation (15). □

Theorem 5 *Suppose that on some interval I equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Assume that there exists $A > 0$ such that*

$$g_{1-l} \geq A, \quad n = 0, 1, \ldots. \tag{20}$$

Then if $\vec{x}_{-k}, \ldots, \vec{x}_0 \in I$

$$\lim_{n\to\infty} \vec{x}_n = \vec{L},$$

where $\vec{L} \in I^p$ is a constant vector

Proof. Proof follows from Theorem 4 in [4] applied to equation (15). □

Theorem 6 *Suppose that on some interval $I \subset \mathbb{R}$ equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Assume that there exists $A > 0$ such that for some $j \in \{2 - l, \ldots, k - 1\}$*

$$g_j \geq A, g_{j+1} \geq A, \quad n = 0, 1, \ldots. \tag{21}$$

If $\vec{x}_{l-1}, \ldots, \vec{x}_{-k} \in I$, then

$$\lim_{n\to\infty} \vec{x}_n = \vec{L},$$

where $\vec{L} \in I^p$ is a constant vector

Proof. Proof follows from Theorem 5 in [4] applied to equation (15). □

4 Examples

In this section we present some examples that illustrate our results.

Example 1 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & b_n \\ c_n & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \ldots,$$

where $a, d > 0, b_n, c_n \geq 0, x_0, y_0 \geq 0, n = 0, 1, \ldots$, converges to the zero equilibrium if $\max\{a + U_c, d + U_b\} < 1$ is satisfied, where U_b and U_c are upper bounds of sequences $\{b_n\}$ and $\{c_n\}$ respectively. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have

$$\|g_0\| = \left\| \begin{bmatrix} \frac{a}{1+x_n} & b_n \\ c_n & \frac{d}{1+y_n} \end{bmatrix} \right\| = \max\left\{ \frac{a}{1+x_n} + c_n, \frac{d}{1+y_n} + b_n \right\} \leq \max\{a + U_c, d + U_b\} < 1,$$

that is $U_c < 1 - a, U_b < 1 - d$, and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable. If we use L_2 norm we have that the zero equilibrium is globally asymptotically stable if $\max\{a + U_b, d + U_c\} < 1$ is satisfied.

Example 2 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & b \\ c & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \ldots, \tag{22}$$

where $a, b, c, d > 0, x_0, y_0 \geq 0$, converges to the zero equilibrium if $\max\{a + c, b + d\} < 1$ is satisfied. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have that

$$\|g_0\| = \left\| \begin{bmatrix} \frac{a}{1+x_n} & b \\ c & \frac{d}{1+y_n} \end{bmatrix} \right\| = \max\left\{ \frac{a}{1+x_n} + c, \frac{d}{1+y_n} + b \right\} \leq \max\{a + c, b + d\} < 1$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable. If we use L_2 norm we have that $\max\{a + b, c + d\} < 1$ implies that the zero equilibrium is globally asymptotically stable.

Next, consider the positive equilibrium $E(\bar{x}, \bar{y})$. Then we have that the positive equilibrium $E(\bar{x}, \bar{y})$ of system (22) satisfies the system

$$\begin{aligned} \bar{x} &= a\tfrac{\bar{x}}{1+\bar{x}} + b\bar{y} \\ \bar{y} &= c\bar{x} + d\tfrac{\bar{y}}{1+\bar{y}}. \end{aligned} \tag{23}$$

which implies

$$\begin{aligned} \bar{x}\tfrac{1+\bar{x}-a}{1+\bar{x}} &= b\bar{y} \\ \bar{y}\tfrac{1+\bar{y}-d}{1+\bar{y}} &= c\bar{x}. \end{aligned}$$

Thus the positive equilibrium exists if

$$\bar{x} > a - 1, \bar{y} > d - 1. \tag{24}$$

Linearizing system (22) about the positive equilibrium E gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{(1+\bar{x})(1+x_n)} & b \\ c & \frac{d}{(1+\bar{y})(1+y_n)} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, n = 0, 1, \ldots, \tag{25}$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition for global asymptotic stability of $E(\bar{x}, \bar{y})$ to be

$$\bar{x} > \frac{a + c - 1}{1 - c} \quad \text{if} \quad c < 1 < a + c, \quad \bar{y} > \frac{b + d - 1}{1 - b} \quad \text{if} \quad b < 1 < b + d.$$

If we use L_2 norm we obtain sufficient condition for global asymptotic stability of $E(\bar{x}, \bar{y})$ to be

$$\bar{x} > \frac{a+b-1}{1-b} \quad \text{if} \quad b < 1 < a+b, \quad \bar{y} > \frac{c+d-1}{1-c} \quad \text{if} \quad c < 1 < c+d.$$

Example 3 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \ldots, \tag{26}$$

where $a, b, c, d > 0, x_0, y_0 \geq 0, n = 0, 1, \ldots$, converges to the zero equilibrium if $\max\{a+c, b+d\} < 1$ is satisfied. Indeed, in this case if $\|x\|_1$ denotes the L_1 norm we have

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \right\|_1 = \max\left\{ \frac{a}{1+x_n} + \frac{c}{1+x_n}, \frac{b}{1+y_n} + \frac{d}{1+y_n} \right\} \leq \max\{a+c, b+d\} < 1$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

In the case if $\|x\|_2$ denotes the L_2 norm we have

$$\|g_0\|_2 = \left\| \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \right\|_2 = \max\left\{ \frac{a}{1+x_n} + \frac{b}{1+y_n}, \frac{c}{1+x_n} + \frac{d}{1+y_n} \right\} \leq \max\{a+b, c+d\} < 1.$$

In this case the condition for global asymptotic stability of the zero equilibrium becomes $\max\{a+b, c+d\} < 1$.

Now, consider global attractivity of the positive equilibrium $E(\bar{x}, \bar{y})$ of system (26). The positive equilibrium of system (26) satisfies the system

$$\begin{aligned} \bar{x} &= a\tfrac{\bar{x}}{1+\bar{x}} + b\tfrac{\bar{y}}{1+\bar{y}} \\ \bar{y} &= c\tfrac{\bar{x}}{1+\bar{x}} + d\tfrac{\bar{y}}{1+\bar{y}}. \end{aligned} \tag{27}$$

Adding two equations in (27) we obtain

$$\bar{x} + \bar{y} = (a+c)\frac{\bar{x}}{1+\bar{x}} + (b+d)\frac{\bar{y}}{1+\bar{y}},$$

which implies

$$\frac{\bar{x}}{1+\bar{x}}(1+\bar{x}-a-c) = \frac{\bar{y}}{1+\bar{y}}(b+d-1-\bar{y})$$

and so we obtain that the positive equilibrium satisfies

$$\bar{x} > a+c-1 \Leftrightarrow \bar{y} < b+d-1. \tag{28}$$

Linearizing system (26) about the positive equilibrium E gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{(1+\bar{x})(1+x_n)} & \frac{b}{(1+\bar{y})(1+y_n)} \\ \frac{c}{(1+\bar{x})(1+x_n)} & \frac{d}{(1+\bar{y})(1+y_n)} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, n = 0, 1, \ldots,$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition

$$\bar{x} > a+c-1, \bar{y} > b+d-1. \tag{29}$$

is sufficient for the global asymptotic stability of the positive equilibrium solution. The condition (29) contradicts condition (28). If we use L_2 norm we obtain sufficient condition for the global asymptotic stability of the positive equilibrium solution to be

$$\begin{aligned} b\bar{x} + a\bar{y} &< 1-a-b \\ d\bar{x} + c\bar{y} &< 1-c-d. \end{aligned}$$

Example 4 Every solution of the vector equation in $\mathbb{R}^n$

$$\vec{x}_{n+1} = A_n \vec{x}_n \tag{30}$$

where

$$\vec{x}_n = \begin{bmatrix} x_n^1 \\ x_n^2 \\ \vdots \\ x_n^k \end{bmatrix}, \quad A_n = \begin{bmatrix} \frac{a_{11}}{1+x_n^1} & \frac{a_{12}}{1+x_n^2} & \cdots & \frac{a_{1k}}{1+x_n^k} \\ \frac{a_{21}}{1+x_n^1} & \frac{a_{22}}{1+x_n^2} & \cdots & \frac{a_{2k}}{1+x_n^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+x_n^1} & \frac{a_{k2}}{1+x_n^2} & \cdots & \frac{a_{kk}}{1+x_n^k} \end{bmatrix}$$

where $a_{ij} > 0, i, j = 0, 1, \ldots \quad x_0, y_0 \geq 0, n = 0, 1, \ldots,$, converges to the zero equilibrium if

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a_{11}}{1+x_n^1} & \frac{a_{12}}{1+x_n^2} & \cdots & \frac{a_{1k}}{1+x_n^k} \\ \frac{a_{21}}{1+x_n^1} & \frac{a_{22}}{1+x_n^2} & \cdots & \frac{a_{2k}}{1+x_n^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+x_n^1} & \frac{a_{k2}}{1+x_n^2} & \cdots & \frac{a_{kk}}{1+x_n^k} \end{bmatrix} \right\|_1$$

$$\begin{aligned} &= \max\left\{ \tfrac{a_{11}}{1+x_n^1} + \tfrac{a_{21}}{1+x_n^1} + \ldots + \tfrac{a_{k1}}{1+x_n^1}, \ldots, \tfrac{a_{1k}}{1+x_n^1} + \tfrac{a_{2k}}{1+x_n^1} + \ldots + \tfrac{a_{kk}}{1+x_n^1} \right\} \\ &\leq \max\{a_{11} + a_{21} + \ldots + a_{k1}, \ldots, a_{1k} + a_{2k} + \ldots + a_{kk}\} \\ &= \max_{1 \leq j \leq n} \{\sum_{i=1}^{k} a_{ij}\} < 1, \end{aligned}$$

which follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

Now, consider global attractivity of the positive equilibrium of system (30). The positive equilibrium satisfies the system

$$(A_n(\vec{\bar{x}}) - \mathbf{I})\vec{\bar{x}} = \vec{0},$$

where

$$A_n(\vec{\bar{x}}) = \begin{bmatrix} \frac{a_{11}}{1+\bar{x}^1} & \frac{a_{12}}{1+\bar{x}^2} & \cdots & \frac{a_{1k}}{1+\bar{x}^k} \\ \frac{a_{21}}{1+\bar{x}^1} & \frac{a_{22}}{1+\bar{x}^2} & \cdots & \frac{a_{2k}}{1+\bar{x}^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+\bar{x}^1} & \frac{a_{k2}}{1+\bar{x}^2} & \cdots & \frac{a_{kk}}{1+\bar{x}^k} \end{bmatrix}.$$

Linearizing system (30) about the positive equilibrium E gives the following system

$$\vec{u}_{n+1} = \begin{bmatrix} \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{12}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} \\ \frac{a_{21}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{22}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{2k}}{(1+\bar{x})(1+x_n^k)} \\ \vdots & & & \\ \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{k2}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \end{bmatrix} \vec{u}_n, \quad n = 0, 1, \ldots,$$

where $\vec{u}_n = \vec{x}_n - \vec{\bar{x}}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{12}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} \\ \frac{a_{21}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{22}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{2k}}{(1+\bar{x})(1+x_n^k)} \\ \vdots & & & \\ \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{k2}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \end{bmatrix} \right\|_1$$

$$\begin{aligned}
&= \max\left\{\tfrac{a_{11}}{(1+\bar{x})(1+x_n^1)} + \ldots + \tfrac{a_{k1}}{(1+\bar{x})(1+x_n^1)}, \ldots, \tfrac{a_{1k}}{(1+\bar{x})(1+x_n^k)} + \tfrac{a_{2k}}{(1+\bar{x})(1+x_n^k)} + \ldots + \tfrac{a_{kk}}{(1+\bar{x})(1+x_n^k)}\right\} \\
&\leq \max\{\tfrac{1}{1+\bar{x}}(a_{11} + a_{21} + \ldots + a_{k1}, \ldots, a_{1k} + a_{2k} + \ldots + a_{kk})\} \\
&= \tfrac{1}{1+\bar{x}} \quad \max_{1\leq j\leq n}\{\sum_{i=1}^{k} a_{ij}\} \\
&< 1
\end{aligned}$$

implies the global asymptotic stability of the positive equilibrium solution. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition for the global asymptotic stability of the positive equilibrium solution is

$$1 + \bar{x} > \sum_{i=1}^{k} a_{ij} \Longleftrightarrow \bar{x} > \sum_{i=1}^{k} a_{ij} - 1.$$

Example 5 The cooperative system

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \ldots, \tag{31}$$

where $a, b, c, d > 0$, $x_0, y_0 \geq 0$ was considered in [1]. The equilibrium solutions are the zero equilibrium $E_0(0, 0)$ and when $a < 1, d < 1$ the unique positive equilibrium solution $E_+(\bar{x}, \bar{y})$, is given as

$$\bar{x} = \frac{b}{1-a}\frac{\bar{y}}{1+\bar{y}}, \quad \bar{y} = \frac{bc - (1-d)(1-a)}{(1-d)(b+1-a)},$$

when

$$(1 - a)(1 - d) < bc. \tag{32}$$

The local stability of system (31) is described with the following result, see [1]

Claim 1 *Consider system* (31).

1.) *The positive equilibrium $E_+(\bar{x}, \bar{y})$ of system* (31) *is locally asymptotically stable when* (32) *holds.*

2.) *The zero equilibrium $E_0(0, 0)$ of system* (31) *is locally asymptotically stable if $bc < (1 - a)(1 - d)$; it is a saddle point if $bc > (1 - a)(1 - d)$; it is a nonhyperbolic equilibrium if $bc = (1 - a)(1 - d)$.*

The global dynamics of system (31) is described with the following result, see [1]:

Theorem 7 *Consider system* (31).

1.) *If $a \geq 1$ then $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$ if $d \geq 1$ and $\lim_{n\to\infty} y_n = \frac{c}{1-d}$, if $d < 1$.*

2.) *If $d \geq 1$ then $\lim_{n\to\infty} y_n = \infty$ and $\lim_{n\to\infty} x_n = \infty$ if $a \geq 1$ and $\lim_{n\to\infty} x_n = \frac{b}{1-a}$, if $a < 1$.*

3.) *The positive equilibrium $E_+(\bar{x}, \bar{y})$ of system* (31) *is globally asymptotically stable when* (32) *holds.*

4.) *The zero equilibrium $E_+(\bar{x}, \bar{y})$ of system* (31) *is globally asymptotically stable when $a < 1, d < 1$ and*

$$bc \leq (1 - a)(1 - d) \tag{33}$$

holds.

Theorem 2 and Corollary 1 implies that any of two conditions $\max\{a+c, b+d\} < 1$ or $\max\{a+b, c+d\} < 1$ provides the global asymptotic stability of the zero equilibrium. Both of these conditions imply (33) which is clearly the necessary and sufficient condition for the global asymptotic stability of the zero equilibrium..

Linearizing system (31) about the positive equilibrium $E(\bar{x},\bar{y})$ gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} a & \frac{b}{(1+\bar{y})(1+y_n)} \\ \frac{c}{(1+\bar{x})(1+x_n)} & d \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \quad n = 0,1,\ldots,$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 or L_2 norm, we obtain that the condition

$$\max\left\{a + \frac{c}{1+\bar{x}}, \frac{b}{1+\bar{y}} + d\right\} < 1 \quad \text{or} \quad \max\left\{a + \frac{b}{1+\bar{y}}, \frac{c}{1+\bar{x}} + d\right\} < 1 \tag{34}$$

implies that the positive equilibrium $E(\bar{x},\bar{y})$ is globally asymptotically stable. Condition (34) implies condition (32) which is clearly the necessary and sufficient condition for the global asymptotic stability of the positive equilibrium.

Example 6 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{an}{1+n^2} & \frac{cn}{1+n^3} \\ \frac{bn}{1+n^2} & \frac{dn}{1+n^3} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \frac{An}{1+n} & \frac{Cn}{1+n^2} \\ \frac{Bn}{1+n} & \frac{Dn}{1+n^2} \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, n = 0,1,\ldots,$$

where $a,b,c,d,A,B,C,D > 0, x_{-1}, y_{-1}, x_0, y_0 \geq 0, n = 0,1,\ldots,$ converges to the zero equilibrium if $\max\{\frac{a+b}{2}, \frac{2(c+d)}{32^{1/3}}\} + \max\{A+B, \frac{C+D}{2}\} < 1$ is satisfied. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have

$$\|g_0\| = \left\|\begin{bmatrix} \frac{an}{1+n^2} & \frac{cn}{1+n^3} \\ \frac{bn}{1+n^2} & \frac{dn}{1+n^3} \end{bmatrix}\right\| = \max\left\{\frac{(a+b)n}{1+n^2}, \frac{(c+d)n}{1+n^3}\right\} \leq \max\left\{\frac{a+b}{2}, \frac{2(c+d)}{32^{1/3}}\right\}$$

and

$$\|g_1\| = \left\|\begin{bmatrix} \frac{An}{1+n} & \frac{Cn}{1+n^2} \\ \frac{Bn}{1+n} & \frac{Dn}{1+n^2} \end{bmatrix}\right\| = \max\left\{\frac{(A+B)n}{1+n}, \frac{(C+D)n}{1+n^2}\right\} \leq \max\left\{A+B, \frac{C+D}{2}\right\}$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

Example 7 The vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{ax_n}{1+x_n} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \frac{a}{1+x_n} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, n = 0,1,\ldots \tag{35}$$

is equivalent to the system

$$\begin{aligned} x_{n+1} &= \tfrac{ax_n}{1+x_n} x_n + \tfrac{a}{1+x_n} x_{n-1} \\ y_{n+1} &= \tfrac{ax_n}{1+x_n} y_n + \tfrac{a}{1+x_n} y_{n-1}, \quad n = 0,1,\ldots, \end{aligned}$$

where $a > 0$. Since $g_0 + g_1 = a$ for all $n = 0,1,\ldots$ we have the following result which proof follows from Theorems 2, 3, 5 and Corollary 1.

Proposition 1 *The following trichotomy holds for equation* (35)*:*

(a) *if $a < 1$ then the zero equilibrium of* (35) *is globally asymptotically stable.*

(b) *if $a = 1$ then every nonnegative constant vector $\vec{L}$ is an equilibrium of* (35) *and every solution of* (35) *converges to some constant vector.*

(a) *if $a > 1$ then every set of positive (resp. negative) initial conditions generates the solution which component-wise tends to ∞ (resp. $-\infty$).*

Proposition 1 can be extended to the case of corresponding vector equation in $\mathbb{R}^p$.

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The Naimark-Sacker bifurcation and symptotic approximation of the invariant curve of a certain difference equation

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Abstract

We compute the direction of the Naimark-Sacker bifurcation for the difference equation $x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2}$ where p is a positive number and the initial conditions x_{-1} and x_0 are positive numbers. Furthermore, we give the asymptotic approximation of the invariant curve.

Keywords: difference equation, Naimark-Sacker bifurcation, normal form. invariant curve, stability.
AMS 2010 Mathematics Subject Classification: 39A10, 39A20, 65L20

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1 Introduction and Preliminaries

In this paper we consider the difference equation

$$x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{1}$$

where the parameter a is positive number and the initial conditions x_{-1} and x_0 are positive numbers. Clearly equation (1) has the unique equilibrium point $\bar{x} = p + 1$. Linear fractional version of equation (1)

$$x_{n+1} = p + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \tag{2}$$

was considered in [3], where we proved that the unique equilibrium $\bar{x} = p+1$ of equation (2) is globally asymptotically stable. Introduction of quadratic terms into equation (2) changes local stability analysis and consequently the global dynamics as well. In particular, quadratic terms introduces the possibility of Naimark-Sacker bifurcation and the existence of locally stable periodic solution, see [6] for several similar examples.

The linearized equation of equation (2) at the equilibrium point $\bar{x} = p + 1$ is

$$z_{n+1} = \frac{2}{p+1} z_n - \frac{2}{p+1}, \quad n = 0, 1, \ldots,$$

with the characteristic equation

$$\lambda^2 - \frac{2}{p+1}\lambda + \frac{2}{p+1} = 0,$$

and the characteristic roots

$$\lambda_{\pm} = \frac{1 \pm i\sqrt{2p+1}}{p+1}.$$

Since

$$|\lambda_{\pm}| = \sqrt{\frac{2}{p+1}}$$

it is clear that that the equilibrium point $\bar{x} = p + 1$ is asymptotically stable if $p > 1$, non-hyperbolic if $p = 1$ and unstable if $p < 1$. In all cases the eigenvalues are complex conjugate numbers which indicates the presence of the Naimark-Sacker bifurcation at $p = 1$. We will prove that indeed the equilibrium point $\bar{x} = p + 1$ is globally asymptotically stable if $p > \sqrt{2}$ and that the Naimark-Sacker bifurcation takes the place at $p = 1$. Our tool in proving global asymptotic stability of equation (2) is the result in [3, 5]. We conjecture that the equilibrium point $\bar{x} = p + 1$ is globally asymptotically stable if $a > 1$. Furthermore, we give some numeric values of parameter a with corresponding periodic solutions. Our bifurcation diagram indicates a complicated behavior and possible chaos for the values $p < 1$.

Now, for the sake of completness we give the basic facts about the Naimark-Sacker bifurcation.

The Hopf bifurcation is well known phenomenon for a system of ordinary differential equations in two or more dimension, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point crosses the imaginary axis, so

that the fixed point changes its behavior from stable to unstable and a limit cycle appears. In the discrete setting, the Naimark-Sacker bifurcation is the discrete analogue of the Hopf bifurcation.

The Naimark-Sacker bifurcation occurs for a discrete system depending on a parameter, λ say, with a fixed point whose Jacobian has a pair of complex conjugate $\mu(\lambda)$, $\bar{\mu}(\lambda)$ which cross the unit transversally at $\lambda = \lambda_0$.

The following result is referred as the Neimark-Sacker bifurcation Theorem [1, 4, 7, 8, 11].

Theorem 1 (Naimark-Sacker bifurcation) *Let*

$$\mathbf{F} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to \mathbf{F}(\lambda, \mathbf{x})$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, \mathbf{0}) = 0$ *for λ near some fixed λ_0;*

(ii) $DF(\lambda, \mathbf{0})$ *has two non-real eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with $|\mu(\lambda_0)| = 1$;*

(iii) $\frac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) < 0$ *at $\lambda = \lambda_0$ (transversality condition);*

(iv) $\mu^k(\lambda_0) \neq 1$ *for $k = 1, 2, 3, 4$. (nonresonance condition).*

Then there is a smooth λ-dependent change of coordinate bringing F into the form

$$F(\lambda, \mathbf{x}) = \mathcal{F}(\lambda, \mathbf{x}) + O(\| \mathbf{x} \|^5)$$

and there are smooth function $a(\lambda)$, $b(\lambda)$, and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} |\mu(\lambda)|r + a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{3}$$

If $a(\lambda_0) < 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda > \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda < \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.
If $a(\lambda_0) > 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then α-limit set of x_0 is the origin if $\lambda < \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $\mathbf{F}(\lambda_0, \mathbf{x})$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda_0) = \alpha(\lambda_0) + i\beta(\lambda_0)$ and $\bar{\mu}(\lambda_0) = \alpha(\lambda_0) - i\beta(\lambda_0)$ satisfying $\alpha(\lambda_0)^2 + \beta(\lambda_0)^2 = 1$ and $\beta(\lambda_0) \neq 0$. Assume that

$$\mathbf{F}(\lambda_0, \mathbf{x}) = \mathbf{A}(\lambda_0)\mathbf{x} + \mathbf{G}(\lambda_0, \mathbf{x}) \tag{4}$$

where $\mathbf{A}$ is Jacobian matrix of $\mathbf{F}$ evaluated at fixed point $(0, 0)$, and

$$\mathbf{G}(\lambda_0, \mathbf{x}) := \begin{pmatrix} g_1(\lambda_0, x_1, x_2) \\ g_2(\lambda_0, x_1, x_2) \end{pmatrix}.$$

Here we donate $\mu(\lambda_0) = \mu$, $\mathbf{A}(\lambda_0) = \mathbf{A}$ and $\mathbf{G}(\lambda_0, x) = \mathbf{G}(\mathbf{x})$. We let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of A associated with μ satisfying

$$\mathbf{A}\mathbf{q} = \mu\mathbf{q}, \quad \mathbf{p}A = \mu\mathbf{p}, \quad \mathbf{p}\mathbf{q} = 1$$

and $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$. Assume that

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) + O(|z|^3)$$

and

$$\begin{array}{l} \mathbf{K}_{20} = (\mu^2 I - A)^{-1}\mathbf{g}_{20} \\ \mathbf{K}_{11} = (I - A)^{-1}\mathbf{g}_{11} \quad . \\ \mathbf{K}_{02} = (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} \end{array} \tag{5}$$

Let

$$\begin{aligned} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}(\mathbf{K}_{20}\xi^2 + 2\mathbf{K}_{11}\xi\bar{\xi} + \mathbf{K}_{02}\bar{\xi}^2)\right) \\ = \frac{1}{2}(\mathbf{g}_{20}\xi^2 + 2\mathbf{g}_{11}\xi\bar{\xi} + \mathbf{g}_{02}\bar{\xi}^2) \\ + \frac{1}{6}(\mathbf{g}_{30}\xi^3 + 3\mathbf{g}_{21}\xi^2\bar{\xi} + 3\mathbf{g}_{12}\xi\bar{\xi}^2 + \mathbf{g}_{03}\bar{\xi}^3) + O(|\xi|^4), \end{aligned} \tag{6}$$

then

$$a(\lambda_0) = \frac{1}{2}Re(\mathbf{p}\mathbf{g}_{21}\bar{\mu}).$$

Corollary 1 ([9]) *Assume $a(\lambda_0) \neq 0$ and $\lambda = \lambda_0 + \eta$ where η is a sufficient small parameter. If $\bar{\mathbf{x}}$ is fixed point of F then invariant curve $\Gamma(\lambda)$ from Theorem 1 can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \bar{\mathbf{x}} + 2\rho_0 Re\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(Re\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right),$$

where

$$d = \frac{d}{d\eta}|\mu(\lambda)|\Big|_{\lambda=\lambda_0}, \quad \rho_0 = \sqrt{-\frac{d}{a}\eta}, \quad \theta \in \mathbb{R}.$$

Here "Re" represents the real parts of those complex numbers.

The second section of the paper gives global asymptotic stability result for the values of parameter $p > \sqrt{2}$ and the third section gives the reduction to the normal form and computation of the coefficients of the Naimark-Sacker bifurcation and the asymptotic approximation of the invariant curve. Our computational method is based on the computational algorithm developed in [9] rather than more often used computational algorithm in [10]. The advantage of the computational algorithm of [9] lies in the fact that this algorithm computes also the approximate equation of the invariant curve in Naimark-Sacker theorem, which is not provided by Wan's algorithm. Here we give numeric and visual eveidence that the approximate equation of the invariant curve is accurate. See Figure 4.

2 Global Asymptotic Stability

We use the method of embedding [2]. By substituting

$$x_n = p + \left(\frac{x_{n-1}}{x_{n-2}}\right)^2$$

in equation (1) we get:

$$x_{n+1} = p + \left(\frac{p}{x_{n-1}} + \frac{x_{n-1}}{x_{n-2}^2}\right)^2.$$

Now by substituting for x_{n-1} in the term $\frac{x_{n-1}}{x_{n-2}^2}$ of the last equation we we obtain

$$x_{n+1} = p + \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2. \tag{7}$$

From equation (7) we observe that $p<x_n<p+(1+\frac{1}{p}+\frac{1}{p^2})^2$ for $n \geq 4$.

Also from (1) and (7) we have:

$$\begin{cases} x_{n+1} - p = \left(\frac{x_n}{x_{n-1}}\right)^2 \\ x_{n+1} - p = \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2 \end{cases}.$$

Consequently

$$\left(\frac{x_n}{x_{n-1}}\right)^2 = \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2,$$

which implies:

$$x_{n+1} = p + \frac{px_n}{x_{n-1}^2} + \frac{x_n}{x_{n-2}^2}. \tag{8}$$

Replacing x_n in (8) by $p + \left(\frac{x_{n-1}}{x_{n-2}}\right)^2$ we obtain the equation

$$x_{n+1} = p + \frac{a^2}{x_{n-1}^2} + \frac{p + x_n}{x_{n-2}^2}. \tag{9}$$

Observe now that every solution of equation (1) is also a solution of equation (9), with initial values x_{-2}, x_{-1} and $x_0 = p + (\frac{x_0}{x_{-1}})^2$.

Observe also that it is of the form $x_{n+1} = f(x_n, x_{n-1}, x_{n-2})$ where :

$$f(u,v,w) = p + \frac{p^2}{v^2} + \frac{p+u}{w^2}$$

.

Theorem 2 *If $p > \sqrt{2}$ then the equilibrium of equation (1) is globally asymptotically stable.*

Proof. First we show that every interval I of the form $[p, \mathcal{U}]$ where $\mathcal{U} \geq \frac{p(p^2+p+1)}{(p^2-1)}$ with $p > 1$ is invariant for the function f.

Let $\mathcal{U} > p$ then $I = [p, \mathcal{U}]$ is invariant if and only if for all $u, v, w \in I, f(u,v,w) \in I$ that is:

$$p \leq p + \frac{p^2}{v^2} + \frac{p+u}{w^2} \leq \mathcal{U}.$$

As $p \leq u, v, w \leq \mathcal{U}$ we have that: $p \leq f(u, v, w) \leq p + 1 + \frac{1}{p} + \frac{\mathcal{U}}{p^2}$. We also know that if $\mathcal{U}$ satisfies: $p + 1 + \frac{1}{p} + \frac{\mathcal{U}}{p^2} \leq \mathcal{U}$ then we have

$$f(u, v, w) \leq \mathcal{U}.$$

It follows that given $p > 1$ such $\mathcal{U}$ exists and therefore I is invariant for f where $\mathcal{U} \geq \frac{p(p^2+p+1)}{(p^2-1)}$. In the following we may assume $p > 1$ and $\mathcal{U} = \frac{p(p^2+p+1)}{(p^2-1)}$, so I is invariant by f.

Next, we prove that I is an attracting interval, that is every solution of equation (8) must enter the interval I. Observe that given the initial values x_{-2}, x_{-1} and x_0 for equation (8), we have $x_n > p$ for $n \geq 1$.

Now if $x_3 \leq \mathcal{U}$ then $x_n \in [p, \mathcal{U}]$ for all $n \geq 3$. Otherwise, from equation (4) given that x_{n-2} , $x_{n-3} > p$ we have

$$x_n < p + 1 + \frac{1}{p} + \frac{x_{n-1}}{p^2},$$

that is if we set $A = p + 1 + \frac{1}{p}$

$$x_n < A + \frac{x_{n-1}}{p^2}.$$

Thus by induction we can conclude that

$$x_n < A \frac{1 - (\frac{1}{p^2})^{n-3}}{1 - \frac{1}{p^2}} + \frac{x_3}{(p^2)^{n-3}}. \tag{10}$$

It is straightforward to check that when $x_3 > \mathcal{U}$ the right hand side of (10) is a decreasing sequence that converges to $A\ (\frac{1}{1-\frac{1}{p^2}})$. This limit is in fact $\mathcal{U} = \frac{p(p^2+p+1)}{(p^2-1)}$. It follows that there must exist $k > 3$ such that: $a < x_k < \mathcal{U}$ Otherwise x_n must converge to $\mathcal{U}$ which is impossible.

Thus we have $x_{k-1}, x_{k-2} > p$ and $x_k \leq \mathcal{U}$, hence $x_{k+1} \in [a, \mathcal{U}]$ it follows by induction that $x_n \in [p, \mathcal{U}]$ for $n \geq k$.

Consequently every solution of equation (8) must enter the interval $[p, \mathcal{U}]$.

Now that we have an invariant and attracting interval we check the conditions of Theorem A.0.5 [3]:

$$\begin{cases} f(M, m, m) = M \\ f(m, M, M) = m \end{cases} \Leftrightarrow \begin{cases} M = p + \frac{p^2+p+M}{m^2} \\ m = p + \frac{p^2+p+m}{M^2} \end{cases}.$$

From the second equation we get

$$M^2 = \frac{p^2 + p + m}{m - p}. \tag{11}$$

On the other hand the system is equivalent to:

$$\begin{cases} (M - p)m^2 = p^2 + p + M \\ (m - p)M^2 = p^2 + p + m \end{cases} \Leftrightarrow \begin{cases} Mm^2 = pm^2 + p^2 + p + M \\ mM^2 = pM^2 + p^2 + p + m \end{cases}$$

By subtracting the second equation from the first we obtain:

$$Mm(m-M) = p(m-M)(m+M) - (m-M)$$

and given that $m \neq M$ we have:

$$Mm = p(m+M) - 1$$

which implies:

$$M = \frac{pm-1}{m-p}. \tag{12}$$

Equations (11) and (12) yield

$$\frac{(pm-1)^2}{(m-p)^2} = \frac{p^2+p+m}{m-p},$$

which implies:

$$(pm-1)^2 = (p^2+p+m)(m-p).$$

This leads to the following quadratic equation:

$$m^2(p^2-1) - m(p^2+2p) + p^2(p+1) + 1 = 0,$$

which discriminant is

$$\Delta = (p^2+2p)^2 - 4(p^2-1)(p^2(p+1)+1)$$

and

$$\Delta = -4p^5 - 3p^4 + 8p^3 + 4p^2 + 4 = (\sqrt{2}-p)(4p^4 + (3+4\sqrt{2})p^3 + 3\sqrt{2}p^2 + 2p + 2\sqrt{2}).$$

It is clear that when $a > \sqrt{2}$ there is no real solutions. and when $p = \sqrt{2}$ there is one unique solution $m = p+1 = M$. Consequently if $a \geq \sqrt{2}$ the conditions of Theorem A.0.5 [3] or Theorem 1 [5] are fully satisfied and therefore every solution must converge to the unique equilibrium $(p+1)$ □

Conjecture 1 *The equilibrium point* $\bar{x} = p+1$ *of equation* (2) *is globally asymptotically stable if* $p > 1$.

Remark 1 It could have been easier to prove the fact if we restrict the set of solutions of equation (4) to the ones satisfied by equation (1) as the solutions must oscillate about the equilibrium $(p+1)$ that is there exist k such that: $p < x_k < p+1 < \mathcal{U}$.

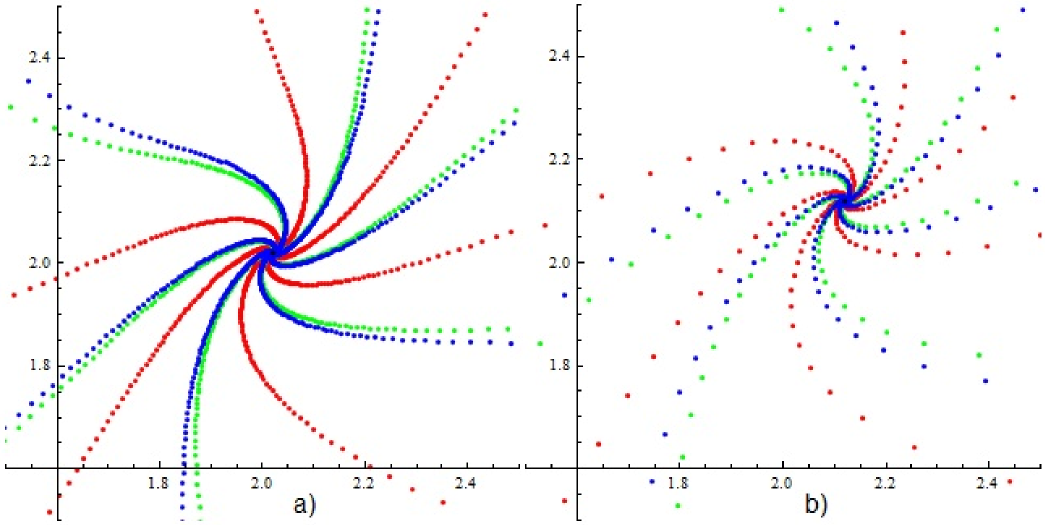

Figure 1: a) Phase diagrams when $n = 10,000$ and a) $p = 1.02$ b) $p = 1.12$

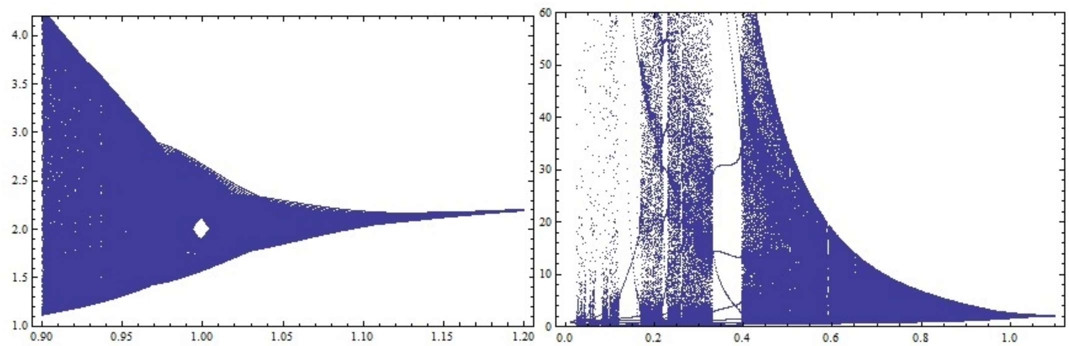

Figure 2: Bifurcation diagrams in $(p - x)$ plane.

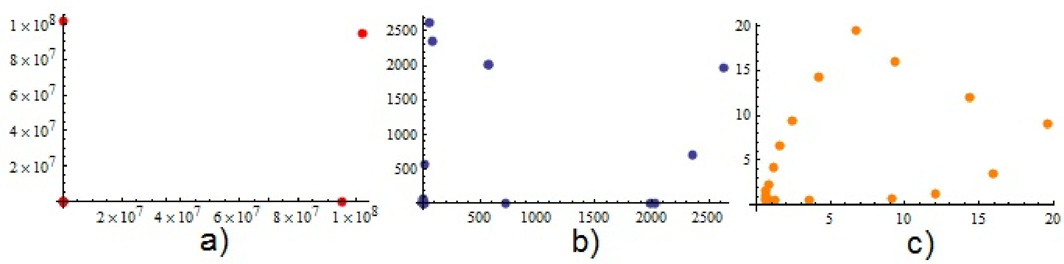

Figure 3: Periodic orbit for a) $p = 0.01$ b) $p = 0.15$ c) $p = 0.5901$ (See Table 2).

3 Reduction to the normal form

If we make a change of variable $y_n = x_n - \bar{x}$, then the transformed equation is given by

$$y_{n+1} = \frac{(p + y_n + 1)^2}{(p + y_{n-1} + 1)^2} - 1, \quad n = 0, 1, \ldots. \tag{13}$$

a	Period of the sol.	Solution
0.01	8	$\{0.877631, 0.01, 0.0101298, 1.03613, 10462.3, 1.01959 \times 10^8, 9.49713 \times 10^7, 0.877631\}$
0.15	20	$\{574.846, 2023.71, 12.5435, 0.150038, 0.150143, 1.1514, 58.9583, 2622.2, 1978.22, 0.719138, 0.15, 0.193507, 1.81422, 88.0493, 2355.59, 715.88, 0.242359, 0.15, 0.533058, 12.7789\}$
0.5901	19	$\{0.804816, 0.597988, 1.14217, 4.23826, 14.3595, 12.0691, 1.29653, 0.60164, 0.805431, 2.38228, 9.33854, 15.9565, 3.50965, 0.638479, 0.623195, 1.5428, 6.71883, 19.5558, 9.06166\}$

Table 1: Periodic solutions for some values of p.

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots$$

and write Eq.(1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{(p+v_n+1)^2}{(p+u_n+1)^2} - 1. \end{aligned} \tag{14}$$

Let F be the corresponding map defined by:

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{(p+v+1)^2}{(p+u+1)^2} - 1 \end{pmatrix}. \tag{15}$$

Then $\mathbf{F}$ has the unique fixed point $(0,0)$ and the Jacobian matrix of $\mathbf{F}$ at $(0,0)$ is given by

$$Jac_{\mathbf{F}}(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{2}{p+1} & \frac{2}{p+1} \end{pmatrix}$$

It is easy to see that

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{p+1} & \frac{2}{p+1} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix}, \tag{16}$$

where

$$\mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{(p+v+1)^2}{(p+u+1)^2} + \frac{2u}{p+1} - \frac{2v}{p+1} - 1 \end{pmatrix}.$$

The eigenvalues of $Jac_{\mathbf{F}}(0,0)$ are $\mu(p)$ and $\overline{\mu(p)}$ where

$$\mu(p) = \frac{1 + i\sqrt{2p+1}}{p+1}, \quad |\mu(p)| = \sqrt{\frac{2}{p+1}}.$$

One can prove that for $p = p_0 = 1$ we obtain $\mu(p_0)| = 1$ and

$$\mu(p_0) = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \mu^2(p_0) = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \mu^3(p_0) = -1, \quad \mu^4(p_0) = -\frac{1}{2} - \frac{i\sqrt{3}}{2},$$

from which follows that $\mu^k(p_0) \neq 1$ for $k = 1, 2, 3, 4$. Furthermore, we get

$$\frac{d}{dp}|\mu(p)| = -\frac{1}{\sqrt{2}}\left(\frac{1}{p+1}\right)^{3/2}, \quad \left.\frac{d|\mu(p)|}{dp}\right|_{p=p_0} = -\frac{1}{4} < 0.$$

The eigenvectors of corresponding to $\mu(p)$ and $\overline{\mu(p)}$ are $\mathbf{q}(p)$ and $\overline{\mathbf{q}(p)}$, where

$$\mathbf{q}(p) = \left(\frac{1 - i\sqrt{2p+1}}{p+1}, 1\right)^T.$$

Substituting $p = p_0 = 1$ into (16) we get

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{17}$$

where

$$\mathbf{A} = Jac_{\mathbf{F}}(0,0)|_{p=1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ \frac{(v+2)^2}{(u+2)^2} + u - v - 1 \end{pmatrix}.$$

Hence, for $p = p_0$ system (14) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \mathbf{G}\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{18}$$

Define the basis of $\mathbb{R}^2$ by $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$, where $\mathbf{q} = \mathbf{q}(p_0)$, then we can represent (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}z + \bar{\mathbf{q}}\bar{z}) = \begin{pmatrix} \frac{1}{2}\left(1 + i\sqrt{3}\right)\bar{z} + \frac{1}{2}\left(1 - i\sqrt{3}\right)z \\ \bar{z} + z \end{pmatrix}.$$

By using this, we have

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{(\bar{z}+z+2)^2}{\left(\frac{1}{2}\left(1+i\sqrt{3}\right)\bar{z}+\frac{1}{2}\left(1-i\sqrt{3}\right)z+2\right)^2} + \frac{1}{2}\left(-1 + i\sqrt{3}\right)\bar{z} - \frac{1}{2}\left(1 + i\sqrt{3}\right)z - 1 \end{pmatrix} \tag{19}$$

Thus we obtain that

$$\begin{aligned}
\mathbf{g}_{20} &= \left.\frac{\partial^2}{\partial z^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\right|_{z=0} = \begin{pmatrix} 0 \\ \frac{1}{4}i\left(\sqrt{3} + 5i\right) \end{pmatrix} \\
\mathbf{g}_{11} &= \left.\frac{\partial^2}{\partial z \partial \bar{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\right|_{z=0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\mathbf{g}_{02} &= \left.\frac{\partial^2}{\partial \bar{z}^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\right|_{z=0} = \begin{pmatrix} 0 \\ -\frac{1}{4}i\left(\sqrt{3} - 5i\right) \end{pmatrix},
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\mathbf{K}_{20} &= (\mu^2 I - A)^{-1}\mathbf{g}_{20} = \begin{pmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{4} \\ \frac{5}{8} - \frac{i\sqrt{3}}{8} \end{pmatrix} \\
\mathbf{K}_{11} &= (I - A)^{-1}\mathbf{g}_{11} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{K}_{02} &= (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} = \overline{\mathbf{K}_{20}}
\end{aligned} \tag{21}$$

By using $\mathbf{K}_{20}$, $\mathbf{K}_{11}$ and $\mathbf{K}_{02}$ we have that

$$\mathbf{g}_{21} = \frac{\partial^3}{\partial z^2 \partial \bar{z}} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}\mathbf{K}_{20} z^2 + \mathbf{K}_{11} z\bar{z} + \frac{1}{2}\mathbf{K}_{02}\bar{z}^2\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ -\frac{i\sqrt{3}}{8} \end{pmatrix}. \tag{22}$$

It is easy to see that $\mathbf{pA} = \mu\mathbf{p}$ and $\mathbf{pq} = 1$ where

$$\mathbf{p} = \left(\frac{i}{\sqrt{3}}, \frac{1}{6}\left(3 - i\sqrt{3}\right)\right)$$

and

$$a(p_0) = \frac{1}{2} Re(\mathbf{p}\mathbf{g}_{21}\bar{\mu}) = -\frac{1}{16} < 0.$$

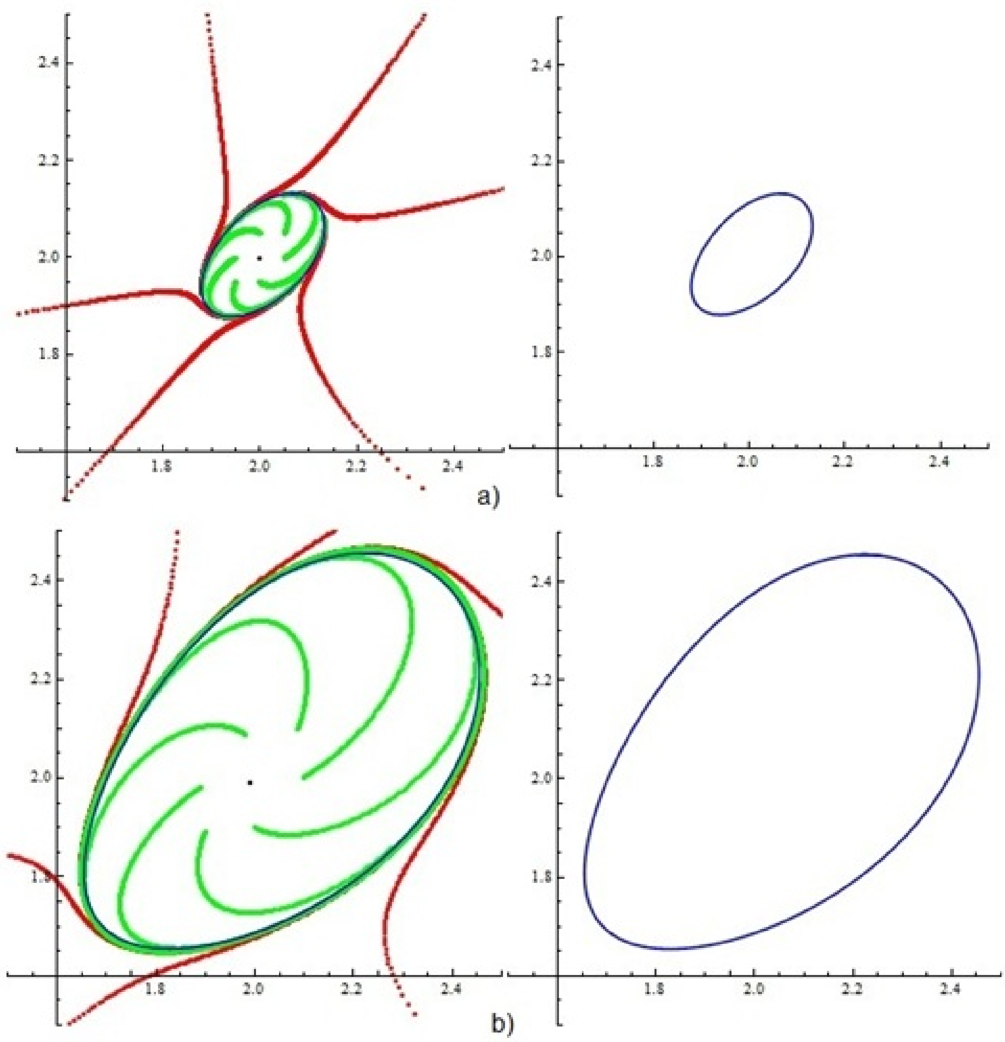

Figure 4: Trajectories and invariant curve for a) $p = 0.999$ b) $p = 0.99$.

Thus we prove the following result:

Theorem 3 *Let $\bar{x} = p + 1$. Then there is a neighborhood U of the equilibrium point $\bar{x}$ and a $\rho > 0$ such that for $|p - 1| < \rho$ and $x_0, x_{-1} \in U$, then ω-limit set of solution of Eq(1), with initial condition x_0, x_{-1} is equilibrium point $\bar{x}$ if $p > 1$ and belongs to a closed invariant C^1 curve $\Gamma(p)$ encircling the equilibrium point $\bar{x}$ if $p < 1$. Furthermore, $\Gamma(1) = 0$ and invariant curve $\Gamma(p)$ can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} p + 1 + 2\sqrt{1-p}\left(\sqrt{3}\sin\theta + \cos\theta\right) - (p-1)\left(\sqrt{3}\sin 2\theta - 2\cos 2\theta + 4\right) \\ p + 1 + 4\sqrt{1-p}\cos\theta - \frac{1}{2}(p-1)\left(\sqrt{3}\sin 2\theta + 5\cos 2\theta + 8\right) \end{pmatrix}$$

Proof. The proof follows from above discussion and Theorem 1 and Corollary 1. □

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Triple reverse order law for Moore-Penrose inverse of operator product *

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Abstract

In this paper, we study the reverse order law for the Moore-Penrose inverse of an operator product $T_1T_2T_3$. In particular, using the matrix form of a bounded linear operator we derive some necessary and sufficient conditions for the reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$. Moreover, some finite dimensional results are extended to infinite dimensional settings.

Keywords: Moore-Penrose inverse; Reverse order law; Bounded linear operator; Operator product; Hilbert space.

AMS(MOS) Subject Classifications: 47A05; 15A09; 15A24.

1 Introduction

Throughout this paper, "an operator" means "a bounded linear operator over Hilbert space". Let $\mathbb{H}$, $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ denote arbitrary Hilbert spaces. We use $L(\mathbb{H}, \mathbb{K})$ to denote the set of all bounded linear operators from $\mathbb{H}$ to $\mathbb{K}$. Especially, $L(\mathbb{H})$=$L(\mathbb{H}, \mathbb{H})$. For an operator $T \in L(\mathbb{H}, \mathbb{K})$, the symbols $R(T)$, $N(T)$ and T^* denote the range, the null-space and the adjoint of T, respectively. I denotes the unit operator over Hilbert space and O is the zero operator over Hilbert space. An operator $T \in L(\mathbb{H})$ is a Hermitian operator if and only if $T^* = T$. An operator $T \in L(\mathbb{H})$ is an invertible operator if and only if there is a operator $U \in L(\mathbb{H})$, such that $TU = UT = I$. If such operator U exists, we denotes it by T^{-1}.

Recall that an operator $X \in L(\mathbb{K}, \mathbb{H})$ is called the Moore-Penrose inverse of $T \in L(\mathbb{H}, \mathbb{K})$, if X satisfies the following four operator equations [16],

$$(1)\ TXT = T,\quad (2)\ XTX = X,\quad (3)\ (TX)^* = TX,\quad (4)\ (XT)^* = XT.$$

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If such operator X exists then it is unique and is denoted by $T^\dagger$. It is well known that the Moore-Penrose inverse of T exists if and only if $R(T)$ is closed [5, 8].

For a subset $\{i, j, \cdots, k\}$ of the set $\{1, 2, 3, 4\}$, the set of operators satisfying the equations (i), (j), $\cdots$, (k) from among equations (1)-(4) is denoted by $T\{i, j, \cdots, k\}$. An operator in $T\{i, j, \cdots, k\}$ is called an $\{i, j, \cdots, k\}$-inverse of T and is denoted by $T^{(i,j,\cdots,k)}$. For example, an operator X of the set $T\{1\}$ is called a $\{1\}$-inverse or a g-inverse of T and denoted by $X = T^{(1)}$. One usually denotes any $\{1, 3\}$-inverse of the set $T\{1, 3\}$ as $T^{(1,3)}$ which is also called a least squares g-inverse of T. Any $\{1, 4\}$-inverse of the set $T\{1, 4\}$ is denoted by $T^{(1,4)}$ which is also called a minimum norm g-inverse of T. The unique $\{1, 2, 3, 4\}$-inverse of T is the Moore-Penrose inverse of T. We refer the reader to [1, 14] for basic results on the generalized inverses of bounded linear operators.

If s is a semigroup with the unit 1 and if $a_i \in s$, $i = 1, 2, 3$, are invertible, then the equality $(a_1a_2a_3)^{-1} = a_3^{-1}a_2^{-1}a_1^{-1}$ is called the reverse order law for the ordinary inverse. Let T_i, $i = 1, 2, 3$, be three operators over Hilbert space such that the product $T_1T_2T_3$ is meaningful. If each of the three operators is invertible, then the product $T_1T_2T_3$ is invertible too, and the ordinary inverse of $T_1T_2T_3$ satis es the reverse order law $(T_1T_2T_3)^{-1} = T_3^{-1}T_2^{-1}T_1^{-1}$. However, this so-called reverse order law is not necessarily true for other kind generalized inverses. An interesting problem is, for given $\{i, j, \cdots, k\}$-inverses and operators T_i, $i = 1, 2, 3$, with $T_1T_2T_3$ is meaningful, when

$$(T_1T_2T_3)\{i, j, \cdots, k\} = T_3\{i, j, \cdots, k\}T_2\{i, j, \cdots, k\}T_1\{i, j, \cdots, k\}?$$

The reverse order laws for generalized inverses of operator product yield a class of interesting problems that are fundamental in the theory of generalized inverses of operator, see [1, 10, 21]. Theory and computations of the reverse order laws for generalized inverses of operator product are important subjects in many branches of applied science, such as nonlinear control theory, operator theory, operator algebra, global analysis and approximation theory, see [1, 6, 20, 21]. Suppose T_i, $i = 1, 2, 3$, and are bounded linear operators over Hilbert space. The least squares technique (LS):

$$\min_Y \|(T_1T_2T_3)Y \qquad \|_2,$$

is used in many practical scienti c problems. Any solution Y of the above LS problem can be expressed as $Y = (T_1T_2T_3)^{(1,3)}$. If the LS problem is consistent, then the minimum norm solution Y has the form $Y = (T_1T_2T_3)^{(1,4)}$. The unique minimal norm least square solution Y of the LS problem is $Y = (T_1T_2T_3)^\dagger$. One such problem concerned with the above LS problem is, under what conditions, $(T_1T_2T_3)^{(i,j,\cdots,k)} = T_3^{(i,j,\cdots,k)}T_2^{(i,j,\cdots,k)}T_1^{(i,j,\cdots,k)}$?

Since the middle 1960s, the reverse order law for generalized inverses have attracted considerable attention, and a signi cant number of paper treat the su cient or equivalent conditions such that the reverse order law holds in some sense. It is a classical result of Greville [10], that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$, in this case when A and B are complex matrices. This result is extended to bounded linear operators on Hilbert space, by Bouldin [2] and Izumino [12]. In [13] the reverse order law for the Moore-Penrose

inverse is proved in rings with involutions. In [4] D.S.Cvetkovic-IIic studied this reverse order law in C^*-algebra. Then, in [7], the reverse order law for the Moore-Penrose inverse is obtained as a consequence of some set equalities. The reader can nd some interesting and related results in [7, 15, 17, 18, 19, 22].

In 1986, R.E.Hartwig [11] rst discussed the reverse order law for Moore-Penrose inverse of three matrices product. In the paper [9] D.S. Djordjevic et al., extended the results of [11] to the bounded linear operators on Hilbert space, using some algebraic method. In this paper, we revisit this reverse order law by applying the technique of matrix form of bounded linear operators [3]. Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$ such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. Then using the technique of matrix form of a bounded linear operator [3] and the solving operator equations, we will revisit the following reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$. Some new simpler equivalent conditions for this reverse order law are obtained.

We rst mention the following results, which will be used in this paper.

Lemma 1.1. [3, 7, 8] *Let $T \in L(\mathbb{H}, \mathbb{K})$ have a closed range. Let H_1 and H_2 be closed and mutually orthogonal subspace of $\mathbb{H}$, such that $H_1 \bigoplus H_2 = \mathbb{H}$. Let K_1 and K_2 be closed and mutually orthogonal subspace of $\mathbb{K}$, such that $\mathbb{K} = K_1 \bigoplus K_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of subspaces $\mathbb{H} = H_1 \bigoplus H_2 = R(T^*) \bigoplus N(T)$ and $\mathbb{K} = K_1 \bigoplus K_2 = R(T) \bigoplus N(T^*)$:*

(1) $T = \begin{pmatrix} T_{11} & T_{12} \\ O & O \end{pmatrix} : \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} T_{11}^* E^{-1} & O \\ T_{12}^* E^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$,
where $E = T_{11}T_{11}^* + T_{12}T_{12}^*$ *is invertible on* $R(T)$;

(2) $T = \begin{pmatrix} T_{11} & O \\ T_{21} & O \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} F^{-1}T_{11}^* & F^{-1}T_{12}^* \\ O & O \end{pmatrix} : \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}$,
where $F = T_{11}^*T_{11} + T_{21}^*T_{21}$ *is invertible on* $R(T^*)$;

(3) $T = \begin{pmatrix} T_{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} T_{11}^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}$,
where T_{11} *is invertible.*

Lemma 1.2. [1] *Let $T \in L(\mathbb{H}, \mathbb{K})$ and $N \in L(\mathbb{K}, \mathbb{H})$ have closed ranges. Then,*

(1) $TT^\dagger N = N \Leftrightarrow R(N) \subseteq R(T)$;

(2) $NT^\dagger T = N \Leftrightarrow R(N^*) \subseteq R(T^*)$.

2 The triple reverse order law for Moore-Penrose inverse of operator product

Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. In this section, we will give necessary and su cient conditions for the triple reverse

order law of the Moore-Penrose inverse of the operator product $T_1T_2T_3$. First of all let us de ne

$$E = T_1^{\dagger}T_1, \quad F = T_3T_3^{\dagger}, \quad P = ET_2F, \quad Q = FT_2^{\dagger}E, \quad M = T_1T_2T_3, \quad G = T_3^{\dagger}T_2^{\dagger}T_1^{\dagger}. \tag{2.1}$$

In terms of these, we get the following results.

Theorem 2.1. *Let $T_1 \in L(\mathbb{J},\mathbb{K})$, $T_2 \in L(\mathbb{I},\mathbb{J})$ and $T_3 \in L(\mathbb{H},\mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. Then the following statements are equivalent:*

(1) $(T_1T_2T_3)^{\dagger} = T_3^{\dagger}T_2^{\dagger}T_1^{\dagger}$;

(2) $Q \in P\{1,2\}$, *and* $T_1^*T_1PQ$, $QPT_3T_3^*$ *are two Hermitian operators*;

(3) $MGM = G$, *and* $GMG = G$, *and* $(MG)^* = MG$, *and* $(GM)^* = GM$.

Proof. (1)$\Leftrightarrow$ (3): Obvious.

Next, we will prove (2)$\Leftrightarrow$ (3). From Lemma 1.1, we know that the operators T_1, T_2, T_3, $T_1T_2T_3$ and $T_3^{\dagger}T_2^{\dagger}T_1^{\dagger}$ have the following matrix form with respect to the orthogonal sum of subspaces:

$$T_1 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix}, \tag{2.2}$$

$$T_1^{\dagger} = \begin{pmatrix} (T_1^{11})^*D^{-1} & O \\ (T_1^{12})^*D^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.3}$$

where $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ is invertible on $R(T_1)$.

$$T_2 = \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.4}$$

$$T_2^{\dagger} = \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \tag{2.5}$$

where T_2^{11} is invertible.

$$T_3 = \begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \tag{2.6}$$

$$T_3^{\dagger} = \begin{pmatrix} S^{-1}(T_3^{11})^* & S^{-1}(T_3^{21})^* \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}, \tag{2.7}$$

where $S = (T_3^{11})^*T_3^{11} + (T_3^{21})^*T_3^{21}$ is invertible on $R(T_3^*)$.

Let $M = T_1T_2T_3$ and $G = T_3^{\dagger}T_2^{\dagger}T_1^{\dagger}$, then form (2.2)$\sim$(2.7), we have

$$M = T_1T_2T_3 = \begin{pmatrix} T_1^{11}T_2^{11}T_3^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \tag{2.8}$$

and

$$G = T_3^\dagger T_2^\dagger T_1^\dagger = \begin{pmatrix} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}. \tag{2.9}$$

According to the formulas (2.1)$\sim$(2.7), we have

$$Q = \begin{pmatrix} T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} & T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{12} \\ T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} & T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{12} \end{pmatrix} \tag{2.10}$$

and

$$P = \begin{pmatrix} (T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{11})^* & (T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{21})^* \\ (T_1^{12})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{11})^* & (T_1^{12})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{21})^* \end{pmatrix}. \tag{2.11}$$

From (2.2), (2.6), (2.10) and (2.11), we get

$$T_1^* T_1 PQ = \begin{pmatrix} {}_{11} & {}_{12} \\ {}_{21} & {}_{22} \end{pmatrix}, \text{where} \tag{2.12}$$

$$\begin{aligned}
{}_{11} &= (T_1^{11})^* T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11}, \\
{}_{12} &= (T_1^{11})^* T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{12}, \\
{}_{21} &= (T_1^{12})^* T_1^{11} T_2^{11} T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11}, \\
{}_{22} &= (T_1^{12})^* T_1^{11} T_2^{11} T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{12},
\end{aligned}$$

and

$$QPT_3T_3^* = \begin{pmatrix} {}_{11} & {}_{12} \\ {}_{21} & {}_{22} \end{pmatrix}, \text{where} \tag{2.13}$$

$$\begin{aligned}
{}_{11} &= T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} (T_3^{11})^*, \\
{}_{12} &= T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} (T_3^{21})^*, \\
{}_{21} &= T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} (T_3^{11})^*, \\
{}_{22} &= T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} (T_3^{21})^*.
\end{aligned}$$

Combining (2.8) with (2.9), we know that $G = M^\dagger$ (i.e. $T_3^\dagger T_2^\dagger T_1^\dagger = (T_1T_2T_3)^\dagger$), if and only if

$$(I)\ MGM = M, \quad (II)\ GMG = G, \quad (III)\ (MG)^* = MG, \quad (IV)\ (GM)^* = GM. \tag{2.14}$$

From the formulas (2.10)$\sim$(2.13), we know that the statement (2) of Theorem 2.1 can be rewrited as

$$(a)\ PQP = P,\ (b)\ QPQ = Q,\ (c)\ (T_1^*T_1PQ)^* = T_1^*T_1PQ,\ (d)\ (QPT_3T_3^*)^* = QPT_3T_3^*. \tag{2.15}$$

In the rest of this section, we will prove (2.14) is equivalent to (2.15). That is the conditions (2) in Theorem 2.1 is equal to the conditions (3) in Theorem 2.1.

(I)$\Leftrightarrow$(a): From (2.8) and (2.9), we have

$$\begin{aligned} MGM &= (T_1T_2T_3)(T_3^{\dagger}T_2^{\dagger}T_1^{\dagger})(T_1T_2T_3) \\ &= \begin{pmatrix} T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11} & O \\ O & O \end{pmatrix}. \end{aligned} \tag{2.16}$$

Then from (2.8) and (2.16), we know that the inclusion $MGM = M$ is equivalent to

$$T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11} = T_1^{11}T_2^{11}T_3^{11}. \tag{2.17}$$

By the formulas (2.10) and (2.11), we have

$$PQP = \begin{pmatrix} {}_{11} & {}_{12} \\ {}_{21} & {}_{22} \end{pmatrix}, \text{where} \tag{2.18}$$

$$\begin{aligned} {}_{11} &= (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \\ {}_{12} &= (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*, \\ {}_{21} &= (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \\ {}_{22} &= (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*. \end{aligned}$$

From (2.11) and (2.18), we know that the inclusion $PQP = P$ is equivalent to

$$\begin{aligned} & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^* \\ = \ & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \end{aligned} \tag{2.19}$$

$$\begin{aligned} & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^* \\ = \ & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*, \end{aligned} \tag{2.20}$$

$$\begin{aligned} & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^* \\ = \ & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \end{aligned} \tag{2.21}$$

$$\begin{aligned} & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^* \\ = \ & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*. \end{aligned} \tag{2.22}$$

If the equation (2.17) holds, we have the equations (2.19)$\sim$(2.22) hold too. That is (I)$\Rightarrow$(a).

On the other hand, if the equations (2.19)$\sim$(2.22) hold, we have

$$\begin{aligned} & T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11} \\ = \ & T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11}, \end{aligned} \tag{2.23}$$

$$
\begin{aligned}
&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}\\
&=T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}, \quad (2.24)
\end{aligned}
$$

$$
\begin{aligned}
&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11}\\
&=T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11}, \quad (2.25)
\end{aligned}
$$

$$
\begin{aligned}
&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}\\
&=T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}. \quad (2.26)
\end{aligned}
$$

Combining (2.23), (2.24) with the de nition of S in (2.7), we have

$$
\begin{aligned}
&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}\\
=\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}. \quad (2.27)
\end{aligned}
$$

Combining (2.25), (2.26) with the de nition of D in (2.3), we have

$$
\begin{aligned}
&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}\\
=\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}. \quad (2.28)
\end{aligned}
$$

From the results in (2.27) and (2.28), we have

$$
T_1^{11}T_2^{11}T_3^{11}=T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}. \quad (2.29)
$$

That is (a)⇒(I).

(II)⇔(b): With the same method of the proof of (I)⇔(a), the condition $GMG = G$ is easily seen to be equivalent to $QPQ = Q$.

(III)⇔(c): From (2.8) and (2.9), we have

$$
MG=(T_1T_2T_3)(T_3^\dagger T_2^\dagger T_1^\dagger)=\begin{pmatrix} T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1} & O\\ O & O\end{pmatrix}. \quad (2.30)
$$

Since S and D are Hermitian operators, then the inclusion $(MG)^* = MG$ is equivalent to

$$
T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}=D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*. \quad (2.31)
$$

By the formulas (2.12), we have that the inclusion $(T_1^*T_1PQ)^* = T_1^*T_1PQ$ is equivalent to

$$
\begin{aligned}
&(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}\\
=\;&(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11}, \quad (2.32)
\end{aligned}
$$

$$
\begin{aligned}
&(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}\\
=\;&(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}, \quad (2.33)
\end{aligned}
$$

$$\begin{aligned}&(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}\\ =\;&(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11},\end{aligned}\tag{2.34}$$

$$\begin{aligned}&(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}\\ =\;&(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}.\end{aligned}\tag{2.35}$$

If the equation (2.31) holds, we have the equations (2.32)$\sim$(2.35) hold too. That is (III)$\Rightarrow$(c).

On the other hand, if the equations (2.32)$\sim$(2.35) hold, we have

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}(T_1^{11})^*\\ =\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11}(T_1^{11})^*,\end{aligned}\tag{2.36}$$

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}(T_1^{12})^*\\ =\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}(T_1^{12})^*,\end{aligned}\tag{2.37}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}(T_1^{11})^*\\ =\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11}(T_1^{11})^*,\end{aligned}\tag{2.38}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}(T_1^{12})^*\\ =\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}(T_1^{12})^*.\end{aligned}\tag{2.39}$$

Combining (2.36), (2.37) with the de nition of $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ in (2.3), we have

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*\\ =\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D.\end{aligned}\tag{2.40}$$

Combining (2.38), (2.39) with the de nition of D, we have

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*\\ =\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D.\end{aligned}\tag{2.41}$$

Finally, from (3.40), (3.41) and the de nition of D, we have

$$DT_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* = T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D.\tag{2.42}$$

Since $D = (T_1^{11})(T_1^{11})^* + (T_1^{12})(T_1^{12})^*$ is invertible on $R(T_1)$, then (2.42) can be rewrited as

$$T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1} = D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*.\tag{2.43}$$

That is (c)$\Rightarrow$(III).

(IV)$\Leftrightarrow$(d): With the same method of the proof of (III)$\Leftrightarrow$(c), we can get the result that the condition $(GM)^* = GM$ is equivalent to $(QPT_3T_3^*)^* = QPT_3T_3^*$ without the proof.

From the above proof, the formulas (2.14) is equivalent to (2.15). We then complete the proof of the theorem. ■

Be the same as (2.1), $Q = FT_2^\dagger E$ and $P = ET_2F$, next we will derive some other equivalent conditions for the triple reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$.

Theorem 2.2. *Let* $T_1 \in L(\mathbb{J}, \mathbb{K})$*,* $T_2 \in L(\mathbb{I}, \mathbb{J})$ *and* $T_3 \in L(\mathbb{H}, \mathbb{I})$*, such that* T_1*,* T_2*,* T_3 *and* $T_1T_2T_3$ *have closed ranges. Then the following statements are equivalent:*

(1) $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$;

(2) $Q \in P\{1, 2\}$ and $T_1^*T_1PQ$, $QPT_3T_3^*$ are two Hermitian operators;

(3) $Q \in P\{1\}$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$;

(4) $(PQ)^2 = PQ$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$.

Proof. (1)⇔(2): By the results in Theorem 2.1, we know that (1)⇔(2).

(2)⇒(3): According to the de nitions of the generalized inverses of operators, we have

$$Q \in P\{1, 2\} \Rightarrow Q \in P\{1\}. \tag{2.44}$$

By the de nitions of the ranges of operators and the formula (2.44), we have

$$R(T_1^*T_1P) = R(T_1^*T_1PQP) \subseteq R(T_1^*T_1PQ) \subseteq R(T_1^*T_1P). \tag{2.45}$$

That is

$$R(T_1^*T_1P) = R(T_1^*T_1PQ). \tag{2.46}$$

If $T_1^*T_1PQ$ is a Hermitian operator, then

$$R(T_1^*T_1P) = R(T_1^*T_1PQ) = R(Q^*P^*T_1^*T_1) = R(Q^*P^*T_1^\dagger T_1). \tag{2.47}$$

Since $Q^*P^*T_1^\dagger T_1 = Q^*P^*$, then from (2.44) and (2.47), we have

$$R(T_1^*T_1P) = R(Q^*P^*T_1^\dagger T_1) = R(Q^*P^*) = R(Q^*). \tag{2.48}$$

Similarly, if $QPT_3T_3^*$ is a Hermitian operator, we have

$$R(T_3^*T_3P^*) = R(T_3^*T_3P^*Q^*) = R(QPT_3T_3^*) = R(QP) = R(Q). \tag{2.49}$$

Combining (2.44), (2.48) with (2.49), we have the result (2)⇒(3).

(3)⇒(4): Obvious.

(4)⇒(2): Firstly, we will prove that if the statement (4) in Theorem 2.2 is true, then $PQP = P$. Since $P = PT_3T_3^\dagger$ and $R(T_3T_3^*P^*) = R(Q)$, then we have

$$R(P) = R(PT_3) = R(PT_3T_3^*P^*) = R(PQ). \tag{2.50}$$

Combining (2.50) with $(PQ)^2 = PQ$, we have

$$PQP = P \text{ and } (QP)^2 = QP. \tag{2.51}$$

Secondly, we will prove that if the statement (4) in Theorem 2.2 is true, then $QPQ = Q$. From the statement (4) in Theorem 2.2 and the de nitions of Q and P, we have

$$\begin{aligned} R(Q^*) &= R(T_1^*T_1P) = R(T_1^*T_1PP^*T_1^*T_1) = R(T_1^*T_1PP^*T_1^\dagger T_1) \\ &= R(T_1^*T_1PP^*) = R(Q^*P^*). \end{aligned} \tag{2.52}$$

Combining (2.52) with $(Q^*P^*)^2 = Q^*P^*$, we have

$$Q^*P^*Q^* = Q^* \text{ i.e. } QPQ = Q. \tag{2.53}$$

Thirdly, we will prove that if the statement (4) in Theorem 2.2 is true, then $T_1^*T_1PQ$ is a Hermitian operator. Since $R(T_1^*T_1P) = R(Q^*)$ and $R(Q^*P^*) = R(Q^*)$, then we have

$$Q^*P^*T_1^*T_1P = T_1^*T_1P. \tag{2.54}$$

From (2.54), we have

$$Q^*P^*T_1^*T_1PQ = T_1^*T_1PQ = (T_1^*T_1PQ)^*. \tag{2.55}$$

Fourthly, we will prove that if the statement (4) in Theorem 2.2 is true, then $QPT_3T_3^*$ is a Hermitian operator. Since $R(T_3T_3^*P^*) = R(Q)$ and $QPQ = Q$, then we have

$$R(QP) = R(Q) \text{ and } QPT_3T_3^*P^* = T_3T_3^*P^*. \tag{2.56}$$

From (2.56), we have

$$QPT_3T_3^*P^*Q^* = T_3T_3^*P^*Q^* = (QPT_3T_3^*)^* = QPT_3T_3^*. \tag{2.57}$$

Combining the formulas (2.51), (2.53), (2.55) with (2.57), we immediately obtain the result (4)⇒(2). We then complete the proof of the theorem. ■

Let us now see how some of the special cases come out of the conditions of Theorem 2.2.

Corollary 2.1. *Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. If $R(T_2) \subseteq R(T_1^*)$ and $R(T_2^*) \subseteq R(T_3)$, then*

$$(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger \Leftrightarrow R(T_1^*T_1T_2) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) \subseteq R(T_2^*).$$

Proof. According to the hypothesis $R(T_2) \subseteq R(T_1^*)$ and $R(T_2^*) \subseteq R(T_3)$ and the results in Lemma 1.2, we have

$$Q = FT_2^\dagger E = T_2^\dagger, \quad P = ET_2F = T_2. \tag{2.58}$$

$\Rightarrow$: If $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$, then from Theorem 2.1 and Theorem 2.2 , we have $(PQ)^2 = PQ$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$. So, we get

$$R(T_1^*T_1T_2) = R((T_2^\dagger)^*) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) = R(T_2^\dagger) \subseteq R(T_2^*). \tag{2.59}$$

$\Leftarrow$: From (2.58), we have $PQP = P$ and $QPQ = Q$. That is

$$Q \in P\{1, 2\}. \tag{2.60}$$

By (2.58), we also have

$$T_1^*T_1PQ = T_1^*T_1T_2T_2^\dagger \text{ and } QPT_3T_3^* = T_2^\dagger T_2T_3T_3^*. \tag{2.61}$$

Combining the hypothesis $R(T_1^*T_1T_2) \subseteq R(T_2)$ with results in Lemma 1.2, we have

$$T_2T_2^\dagger T_1T_1^*T_2T_2^\dagger = T_1T_1T_2^*T_2^\dagger = (T_1T_1T_2^*T_2^\dagger)^*. \tag{2.62}$$

Combining the hypothesis $R(T_3T_3^*T_2) \subseteq R(T_2^*)$ with results in Lemma 1.2, we have

$$T_2^\dagger T_2T_3T_3^*T_2^*(T_2^*)^\dagger = T_3T_3^*T_2^*(T_2^*)^\dagger = (T_3T_3^*T_2^*(T_2^*)^\dagger)^* = T_2^\dagger T_2T_3T_3^* = (T_2^\dagger T_2T_3T_3^*)^*. \tag{2.63}$$

According to the formulas (2.59), (2.60), (2.62), (2.63) and the statement (2) in Theorem 2.2, we immediately obtain the results of Corollary 2.1. ■

Corollary 2.2. *Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_2 and $T_1T_2T_3$ have closed ranges. If $T_1^\dagger T_1 = I$ and $T_3T_3^\dagger = I$ (i.e. T_1 and T_3 are invertible operators), then*

$$(T_1T_2T_3)^\dagger = T_3^{-1}T_2^\dagger T_1^{-1} \Leftrightarrow R(T_1^*T_1T_2) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) \subseteq R(T_2^*).$$

Corollary 2.3. *Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_1, T_2, T_3, $T_1T_2T_3$ and $T_1^\dagger T_1T_2T_3T_3^\dagger$ have closed ranges. If $T_1^\dagger T_1 = T_1$ and $T_3T_3^\dagger = T_3$, then*

$$(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger \Leftrightarrow T_3T_3^\dagger T_2^\dagger T_1^\dagger T_1 = (T_1^\dagger T_1T_2T_3T_3^\dagger)^\dagger.$$

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DIFFERENTIAL EQUATIONS ARISING FROM CERTAIN SHEFFER SEQUENCE

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ABSTRACT. In this paper, we study some differential equations arising from certain Sheffer sequence and investigate some identities for the Sheffer sequence of polynomials which is related to the theory of hyperbolic differential equations.

1. Introduction

A partial differential equation of the second-order

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0,$$

is called hyperbolic if the matrix is

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0, \text{ (see [6])}.$$

The wave equation is an example of a hyperbolic partial differential equation. A sequence $S_n(x)$ is called a Sheffer sequence if the generating function has the form

$$\sum_{k=0}^{\infty} S_k(x)\frac{t^k}{k!} = A(t)e^{xB(t)},$$

where

$$\begin{aligned} A(t) =& A_0 + A_1t + A_2t^2 + \cdots \\ B(t) =& B_1t + B_2t^2 + \cdots , \qquad\qquad \text{with} \quad A_0 \neq 0, \;\; B_0 \neq 0 \text{ (see [12])}. \end{aligned}$$

If $f(t)$ is a delta series and $g(t)$ is an invertible series, there exists a uniquen sequence $S_n(x)$ of Sheffer polynomials such that the orthogonality condition $< g(t)f(t)^k|S_n(x) >= \delta_{n,k}$ holds, where $\delta_{n,k}$ is the Kronecker delta (see [8-11]).

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In this paper, we consider the Sheffer sequence given by the pair $\left(\frac{1}{1+t}, 1-(1+t)^{-2}\right)$, namely

$$F(t,x)=\frac{1}{\sqrt{1-t}}e^{x\left(\frac{1}{\sqrt{1-t}}-1\right)}=\sum_{n=0}^{\infty}h_n(x)\frac{t^n}{n!}. \tag{1.1}$$

In [5], Erdélyi also considered a Sheffer sequence which is related to $h_n(x)$. Indeed, his sequence is given by $g_n(x)=\frac{1}{n!}h_n(x)$. Also, we note that

$$h_n(x)=xe^{-x}\left[\frac{d}{dx^2}\right]^n(x^{2n-1}e^x),\ \text{(see [5])}. \tag{1.2}$$

The polynomials $h_n(x)$ have applications to the theory of hyperbolic differential equations (see [1-4]). From (1.1), by replacing t by $1-e^{-2t}$, we can derive the following equation:

$$\begin{aligned} e^te^{x(e^t-1)} &= \sum_{n=0}^{\infty}(-1)^nh_n(x)\frac{1}{n!}(e^{-2t}-1)^n \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}(-1)^{n+m}h_n(x)2^mS_2(n,m)\right)\frac{t^m}{m!}, \end{aligned} \tag{1.3}$$

where $S_2(n,m)$ is the Stirling number of the second kind.
As is well known, the Bell polynomials are defined by the generating function

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!},\ \text{(see [7])}. \tag{1.4}$$

By (1.3), we get

$$\begin{aligned} e^te^{x(e^t-1)} &= \left(\sum_{l=0}^{\infty}\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!}\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\binom{m}{n}Bel_n(x)\right)\frac{t^m}{m!}. \end{aligned} \tag{1.5}$$

From (1.3) and (1.5), we have

$$\sum_{n=0}^{m}\binom{m}{n}Bel_n(x)=\sum_{n=0}^{m}(-1)^{n+m}h_n(x)2^mS_2(n,m),\ (m\geq 0). \tag{1.6}$$

In this paper, we study some differential equations arising from certain sheffer sequence and investigate some identities for the Sheffer sequence of polynomials which is related to the theory of hyperbolic differential equations.

2. Differential equations arising from certain Sheffer sequence

Let

$$F = F(t,x) = (1-t)^{-\frac{1}{2}} e^{x\left((1-t)^{-\frac{1}{2}}-1\right)} \tag{2.1}$$

Then, we have

$$\begin{aligned} F^{(1)} =& \frac{dF(t,x)}{dt} = (1-t)^{-\frac{1}{2}} e^{x\left((1-t)^{-\frac{1}{2}}-1\right)} \left(\tfrac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) \\ =& \left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) F, \end{aligned} \tag{2.2}$$

$$F^{(2)} = \frac{dF^{(1)}}{dt} = \left(\frac{3}{4}(1-t)^{-2} + \frac{5}{4}x(1-t)^{-\frac{5}{2}} + \frac{1}{4}x^2(1-t)^{-3}\right) F, \tag{2.3}$$

and

$$F^{(3)} = \left(\frac{15}{8}(1-t)^{-3} + \frac{33}{8}x(1-t)^{-\frac{7}{2}} + \frac{12}{8}x^2(1-t)^{-4} + \frac{1}{8}x^3(1-t)^{-\frac{9}{2}}\right) F.$$

Thus, we are let to put

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F, \tag{2.4}$$

where $N = 0, 1, 2, \cdots$.
Taking the derivative of (2.4) with respect to t, we have

$$\begin{aligned} F^{(N+1)} =& \frac{dF^{(N)}}{dt} = \left(\sum_{i=0}^{N}(N+\tfrac{1}{2}i) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i}\right) F \\ &+ \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F^{(1)} \\ =& \left(\sum_{i=0}^{N}(N+\tfrac{1}{2}i) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i}\right) F \\ &+ \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) \left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) F \end{aligned} \tag{2.5}$$

$$= \left(\sum_{i=0}^{N} \left(N+\tfrac{1}{2}i+\tfrac{1}{2}\right) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i} + \sum_{i=0}^{N} \frac{1}{2} a_i(N) x^{i+1} (1-t)^{-N-\frac{3}{2}-\frac{1}{2}i}\right) F$$

$$= \left(\sum_{i=0}^{N} \left(N+\tfrac{1}{2}i+\tfrac{1}{2}\right) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i} + \sum_{i=1}^{N+1} \frac{1}{2} a_{i-1}(N) x^{i} (1-t)^{-N-1-\frac{1}{2}i}\right) F.$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1) x^i (1-t)^{-N-1-\frac{1}{2}i} \right) F. \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain the following recurrence relations:

$$a_0(N+1) = (N+\tfrac{1}{2}) a_0(N), \ \ a_{N+1}(N+1) = \frac{1}{2} a_N(N), \tag{2.7}$$

and

$$a_i(N+1) = \frac{1}{2} a_{i-1}(N) + \left(N + \tfrac{1}{2}i + \tfrac{1}{2}\right) a_i(N), \ \ (1 \le i \le N). \tag{2.8}$$

In addition, we note that

$$F = F^{(0)} = a_0(0) F. \tag{2.9}$$

Thus, by (2.9), we easily get

$$a_0(0) = 1. \tag{2.10}$$

For $N = 1$ in (1.5) and (1.2), it is not difficult to show that

$$\begin{aligned} &\left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2} x (1-t)^{-3/2} \right) F = F^{(1)} \\ &= \left(a_0(1)(1-t)^{-1} + a_1(x) x (1-t)^{-3/2} \right) F. \end{aligned} \tag{2.11}$$

By comparing the coefficients on both sides of (2.11), we easily get

$$a_0(1) = \frac{1}{2}, \quad a_1(1) = \frac{1}{2}. \tag{2.12}$$

From (2.7), we can easily derive the following equations:

$$\begin{aligned} &a_{N+1}(N+1) = \frac{1}{2} a_N(N) = \left(\frac{1}{2}\right)^2 a_{N-1}(N-1) = \cdots = \left(\frac{1}{2}\right)^{N+1}, \\ &a_0(0) = \left(\frac{1}{2}\right)^{N+1}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} a_0(N+1) =& (N+\tfrac{1}{2}) a_0(N) = (N+\tfrac{1}{2})(N-\tfrac{1}{2}) a_0(N-1) = \cdots \\ =& (N+\tfrac{1}{2})(N-\tfrac{1}{2}) \cdots \tfrac{3}{2} \cdot \tfrac{1}{2} a_0(0) = (N+\tfrac{1}{2})_{N+1}, \end{aligned} \tag{2.14}$$

where

$$(x)_n = x(x-1)\cdots(x-n+1), \ \ (n \ge 1), \quad (x)_0 = 1.$$

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The matrix $(a_i(j))$ $(0 \le i, j \le N)$ is given by

$$\big(a_i(j)\big) = \begin{pmatrix} 1 & \frac{1}{2} & \left(\frac{3}{2}\right)_2 & \left(\frac{5}{2}\right)_3 & \cdots & \left(N-\frac{1}{2}\right)_N \\ 0 & \frac{1}{2} & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & \left(\frac{1}{2}\right)^2 & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \left(\frac{1}{2}\right)^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \left(\frac{1}{2}\right)^N \end{pmatrix}$$

For $i = 1, 2, 3$ in (2.8), we have

$$\begin{aligned} a_1(N+1) =& \frac{1}{2} a_0(N) + (N+1) a_1(N) \\ =& \frac{1}{2}\Big(a_0(N) + (N+1)a_0(N-1)\Big) + (N+1)N a_1(N-1) \\ =& \frac{1}{2}\Big(a_0(N) + (N+1)a_0(N-1) + (N+1)N a_0(N-2)\Big) \\ & + (N+1)N(N-1)a_1(N-2) \\ =& \cdots \\ =& \frac{1}{2}\sum_{k=0}^{N-1}(N+1)_k a_0(N-k) + (N+1)_N a_1(1) \\ =& \frac{1}{2}\sum_{k=0}^{N}(N+1)_k a_0(N-k), \end{aligned} \tag{2.15}$$

$$\begin{aligned} a_2(N+1) =& \frac{1}{2}\sum_{k=0}^{N-2}\left(N+\tfrac{3}{2}\right)_k a_1(N-k) + \left(N+\tfrac{3}{2}\right)_{N-1} a_2(2) \\ =& \frac{1}{2}\sum_{k=0}^{N-1}\left(N+\tfrac{3}{2}\right)_k a_1(N-k), \end{aligned}$$

and

$$\begin{aligned} a_3(N+1) =& \frac{1}{2}\sum_{k=0}^{N-3}(N+2)_k a_2(N-k) + (N+2)_{N-2} a_3(3) \\ =& \frac{1}{2}\sum_{k=0}^{N-2}(N+2)_k a_2(N-k). \end{aligned}$$

Continuing this process, we have

$$a_i(N+1) = \frac{1}{2}\sum_{k=0}^{N-i+1}\left(N+\tfrac{1}{2}i+\tfrac{1}{2}\right)_k a_{i-1}(N-k),\ (1 \le i \le N). \tag{2.16}$$

Now, we give explicit expressions for $a_i(N+1)$, $(1 \le i \le N)$. From (2.16), we note that

$$a_1(N+1) = \frac{1}{2}\sum_{k_1=0}^{N}(N+1)_{k_1} a_0(N-k_1) = \frac{1}{2}\sum_{k_1=0}^{N}(N+1)_{k_1}(N-k_1-\tfrac{1}{2})_{N-k_1}, \tag{2.17}$$

$$\begin{aligned} a_2(N+1) &= \frac{1}{2}\sum_{k_2=0}^{N-1}\left(N+\tfrac{3}{2}\right)_{k_2} a_1(N-k_2) \\ &= \left(\frac{1}{2}\right)^2 \sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}\left(N+\tfrac{3}{2}\right)_{k_2}(N-k_2)_{k_1}(N-k_2-k_1-\tfrac{3}{2})_{N-k_2-k_1-1}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} a_3(N+1) = \left(\frac{1}{2}\right)^3 &\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}(N+2)_{k_3}(N-k_3+\tfrac{1}{2})_{k_2} \\ &\times (N-k_3-k_2-1)_{k_1}(N-k_3-k_2-k_1-\tfrac{5}{2})_{N-k_3-k_2-k_1-2}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} a_4(N+1) = \left(\frac{1}{2}\right)^4 &\sum_{k_4=0}^{N-3}\sum_{k_3=0}^{N-3-k_4}\sum_{k_2=0}^{N-3-k_4-k_3}\sum_{k_1=0}^{N-3-k_4-k_3-k_2}(N+\tfrac{5}{2})_{k_4} \\ &\times (N-k_4+1)_{k_3}(N-k_4-k_3-\tfrac{1}{2})_{k_2}(N-k_4-k_3-k_2-2)_{k_1} \\ &\times (N-k_4-k_3-k_2-k_1-\tfrac{7}{2})_{N-k_4-k_3-k_2-k_1-3}. \end{aligned} \tag{2.20}$$

So, we can deduce that, for $1 \le i \le N$,

$$\begin{aligned} &a_i(N+1) \\ &= \left(\frac{1}{2}\right)^i \sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-i+1-k_i}\cdots\sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2}\prod_{l=1}^{i}\left(N+\tfrac{3}{2}l+\tfrac{1}{2}-i-\sum_{j=l+1}^{i}k_j\right)_{k_l} \\ &\times \left(N+\tfrac{1}{2}-i-\sum_{j=1}^{i}k_j\right)_{N+1-i-\sum_{j=1}^{i}k_j}. \end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 1. For $N = 0, 1, 2 \cdots$, the following family of differential equations

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F$$

have a solution

$$F = F(t,x) = (1-t)^{-1/2} e^{x((1-t)^{-1/2}-1)},$$

where

$$a_0(N) = \left(N - \tfrac{1}{2}\right)_N,$$

$$a_i(N) = \left(\frac{1}{2}\right)^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \prod_{l=1}^{i} \left(N + \tfrac{3}{2}l - \tfrac{1}{2} - i - \sum_{j=l+1}^{i} k_j\right)_{k_l}$$
$$\times \left(N - \tfrac{1}{2} - i - \sum_{j=1}^{i} k_j\right)_{N-i-\sum_{j=1}^{i} k_j}.$$

From (1.1), we note that

$$\begin{aligned}
\sum_{k=0}^{\infty} h_{k+N}(x) \frac{t^k}{k!} &= F^{(N)} = \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F \\
&= \sum_{i=0}^{N} a_i(N) x^i \sum_{l=0}^{\infty} \left(N + \tfrac{1}{2}i + l - 1\right)_l \frac{t^l}{l!} \sum_{m=0}^{\infty} h_m(x) \frac{t^m}{m!} \\
&= \sum_{i=0}^{N} a_i(N) x^i \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l h_{k-l}(x)\right) \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l a_i(N) x^i h_{k-l}(x)\right) \frac{t^k}{k!}.
\end{aligned} \tag{2.22}$$

Thus, by comparing the coefficients on both sides of (2.22), we obtain the following theorem.

Theorem 2. For $k, N = 0, 1, 2, \cdots$, we have

$$h_{k+N}(x) = \sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l a_i(N) x^i h_{k-l}(x) \tag{2.23}$$

Letting $k = 0$ in (2.23), we obtain the following corollary.

Corollary 3. For $N = 0, 1, 2, \cdots$, we have

$$h_N(x) = \sum_{i=0}^{N} a_i(N)x^i. \tag{2.24}$$

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Hyers-Ulam stability of the first order inhomogeneous matrix difference equation

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Abstract

We prove Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$ for all integers $i \in \mathbb{Z}$. Moreover, we show Hyers-Ulam stability of the nth order linear difference equation as a corollary.

1 Introduction

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ the set of all complex numbers, of all positive integers, of all nonnegative integers, and the set of all integers, respectively. Given a fixed positive integer n, let $(\mathbb{C}^n, \|\cdot\|_n)$ be a complex normed space, each of whose elements is a column vector, and let $\mathbb{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We choose a norm $\|\cdot\|_{n\times n}$ on $\mathbb{C}^{n\times n}$ which is compatible with $\|\cdot\|_n$, *i.e.*, both norms obey

$$\|\mathbf{AB}\|_{n\times n} \le \|\mathbf{A}\|_{n\times n}\|\mathbf{B}\|_{n\times n} \quad \text{and} \quad \|\mathbf{A}\vec{x}\|_n \le \|\mathbf{A}\|_{n\times n}\|\vec{x}\|_n \tag{1.1}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$.

A matrix difference equation is a difference equation with matrix coefficients in which the value of vector at one point depends on the values of preceding (succeeding) points.

In this paper, we prove Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation

$$\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \tag{1.2}$$

for all integers $i \in \mathbb{Z}$, where the transition matrices $\mathbf{A}(i)$ are nonsingular. More precisely, we prove that if a vector sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_n \le \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the first order matrix difference equation (1.2) such that the bound for $\|\vec{y}_i - \vec{x}_i\|_n$ depends on the sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and the transition

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matrices $\mathbf{A}(i)$ only. Moreover, we investigate Hyers-Ulam stability of the nth order linear inhomogeneous difference equation of the form

$$a(i+1) = p_1(i)a(i) + p_2(i)a(i-1) + \cdots + p_n(i)a(i-n+1) + r(i), \tag{1.3}$$

where $p_j, r : \mathbb{Z} \to \mathbb{C}$ are given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$. We refer the reader to [7, 8, 9, 12, 20] for the exact definition of Hyers-Ulam stability.

2 Preliminaries

In this section, we investigate the general solution to the first order linear inhomogeneous matrix difference equation (1.2) for all integers $i \in \mathbb{Z}$, where

$$\vec{x}_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix} \in \mathbb{C}^n \quad \text{and} \quad \mathbf{A}(i) = \begin{pmatrix} a_{11}(i) & a_{12}(i) & \cdots & a_{1n}(i) \\ a_{21}(i) & a_{22}(i) & \cdots & a_{2n}(i) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(i) & a_{n2}(i) & \cdots & a_{nn}(i) \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

Throughout this paper, we use the following abbreviation.

$$\Phi(n,m) := \begin{cases} \prod_{k=m}^{n-1} \mathbf{A}(k) = \mathbf{A}(n-1)\mathbf{A}(n-2)\cdots\mathbf{A}(m) & (\text{for } n > m), \\ \mathbf{I} & (\text{for } n = m), \end{cases} \tag{2.1}$$

where we set $\Phi(n,m) := \big(\Phi(m,n)\big)^{-1} = \mathbf{A}(n)^{-1}\mathbf{A}(n+1)^{-1}\cdots\mathbf{A}(m-1)^{-1}$ for $n < m$ and $\mathbf{I}$ denotes the identity matrix. Sometimes, we use $\Phi(n)$ and $\Phi^{-1}(m,n)$ instead of $\Phi(n,0)$ and $\big(\Phi(m,n)\big)^{-1}$, respectively.

In the following lemma, we introduce some properties of $\Phi(n,m)$ without proof.

Lemma 2.1 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular. It holds that*

(i) $\Phi(i+1,k) = \mathbf{A}(i)\Phi(i,k)$;

(ii) $\Phi^{-1}(i,k+1) = \mathbf{A}(k)\Phi^{-1}(i,k)$;

(iii) $\mathbf{A}(k-1)^{-1}\Phi^{-1}(i,k) = \Phi^{-1}(i,k-1)$

for all integers $i, k \in \mathbb{Z}$.

In the following lemma, we give the general solution to the first order linear inhomogeneous matrix difference equation (1.2).

Lemma 2.2 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular and the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given. A vector sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$*

is a solution to the first order linear inhomogeneous matrix difference equation (1.2) *if and only if the sequence* $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ *is given in the form of*

$$\vec{x}_i := \begin{cases} \Phi(i,0)\vec{x}_0 + \displaystyle\sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) & (\textit{for } i \ge 0), \\ \Phi^{-1}(0,i)\vec{x}_0 - \displaystyle\sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) & (\textit{for } i < 0), \end{cases} \tag{2.2}$$

where $\vec{x}_0 \in \mathbb{C}^n$ *is an arbitrarily given vector.*

Proof. First, we assume that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is given in the form of (2.2) and we prove that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is a solution to the first order linear inhomogeneous matrix difference equation (1.2).

If i is a nonnegative integer, then it follows from the first formula of (2.2) and Lemma 2.1 (i) that

$$\begin{aligned} \vec{x}_{i+1} &= \Phi(i+1,0)\vec{x}_0 + \sum_{k=0}^{i} \Phi(i+1,k+1)\vec{g}(k) \\ &= \mathbf{A}(i)\Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i} \mathbf{A}(i)\Phi(i,k+1)\vec{g}(k) \\ &= \mathbf{A}(i)\left(\Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k)\right) + \vec{g}(i) \\ &= \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \end{aligned}$$

for any integer $i \ge 0$.

If $i = -1$, then we use (2.2) to get

$$\vec{x}_{i+1} = \vec{x}_0$$

and

$$\vec{x}_i = \vec{x}_{-1} = \Phi^{-1}(0,-1)\vec{x}_0 - \Phi^{-1}(0,-1)\vec{g}(-1) = \mathbf{A}(-1)^{-1}\vec{x}_0 - \mathbf{A}(-1)^{-1}\vec{g}(-1).$$

Hence, we have

$$\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$$

for $i = -1$.

If i is an integer less than -1, then it follows from the second formula of (2.2) and Lemma

2.1 (ii) that

$$\begin{aligned}
\vec{x}_{i+1} &= \Phi^{-1}(0, i+1)\vec{x}_0 - \sum_{k=1}^{-i-1} \Phi^{-1}(i+1+k, i+1)\vec{g}(i+k) \\
&= \mathbf{A}(i)\Phi^{-1}(0, i)\vec{x}_0 - \sum_{k=1}^{-i-1} \mathbf{A}(i)\Phi^{-1}(i+k+1, i)\vec{g}(i+k) \\
&= \mathbf{A}(i)\Phi^{-1}(0, i)\vec{x}_0 - \sum_{j=2}^{-i} \mathbf{A}(i)\Phi^{-1}(i+j, i)\vec{g}(i+j-1) \\
&= \mathbf{A}(i)\Phi^{-1}(0, i)\vec{x}_0 - \sum_{k=1}^{-i} \mathbf{A}(i)\Phi^{-1}(i+k, i)\vec{g}(i+k-1) + \mathbf{A}(i)\Phi^{-1}(i+1, i)\vec{g}(i) \\
&= \mathbf{A}(i)\vec{x}_i + \vec{g}(i)
\end{aligned}$$

for all integers $i < -1$.

Now, we assume that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is a solution to the first order linear inhomogeneous matrix difference equation (1.2) and we prove that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ has the form of (2.2). We can easily show that the first formula of (2.2) holds for $i = 0$. We now assume that the first formula of (2.2) holds for some nonnegative integer i. Then, by using Lemma 2.1 (i), we obtain

$$\begin{aligned}
\vec{x}_{i+1} &= \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \\
&= \mathbf{A}(i)\left(\Phi(i, 0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i, k+1)\vec{g}(k)\right) + \vec{g}(i) \\
&= \Phi(i+1, 0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i+1, k+1)\vec{g}(k) + \vec{g}(i) \\
&= \Phi(i+1, 0)\vec{x}_0 + \sum_{k=0}^{i} \Phi(i+1, k+1)\vec{g}(k)
\end{aligned}$$

by replacing i with $i+1$ in the first formula of (2.2).

Finally, we assume that the sequence $\{\vec{x}_i\}$ is a solution to (1.2) and we will prove that $\vec{x}_i$ is expressed by the second formula of (2.2) for every negative integer i. If we set $i = -1$ in (1.2), then we get

$$\vec{x}_0 = \mathbf{A}(-1)\vec{x}_{-1} + \vec{g}(-1) \quad \text{or} \quad \vec{x}_{-1} = \mathbf{A}(-1)^{-1}\vec{x}_0 - \mathbf{A}(-1)^{-1}\vec{g}(-1),$$

which we obtain from the second formula of (2.2) by setting $i = -1$. We now assume that $\vec{x}_i$ is expressed as the second formula of (2.2) for some negative integer i. Then, it follows from (1.2), the second formula of (2.2), and Lemma 2.1 (iii) that

$$\vec{x}_i = \mathbf{A}(i-1)\vec{x}_{i-1} + \vec{g}(i-1)$$

or

$$\begin{aligned}
\vec{x}_{i-1} &= \mathbf{A}(i-1)^{-1}\vec{x}_i - \mathbf{A}(i-1)^{-1}\vec{g}(i-1) \\
&= \mathbf{A}(i-1)^{-1}\left(\Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i}\Phi^{-1}(i+k,i)\vec{g}(i+k-1)\right) - \mathbf{A}(i-1)^{-1}\vec{g}(i-1) \\
&= \Phi^{-1}(0,i-1)\vec{x}_0 - \sum_{k=0}^{-i}\Phi^{-1}(i+k,i-1)\vec{g}(i+k-1) \\
&= \Phi^{-1}(0,i-1)\vec{x}_0 - \sum_{k=1}^{-i+1}\Phi^{-1}(i+k-1,i-1)\vec{g}(i+k-2),
\end{aligned}$$

which is a consequence of the second formula of (2.2) provided we replace i with $i-1$. □

Remark 2.3 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular and the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given. If vector sequences $\{\vec{x}_{i,h}\}_{i\in\mathbb{Z}}$ and $\{\vec{x}_{i,p}\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ are defined by*

$$\vec{x}_{i,h} := \begin{cases} \Phi(i,0)\vec{x}_0 & (\textit{for } i \geq 0), \\ \Phi^{-1}(0,i)\vec{x}_0 & (\textit{for } i < 0) \end{cases}$$

resp.

$$\vec{x}_{i,p} := \begin{cases} \displaystyle\sum_{k=0}^{i-1}\Phi(i,k+1)\vec{g}(k) & (\textit{for } i \geq 0), \\ \displaystyle -\sum_{k=1}^{-i}\Phi^{-1}(i+k,i)\vec{g}(i+k-1) & (\textit{for } i < 0), \end{cases}$$

then then the sequence $\{\vec{x}_{i,h}\}_{i\in\mathbb{Z}}$ is a solution to the homogeneous difference equation $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i$ corresponding to (1.2) *and the sequence $\{\vec{x}_{i,p}\}_{i\in\mathbb{Z}}$ is a particular solution to the first order linear inhomogeneous matrix difference equation* (1.2)*.*

3 Hyers-Ulam stability of $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$

We now prove our main theorem concerning the Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation (1.2). Obviously, our theorem is a generalization and an improvement of [13, Theorem 2.1].

Theorem 3.1 *Given a fixed positive integer n, let $(\mathbb{C}^n, \|\cdot\|_n)$ and $(\mathbb{C}^{n\times n}, \|\cdot\|_{n\times n})$ be complex normed spaces, whose elements are column vectors resp. $(n\times n)$ complex matrices, with the property* (1.1)*. Assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular, the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given, and that $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ is a sequence of nonnegative real numbers. If a vector sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality*

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_n \leq \varepsilon_{i+1} \tag{3.1}$$

for all $i \in \mathbb{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the first order linear inhomogeneous matrix difference equation (1.2) *such that*

$$\|\vec{y}_i - \vec{x}_i\|_n \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n} + \|\Phi(i,0)\|_{n\times n}\|\vec{y}_0 - \vec{x}_0\|_n & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \big\|\Phi^{-1}(i+k,i)\big\|_{n\times n} + \big\|\Phi^{-1}(0,i)\big\|_{n\times n}\|\vec{y}_0 - \vec{x}_0\|_n & (\text{for } i < 0). \end{cases}$$

Proof. First, we assume that $i \geq 0$. In view of Lemma 2.2, the vector sequence $\{\vec{x}_i\}_{i=0,1,\ldots}$ defined by

$$\vec{x}_i = \Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) \tag{3.2}$$

satisfies the first order linear inhomogeneous matrix difference equation (1.2) for $i \geq 0$.

We now apply the mathematical induction to prove that

$$\vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) = \sum_{k=1}^{i} \Phi(i,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) \tag{3.3}$$

for all integers $i \geq 0$. It is obvious that the equality (3.3) holds for $i = 0$. We assume that the equality (3.3) holds for some integer $i \geq 0$. Then, it follows from Lemma 2.1 (i) and (3.3) that

$$\begin{aligned}
&\vec{y}_{i+1} - \Phi(i+1,0)\vec{y}_0 - \sum_{k=0}^{i} \Phi(i+1,k+1)\vec{g}(k) \\
&= \vec{y}_{i+1} - \mathbf{A}(i)\Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i} \mathbf{A}(i)\Phi(i,k+1)\vec{g}(k) \\
&= \vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i) + \mathbf{A}(i)\left(\vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k)\right) \\
&= \sum_{k=1}^{i} \mathbf{A}(i)\Phi(i,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) + \vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i) \\
&= \sum_{k=1}^{i+1} \Phi(i+1,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big),
\end{aligned}$$

which can be obtained from the equality (3.3) by replacing i with $i+1$. Thus, we conclude by induction that the equality (3.3) holds for all integers $i \geq 0$.

Hence, it follows from (3.1) and (3.3) that

$$\begin{aligned}
&\left\| \vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) \right\|_n \\
&\quad \leq \sum_{k=1}^{i} \|\Phi(i,k)\|_{n\times n} \big\| \vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1) \big\|_n \\
&\quad \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n}
\end{aligned} \tag{3.4}$$

for $i \geq 0$. In view of (3.2) and (3.4), we have

$$\|\vec{y}_i - \Phi(i,0)\vec{y}_0 + \Phi(i,0)\vec{x}_0 - \vec{x}_i\|_n \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n}$$

or

$$\|\vec{y}_i - \vec{x}_i\|_n \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n} + \|\Phi(i,0)\|_{n\times n} \|\vec{y}_0 - \vec{x}_0\|_n$$

for all integers $i \geq 0$.

Now, assume that $i < 0$. By Lemma 2.2, the sequence $\{\vec{x}_i\}_{i=-1,-2,\ldots}$ defined by

$$\vec{x}_i = \Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) \tag{3.5}$$

satisfies the first order linear inhomogeneous matrix difference equation (1.2) for $i < 0$. Using the mathematical induction, we prove that

$$\begin{aligned}
&\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) \\
&\quad = -\sum_{k=i+1}^{0} \Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big)
\end{aligned} \tag{3.6}$$

for all integers $i < 0$. It is obvious that the equality (3.6) holds for $i = -1$. We assume that the equality (3.6) holds for some integer $i < 0$. Then, it follows from Lemma 2.1 (*ii*), (*iii*), and (3.6) that

$$\begin{aligned}
&\vec{y}_{i-1} - \Phi^{-1}(0,i-1)\vec{y}_0 + \sum_{k=1}^{-i+1} \Phi^{-1}(i+k-1,i-1)\vec{g}(i+k-2) \\
&= \vec{y}_{i-1} - \mathbf{A}(i-1)^{-1}\Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i+1} \mathbf{A}(i-1)^{-1}\Phi^{-1}(i+k-1,i)\vec{g}(i+k-2) \\
&= \mathbf{A}(i-1)^{-1}\left(\mathbf{A}(i-1)\vec{y}_{i-1} - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i+1} \Phi^{-1}(i+k-1,i)\vec{g}(i+k-2) \right) \\
&= -\mathbf{A}(i-1)^{-1}\big(\vec{y}_i - \mathbf{A}(i-1)\vec{y}_{i-1} - \vec{g}(i-1)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\ \mathbf{A}(i-1)^{-1}\left(\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=2}^{-i+1}\Phi^{-1}(i+k-1,i)\vec{g}(i+k-2)\right)\\
=\ & -\mathbf{A}(i-1)^{-1}\big(\vec{y}_i - \mathbf{A}(i-1)\vec{y}_{i-1} - \vec{g}(i-1)\big)\\
&-\ \mathbf{A}(i-1)^{-1}\sum_{k=i+1}^{0}\Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big)\\
=\ & -\sum_{k=i}^{0}\mathbf{A}(i-1)^{-1}\Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big)\\
=\ & -\sum_{k=i}^{0}\Phi^{-1}(k,i-1)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big),
\end{aligned}
$$

which can be obtained from the equality (3.6) by replacing i with $i-1$. By induction, we conclude that the equality (3.6) holds for any integer $i<0$.

Therefore, by (3.1) and (3.6), we get

$$
\begin{aligned}
&\left\|\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i}\Phi^{-1}(i+k,i)\vec{g}(i+k-1)\right\|_n\\
&\leq \sum_{k=i+1}^{0}\|\Phi^{-1}(k,i)\|_{n\times n}\big\|\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big\|_n\\
&\leq \sum_{k=i+1}^{0}\varepsilon_k\|\Phi^{-1}(k,i)\|_{n\times n}
\end{aligned}
\tag{3.7}
$$

for any integer $i<0$. Taking (3.5) and (3.7) into account, we get

$$
\|\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \Phi^{-1}(0,i)\vec{x}_0 - \vec{x}_i\|_n \leq \sum_{k=i+1}^{0}\varepsilon_k\|\Phi^{-1}(k,i)\|_{n\times n}
$$

or

$$
\begin{aligned}
\|\vec{y}_i - \vec{x}_i\|_n &\leq \sum_{k=i+1}^{0}\varepsilon_k\big\|\Phi^{-1}(k,i)\big\|_{n\times n} + \big\|\Phi^{-1}(0,i)\big\|_{n\times n}\|\vec{y}_0 - \vec{x}_0\|_n\\
&= \sum_{k=1}^{-i}\varepsilon_{i+k}\big\|\Phi^{-1}(i+k,i)\big\|_{n\times n} + \big\|\Phi^{-1}(0,i)\big\|_{n\times n}\|\vec{y}_0 - \vec{x}_0\|_n
\end{aligned}
$$

for all integers $i<0$. □

4 Applications

In this section, let n be a fixed positive integer. We assume that the nth order linear inhomogeneous difference equation of the form (1.3) is given, where $p_j, r : \mathbb{Z} \to \mathbb{C}$ are given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$.

If we set

$$
\|\mathbf{A}\|_\infty = \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}| \quad \text{and} \quad \|\vec{x}\|_\infty = \max_{1\leq j\leq n}|x_j|
$$

for all $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$, then these norms satisfy the conditions in (1.1).

We now prove Hyers-Ulam stability of the nth order linear inhomogeneous difference equation (1.3).

Theorem 4.1 *Let n be a fixed positive integer and $p_1, \ldots, p_n, r : \mathbb{Z} \to \mathbb{C}$ be given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$. Assume that a sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ of nonnegative numbers is given. If a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality*

$$\big|a(i+1) - p_1(i)a(i) - p_2(i)a(i-1) - \cdots - p_n(i)a(i-n+1) - r(i)\big| \leq \varepsilon_{i+1} \tag{4.1}$$

for all $i \in \mathbb{Z}$, then there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the nth order linear inhomogeneous difference equation (1.3) such that

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_\infty + \|\Phi(i,0)\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \big\|\Phi^{-1}(i+k,i)\big\|_\infty + \big\|\Phi^{-1}(0,i)\big\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

where $\Phi(i,k)$ and $\Phi^{-1}(i,k)$ are defined in (2.1) and (4.2), and where $\vec{y}_0$ and $\vec{x}_0$ are defined in (4.7).

Proof. For any $k \in \{1, 2, \ldots, n-1\}$, we define the complex numbers $b_k(i)$ by

$$\begin{aligned} b_1(i) &= a(i-1), \\ b_2(i) &= b_1(i-1), \\ b_3(i) &= b_2(i-1), \\ &\vdots \\ b_{n-1}(i) &= b_{n-2}(i-1) \end{aligned}$$

for all $i \in \mathbb{Z}$. We further define

$$\mathbf{A}(i) := \begin{pmatrix} p_1(i) & p_2(i) & p_3(i) & \cdots & p_{n-1}(i) & p_n(i) \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{4.2}$$

$$\vec{y}_i := \begin{pmatrix} a(i) \\ b_1(i) \\ b_2(i) \\ \vdots \\ b_{n-1}(i) \end{pmatrix} \quad \text{and} \quad \vec{g}(i) := \begin{pmatrix} r(i) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.3}$$

for all $i \in \mathbb{Z}$, where $\mathbf{A}(i)$ is an $n \times n$ matrix and $\vec{y}_i, \vec{g}(i)$ are $n \times 1$ vectors.

Using these notations and considering (4.1), the sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ satisfies the inequality

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_\infty \leq \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$. Moreover, by the assumption that $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$, we can see that every $\mathbf{A}(i)$ is nonsingular.

According to Theorem 3.1, there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the first order linear inhomogeneous matrix difference equation (1.2) such that

$$\|\vec{y}_i - \vec{x}_i\|_\infty \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_\infty + \|\Phi(i,0)\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \left\|\Phi^{-1}(i+k,i)\right\|_\infty + \left\|\Phi^{-1}(0,i)\right\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0). \end{cases} \tag{4.4}$$

If we set

$$\vec{x}_i := \begin{pmatrix} x_1(i) \\ x_2(i) \\ \vdots \\ x_n(i) \end{pmatrix}, \tag{4.5}$$

then it follows from (1.2) that

$$\begin{aligned} x_1(i+1) &= p_1(i)x_1(i) + p_2(i)x_2(i) + p_3(i)x_3(i) + \cdots + p_n(i)x_n(i) + r(i), \\ x_2(i+1) &= x_1(i), \\ x_3(i+1) &= x_2(i), \\ &\vdots \\ x_n(i+1) &= x_{n-1}(i) \end{aligned} \tag{4.6}$$

for all $i \in \mathbb{Z}$. Moreover, if we define $c(i) := x_1(i)$ for all integers i, then we have

$$\begin{aligned} x_1(i+1) &= c(i+1), \\ x_1(i) &= c(i), \\ x_2(i) &= x_1(i-1) = c(i-1), \\ &\vdots \\ x_n(i) &= x_{n-1}(i-1) = \cdots = x_1(i-n+1) = c(i-n+1). \end{aligned}$$

Hence, by (4.6), the sequence $\{c(i)\}_{i\in\mathbb{Z}}$ is a solution to the nth order linear inhomogeneous difference equation (1.3).

Since

$$\vec{y}_i = \begin{pmatrix} a(i) \\ a(i-1) \\ a(i-2) \\ \vdots \\ a(i-n+1) \end{pmatrix} \quad \text{and} \quad \vec{x}_i = \begin{pmatrix} c(i) \\ c(i-1) \\ c(i-2) \\ \vdots \\ c(i-n+1) \end{pmatrix} \tag{4.7}$$

for all $i \in \mathbb{Z}$, we get

$$|a(i) - c(i)| \le \|\vec{y}_i - \vec{x}_i\|_\infty$$

for all $i \in \mathbb{Z}$. In view of (4.4), we complete the proof of this theorem. □

We now consider the second order linear homogeneous difference equation of the form

$$a(i+1) = p_1(i)a(i) + p_2(i)a(i-1) \tag{4.8}$$

for all $i \in \mathbb{Z}$. The solution of (4.8) is called the (extended) Fibonacci numbers when $p_1(i) = p_2(i) \equiv 1$, $a(0) = 1$, and $a(1) = 1$.

If we substitute $n = 2$, $p_1(i) = 1$, $p_2(i) = 1$, and $r(i) = 0$ for all $i \in \mathbb{Z}$ in Theorem 4.1, then we prove the following corollary concerning Hyers-Ulam stability of the Fibonacci difference equation. However, this corollary shows that Theorem 4.1 is not efficient when the transition matrices $\mathbf{A}(i)$ are constant, *i.e.*, $\mathbf{A}(i) = \mathbf{A}$ for all $i \in \mathbb{Z}$. Nevertheless, we introduce this corollary because its proof includes some new properties of the extended Fibonacci numbers. (In general, it is reasonable to apply [21, Theorem 5] when the transition matrices $\mathbf{A}(i)$ are constant.)

Corollary 4.2 *Assume that a sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ of nonnegative numbers is given. If a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality*

$$|a(i+1) - a(i) - a(i-1)| \le \varepsilon_{i+1} \tag{4.9}$$

for all $i \in \mathbb{Z}$, then there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation, i.e., the difference equation (4.8) with $p_1(i) = p_2(i) \equiv 1$ such that

$$|a(i) - c(i)| \le \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k F(i-k+1) + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\textit{for } i \ge 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} F(k+1) + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\textit{for } i < 0), \end{cases}$$

where $F(i)$ denotes the ith extended Fibonacci number and

$$\|\vec{y}_0 - \vec{x}_0\|_\infty = \max\big\{|a(0) - c(0)|, |a(-1) - c(-1)|\big\}.$$

Proof. If we set

$$\mathbf{A} := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \vec{y}_i := \begin{pmatrix} a(i) \\ a(i-1) \end{pmatrix},$$

then it follows from (4.9) that

$$\|\vec{y}_{i+1} - \mathbf{A}\vec{y}_i\|_\infty \le \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$.

According to Theorem 4.1, there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation (4.8) with $p_1(i) = p_2(i) \equiv 1$ such that

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \big\|\mathbf{A}^{i-k}\big\|_\infty + \big\|\mathbf{A}^i\big\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \big\|\mathbf{A}^{-k}\big\|_\infty + \big\|\mathbf{A}^i\big\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases} \tag{4.10}$$

where $\vec{y}_i$ and $\vec{x}_i$ are defined in (4.7) for all $i \in \mathbb{Z}$.

Here, we introduce some (extended) Fibonacci numbers explicitly.

$$\begin{aligned} \ldots,\ F(-4) = 2,\ F(-3) = -1,\ F(-2) = 1,\ F(-1) = 0, \\ F(0) = 1,\ F(1) = 1,\ F(2) = 2,\ F(3) = 3,\ F(4) = 5,\ \ldots \end{aligned} \tag{4.11}$$

and we prove that

$$F(i)F(i-1) < 0 \tag{4.12}$$

for any integer $i \leq -2$. If the relation (4.12) were not true, then there would exist an integer $i_0 \leq -2$ such that $F(i_0)F(i_0-1) \geq 0$. Then we would have

$$\begin{aligned} -1 &= F(-2)F(-3) \\ &= F(-3)^2 + F(-3)F(-4) \\ &= F(-3)^2 + F(-4)^2 + F(-4)F(-5) \\ &\ \vdots \\ &= F(-3)^2 + F(-4)^2 + \cdots + F(i_0)^2 + F(i_0)F(i_0-1) \\ &\geq 0, \end{aligned}$$

which is a contradiction.

We now prove that

$$|F(i)| = |F(-i-2)| \tag{4.13}$$

for any $i \in \mathbb{Z}$. First, we apply the induction to prove that the equality (4.13) holds for all integers $i \geq 0$. In view of (4.11), it is obvious that the equality (4.13) holds for $i \in \{0, 1, 2\}$. Assume that (4.13) holds for all integers $1 \leq i \leq i_0$, where i_0 is an integer not less than 2. In view of (4.11) and (4.12), we further have

$$\begin{aligned} |F(i_0+1)| &= |F(i_0) + F(i_0-1)| \\ &= |F(i_0)| + |F(i_0-1)| \\ &= |F(-i_0-2)| + |F(-i_0-1)| \\ &= |-F(-i_0-2) + F(-i_0-1)| \\ &= |F(-i_0-3)|, \end{aligned}$$

which can be obtained from (4.13) by replacing i with i_0+1. Hence, we conclude that the equality (4.13) holds for all integers $i \geq 0$.

Now, we apply an induction to prove that the equality (4.13) holds for all integers $i < 0$. In view of (4.11), we easily see that the equality (4.13) holds for $i \in \{-1, -2\}$. Assume that (4.13) holds for all integers $i_0 \leq i \leq -3$, where i_0 is an integer less than -2. Then, by (4.12) and (4.13), we have

$$\begin{aligned}
|F(i_0 - 1)| &= |F(i_0 + 1) - F(i_0)| \\
&= |F(i_0 + 1)| + |F(i_0)| \\
&= |F(-i_0 - 3)| + |F(-i_0 - 2)| \\
&= |F(-i_0 - 3) + F(-i_0 - 2)| \\
&= |F(-i_0 - 1)|,
\end{aligned}$$

which we can obtain from (4.13) by replacing i with $i_0 - 1$. Thus, the equality (4.13) holds for all integers $i < 0$.

Moreover, we apply the mathematical induction to prove

$$\mathbf{A}^i = \begin{pmatrix} F(i) & F(i-1) \\ F(i-1) & F(i-2) \end{pmatrix} \tag{4.14}$$

for any $i \in \mathbb{Z}$. Obviously, the equality (4.14) holds for $i \in \{0, 1\}$. Assume that (4.14) holds for some integer $i \geq 0$. Then, we get

$$\begin{aligned}
\mathbf{A}^{i+1} = \mathbf{A}^i \mathbf{A} &= \begin{pmatrix} F(i) & F(i-1) \\ F(i-1) & F(i-2) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} F(i) + F(i-1) & F(i) \\ F(i-1) + F(i-2) & F(i-1) \end{pmatrix} \\
&= \begin{pmatrix} F(i+1) & F(i) \\ F(i) & F(i-1) \end{pmatrix},
\end{aligned}$$

which can be obtained from (4.14) by replacing i with $i + 1$. Similarly, we prove that the equality (4.14) holds for all negative integers i.

Using (4.13) and (4.14), we prove that

$$\left\| \mathbf{A}^i \right\|_\infty = \begin{cases} F(i+1) & (\text{for } i \geq 0), \\ F(-i+1) & (\text{for } i < 0). \end{cases} \tag{4.15}$$

It is obvious that the first equality of (4.15) is true for $i \in \{0, 1\}$. Assume that $i \geq 2$. Then, considering (4.14) and the fact that $i - 2 \geq 0$, we have

$$\begin{aligned}
\left\| \mathbf{A}^i \right\|_\infty &= \max \left\{ |F(i)| + |F(i-1)|, |F(i-1)| + |F(i-2)| \right\} \\
&= \max \left\{ F(i) + F(i-1), F(i-1) + F(i-2) \right\} \\
&= \max \left\{ F(i+1), F(i) \right\} \\
&= F(i+1)
\end{aligned}$$

for any integer $i \geq 2$.

Now, we prove the equality (4.15) for $i < 0$. It follows from (4.13) and (4.14) that

$$\begin{aligned}
\left\| \mathbf{A}^i \right\|_\infty &= \max \left\{ |F(i)| + |F(i-1)|, |F(i-1)| + |F(i-2)| \right\} \\
&= \max \left\{ |F(-i-2)| + |F(-i-1)|, |F(-i-1)| + |F(-i)| \right\} \\
&= \max \left\{ F(-i-2) + F(-i-1), F(-i-1) + F(-i) \right\} \\
&= \max \left\{ F(-i), F(-i+1) \right\} \\
&= F(-i+1)
\end{aligned}$$

for any integer $i < 0$.

Finally, by (4.10) and (4.15), we have

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k F(i-k+1) + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} F(k+1) + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

which completes our proof. □

According to [16, Theorem 5.1], the following formula is true:

$$\sum_{k=1}^{i} F(k) = F(i+2) - 2 \tag{4.16}$$

for all $i \in \mathbb{N}_0$, where $F(i)$ denotes the ith extended Fibonacci number with the initial values, $F(-1) = 0$, $F(0) = 1$, and $F(1) = 1$.

Remark 4.3 Let ε be an arbitrarily given positive number. Assume that a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality

$$|a(i+1) - a(i) - a(i-1)| \leq \varepsilon$$

for all $i \in \mathbb{Z}$. According to Corollary 4.2 and (4.16), there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation such that

$$|a(i) - c(i)| \leq \begin{cases} F(i+2)\varepsilon - 2\varepsilon + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i > 0), \\ \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i = 0), \\ F(-i+3)\varepsilon - 3\varepsilon + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

where $F(i)$ denotes the ith extended Fibonacci number with the initial values, $F(-1) = 0$, $F(0) = 1$, and $F(1) = 1$, and

$$\|\vec{y}_0 - \vec{x}_0\|_\infty = \max\big\{|a(0) - c(0)|, |a(-1) - c(-1)|\big\}.$$

In particular, under strong additional conditions that $a(-1) = c(-1)$ and $a(0) = c(0)$, the last inequality reduces into

$$|a(i) - c(i)| \leq \begin{cases} F(i+2)\varepsilon - 2\varepsilon & (\text{for } i > 0), \\ 0 & (\text{for } i = 0), \\ F(-i+3)\varepsilon - 3\varepsilon & (\text{for } i < 0). \end{cases}$$

Remark 4.4 The Hyers-Ulam stability of the Fibonacci functional equation has been investigated in [1, 10, 11, 14, 15], while Hyers-Ulam stability of the linear difference equations has been investigated in [1, 2, 3, 5, 17, 18, 19]. It should be remarked that many interesting theorems have been proved in [4, 6] concerning the linear (or nonlinear) recurrences. Especially, Hyers-Ulam stability of the first order matrix difference equations with constant matrix has been proved in [21] in the domain $\mathbb{N}_0$.

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Self Adjoint Operator Ostrowski type Inequalities

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Abstract

We present here several self adjoint operator Ostrowski type inequalities to all directions. These are based in the operator order over a Hilbert space.

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Key Words and Phrases: Self adjoint operator, Hilbert space, Ostrowski inequality.

1 Motivation

In 1938, A. Ostrowski [12] proved the following important inequality:

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty ,$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

In this article we present self adjoint operator Ostrowski type inequalities on a Hilbert space in the operator order.

2 Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set

$C(Sp(A))$ of all continuous functions defind on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [10, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$ (the Banach algebra of all bounded linear operators from H into itself).

Equivalently, we use (see [8], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y\rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \quad\text{and}\quad g_{x,y}(M) = \langle x, y\rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is increasing and right continuous on $[m, M]$.

We have also the formula

$$\langle f(U)x, x\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x\rangle), \quad \forall\, x \in H.$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_{\lambda}.$$

Above, $m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_{\lambda} \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_{\lambda}$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_{\lambda} := \varphi_{\lambda}(U), \ \ \forall\ \lambda \in \mathbb{R},$$

is a projection which reduces U, with

$$\varphi_{\lambda}(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [11], pp. 256-266, and for more detalis see there pp. 157-266. See also [7].

Some more basics are given (we follow [8], pp. 1-5):

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x \rangle \in \mathbb{R}$, $\forall\ x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x \in H: \|x\| = 1} |\langle Ax, x \rangle|.$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x \rangle \leq \langle Bx, x \rangle$, $\forall$ $x \in H$.

In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k | n \geq 0,\ \alpha_k \in \mathbb{C},\ 0 \leq k \leq n \right\}.$$

If $A \in \mathcal{B}(H)$ is selfadjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [8], p. 4, Theorem 7).

Hence it holds

$$\||\varphi(A)|\| = \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\}$$

$$= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|,$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|.$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}.$$

3 Main Results

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, $m < M$; $m, M \in \mathbb{R}$.

In the next we obtain Ostrowski type inequalities in the operator order of $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators from H into itself).

We mention

Theorem 1 *([2], p. 498) Let $f \in C^1([m, M])$, $m < M$, $s \in [m, M]$. Then*

$$\left| \frac{1}{M-m} \int_m^M f(t) \, dt - f(x) \right| \leq \left(\frac{(s-m)^2 + (M-s)^2}{2(M-m)} \right) \|f'\|_\infty . \quad (1)$$

By applying property (P) to (1), we obtain in the operator order the following inequality:

Theorem 2 *Let $f \in C^1([m, M])$. Then*

$$\left| \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) 1_H - f(A) \right| \leq \left(\frac{(A - m1_H)^2 + (M1_H - A)^2}{2(M-m)} \right) \|f'\|_\infty . \tag{2}$$

We mention

Theorem 3 *([1], p. 191, Cerone-Dragomir) Let $f : [m, M] \to \mathbb{R}$ be a continuous on $[m, M]$ and twice differentiable function on (m, M), whose second derivative $f'' : (m, M) \to \mathbb{R}$ is bounded on (m, M). Then*

$$\left| f(s) - \frac{1}{M-m} \int_m^M f(t)\, dt - \left(\frac{f(M) - f(m)}{M-m} \right) \left(s - \frac{m+M}{2} \right) \right| \leq \tag{3}$$

$$\frac{1}{2} \left\{ \left[\frac{\left(s - \frac{m+M}{2} \right)^2}{(M-m)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (M-m)^2 \|f''\|_\infty \leq \frac{\|f''\|_\infty}{6} (M-m)^2 ,$$

$\forall\ s \in [m, M]$.

By applying property (P) to (3), we obtain in the operator order the following inequality:

Theorem 4 *All as in Theorem 3. Then*

$$\left| f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) 1_H - \left(\frac{f(M) - f(m)}{M-m} \right) \left(A - \left(\frac{m+M}{2} \right) 1_H \right) \right| \tag{4}$$

$$\leq \frac{1}{2} \left\{ \left[\frac{\left(A - \left(\frac{m+M}{2} \right) 1_H \right)^2}{(M-m)^2} + \frac{1}{4} 1_H \right]^2 + \frac{1}{12} 1_H \right\} (M-m)^2 \|f''\|_\infty$$

$$\leq \left(\frac{\|f''\|_\infty}{6} (M-m)^2 \right) 1_H .$$

We mention

Theorem 5 *([3], p. 14) Let $f : [m, M] \to \mathbb{R}$ be 3-times differentiable on $[m, M]$. Assume that f''' is bounded on $[m, M]$. Let any $s \in [m, M]$. Then*

$$\left| f(s) - \frac{1}{M-m} \int_m^M f(t)\, dt - \left(\frac{f(M) - f(m)}{M-m} \right) \left(s - \left(\frac{m+M}{2} \right) \right) - \right.$$

5

$$\left(\frac{f'(M)-f'(m)}{2(M-m)}\right)\left[s^2-(m+M)s+\left(\frac{m^2+M^2+4mM}{6}\right)\right]\Bigg| \tag{5}$$
$$\leq \frac{\|f'''\|_\infty}{(M-m)^3} Z(s),$$

where

$$Z(s)=\Big[mMs^4-\frac{1}{3}m^2M^3s+\frac{1}{3}m^3Ms^2-mM^2s^3-\frac{1}{3}m^3M^2s+\frac{1}{3}mM^3s^2$$
$$+m^2M^2s^2-m^2Ms^3-\frac{1}{2}ms^5-\frac{1}{2}Ms^5+\frac{1}{6}s^6+\frac{3}{4}m^2s^4+\frac{3}{4}M^2s^4+\frac{1}{3}M^2m^4-$$
$$\frac{2}{3}m^3s^3-\frac{2}{3}M^3s^3-\frac{1}{3}M^3m^3+\frac{5}{12}m^4s^2+\frac{5}{12}M^4s^2+\frac{1}{3}M^4m^2-$$
$$\frac{2}{15}Mm^5-\frac{2}{15}mM^5-\frac{1}{6}m^5s-\frac{1}{6}M^5s+\frac{m^6}{20}+\frac{M^6}{20}\Big]. \tag{6}$$

Using (P) property and (5), (6) we derive

Theorem 6 *Let $f:[m,M]\to\mathbb{R}$ be 3-times differentiable on $[m,M]$. Assume that f''' is bounded on $[m,M]$. Then*

$$\Bigg|f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H-\left(\frac{f(M)-f(m)}{M-m}\right)\left(A-\left(\frac{m+M}{2}\right)1_H\right)$$
$$-\left(\frac{f'(M)-f'(m)}{2(M-m)}\right)\left[A^2-(m+M)A+\left(\frac{m^2+M^2+4mM}{6}\right)1_H\right]\Bigg| \tag{7}$$
$$\leq \frac{\|f'''\|_\infty}{(M-m)^3} Z(A),$$

where

$$Z(A)=\Big[mMA^4-\frac{1}{3}m^2M^3A+\frac{1}{3}m^3MA^2-mM^2A^3-\frac{1}{3}m^3M^2A+$$
$$\frac{1}{3}mM^3A^2+m^2M^2A^2-m^2MA^3-\frac{1}{2}mA^5-\frac{1}{2}MA^5+\frac{1}{6}A^6+\frac{3}{4}m^2A^4+$$
$$\frac{3}{4}M^2A^4+\left(\frac{1}{3}M^2m^4\right)1_H-\frac{2}{3}m^3A^3-\frac{2}{3}M^3A^3-\left(\frac{1}{3}M^3m^3\right)1_H+$$
$$\frac{5}{12}m^4A^2+\frac{5}{12}M^4A^2+\left(\frac{1}{3}M^4m^2\right)1_H-$$
$$\left(\frac{2}{15}Mm^5\right)1_H-\left(\frac{2}{15}mM^5\right)1_H-\frac{1}{6}m^5A-\frac{1}{6}M^5A+\left(\frac{m^6+M^6}{20}\right)1_H\Big]. \tag{8}$$

Let $f \in AC([m, M])$ (absolutely continuous functions on $[m, M]$), $0 < \alpha < 1$. Denote the right Caputo fractional derivative by $D_{t-}^{\alpha} f$ (see [4], p. 22) and the left Caputo fractional derivative by $D_{*t}^{\alpha} f$ (see [4], p. 78), $\forall\, t \in [m, M]$.

We need

Theorem 7 *([4], p. 44) Let $0 < \alpha < 1$, $f \in AC([m, M])$, and $\left\|D_{t-}^{\alpha} f\right\|_{\infty,[m,t]}$, $\left\|D_{*t}^{\alpha} f\right\|_{\infty,[t,M]} < \infty$, $\forall\, t \in [m, M]$. Then*

$$\left|\frac{1}{M-m}\int_m^M f(z)\,dz - f(t)\right| \leq$$

$$\frac{1}{(M-m)\,\Gamma(\alpha+2)}\left\{\left\|D_{t-}^{\alpha} f\right\|_{\infty,[m,t]}(t-m)^{\alpha+1} + \left\|D_{*t}^{\alpha} f\right\|_{\infty,[t,M]}(M-t)^{\alpha+1}\right\} \leq \tag{9}$$

$$\frac{1}{\Gamma(\alpha+2)}\max\left\{\left\|D_{t-}^{\alpha} f\right\|_{\infty,[m,t]}, \left\|D_{*t}^{\alpha} f\right\|_{\infty,[t,M]}\right\}(M-m)^{\alpha}, \tag{10}$$

$\forall\, t \in [m, M]$.

By property (P) and Theorem 7 we derive

Theorem 8 *Let $0 < \alpha < 1$, $f \in AC([m, M])$, and there exists $K > 0$, such that*

$$\left\|D_{t-}^{\alpha} f\right\|_{\infty,[m,t]}, \left\|D_{*t}^{\alpha} f\right\|_{\infty,[t,M]} \leq K, \quad \forall\, t \in [m, M]. \tag{11}$$

Then

$$\left|\left(\frac{1}{M-m}\int_m^M f(z)\,dz\right)1_H - f(A)\right| \leq$$

$$\frac{K}{(M-m)\,\Gamma(\alpha+2)}\left\{(A - m1_H)^{\alpha+1} + (M1_H - A)^{\alpha+1}\right\} \leq \tag{12}$$

$$\left(\frac{K}{\Gamma(\alpha+2)}(M-m)^{\alpha}\right)1_H. \tag{13}$$

We mention the Fink ([9]) inequality

Theorem 9 *Let $f^{(n-1)}$ be absolutely continuous on $[m, M]$ and $f^{(n)} \in L_{\infty}(m, M)$, $n \in \mathbb{N}$. Then*

$$\left|f(s) + \sum_{k=1}^{n-1} F_k(s) - \frac{n}{M-m}\int_m^M f(t)\,dt\right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty}}{(n+1)!\,(M-m)}\left[(M-s)^{n+1} + (s-m)^{n+1}\right], \quad \forall\, s \in [m, M], \tag{14}$$

where

$$F_k(s) := \left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(m)(s-m)^k - f^{(k-1)}(M)(s-M)^k}{M-m}\right). \tag{15}$$

If $n = 1$, *then* $\sum_{k=1}^{n-1} = 0$.

Inequality (14) is sharp, in the sense that is attained by an optimal f *for any* $s \in [m, M]$.

By property (P) and Theorem 9 we obtain

Theorem 10 *Let* $f^{(n-1)}$ *be absolutely continuous on* $[m, M]$ *and* $f^{(n)} \in L_\infty(m, M)$, $n \in \mathbb{N}$. *Then*

$$\left| f(A) + \sum_{k=1}^{n-1} F_k(A) - \left(\frac{n}{M-m} \int_m^M f(t)\,dt \right) 1_H \right| \leq \tag{16}$$

$$\frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!\,(M-m)} \left[(M1_H - A)^{n+1} + (A - m1_H)^{n+1} \right],$$

where

$$F_k(A) := \left(\frac{n-k}{k!} \right) \left(\frac{f^{(k-1)}(m)\,(A - m1_H)^k - f^{(k-1)}(M)\,(A - M1_H)^k}{M-m} \right). \tag{17}$$

If $n = 1$, *then* $\sum_{k=1}^{n-1} F_k(A) = 0_H$.

We use here the sequence $\{B_k(t),\ k \geq 0\}$ of Bernoulli polynomials which is uniquely determined by the following identities:

$$\begin{array}{l} B_k'(t) = kB_{k-1}(t),\ \ k \geq 1,\ \ B_0(t) = 1 \\ \text{and} \\ B_k(t+1) - B_k(t) = kt^{k-1},\ \ k \geq 0. \end{array} \tag{18}$$

The values $B_k = B_k(0)$, $k \geq 0$ are the known Bernoulli numbers.

We mention

Theorem 11 *([3], p. 23) (see also [5]) Let* $f : [m, M] \to \mathbb{R}$ *be such that* $f^{(n-1)}$, $n \in \mathbb{N}$, *is a continuous function and* $f^{(n)}(t)$ *exists and is finite for all but a countable set of* t *in* (m, M) *and that* $f^{(n)} \in L_\infty([m, M])$.

Dentote by

$$\Delta_n(s) := f(s) - \frac{1}{M-m} \int_m^M f(t)\,dt -$$

$$\sum_{k=1}^{n-1} \frac{(M-m)^{k-1}}{k!} B_k\left(\frac{s-m}{M-m} \right) \left[f^{(k-1)}(M) - f^{(k-1)}(m) \right], \tag{19}$$

$\forall\ s \in [m, M]$.

Then

$$\left|\Delta_n(s)\right| \leq \frac{(M-m)^n}{n!}\left(\sqrt{\frac{(n!)^2}{(2n)!}\left|B_{2n}\right| + B_n^2\left(\frac{s-m}{M-m}\right)}\right)\left\|f^{(n)}\right\|_{\infty}, \tag{20}$$

$\forall\ n \in \mathbb{N}$; $\forall\ s \in [m, M]$.

Using the (P) property and Theorem 11 we derive:

Theorem 12 *All terms and assumptions as in Theorem 11. Denote by*

$$\Delta_n(A) := f(A) - \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H -$$

$$\sum_{k=1}^{n-1}\frac{(M-m)^{k-1}}{k!}B_k\left(\frac{A-m1_H}{M-m}\right)\left[f^{(k-1)}(M) - f^{(k-1)}(m)\right]. \tag{21}$$

Then

$$\left|\Delta_n(A)\right| \leq \frac{(M-m)^n}{n!}\left(\sqrt{\left(\frac{(n!)^2}{(2n)!}\left|B_{2n}\right|\right)1_H + B_n^2\left(\frac{A-m1_H}{M-m}\right)}\right)\left\|f^{(n)}\right\|_{\infty}, \tag{22}$$

$\forall\ n \in \mathbb{N}$.

Denote by (see [3], p. 24)

$$I_4(\lambda) := \begin{cases} \frac{16\lambda^5}{5} - 7\lambda^4 + \frac{14}{3}\lambda^3 - \lambda^2 + \frac{1}{30}, & 0 \leq \lambda \leq \frac{1}{2}, \\ -\frac{16\lambda^5}{5} + 9\lambda^4 - \frac{26\lambda^3}{3} + 3\lambda^2 - \frac{1}{10}, & \frac{1}{2} \leq \lambda \leq 1, \end{cases} \tag{23}$$

which is continuous in $\lambda \in [0, 1]$.

Also denote by

$$B := \left(\frac{A-m1_H}{M-m}\right)$$

and

$$I_4\left(\frac{A-m1_H}{M-m}\right) = I_4(B) =$$

$$\begin{cases} \frac{16}{5}B^5 - 7B^4 + \frac{14}{3}B^3 - B^2 + \frac{1}{30}1_H, & 0_H \leq B \leq \frac{1}{2}1_H, \\ -\frac{16}{5}B^5 + 9B^4 - \frac{26B^3}{3} + 3B^2 - \frac{1}{10}1_H, & \frac{1}{2}1_H \leq B \leq 1_H. \end{cases} \tag{24}$$

We mention

Theorem 13 *([3], p. 25) All terms and assumptions as in Theorem 11, case of $n = 4$. For every $s \in [m, M]$ it holds*

$$\left|\Delta_4(s)\right| \leq \frac{(M-m)^4}{24}I_4(\lambda)\left\|f^{(4)}\right\|_{\infty},$$

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where $I_4(\lambda)$ is given by (23) with

$$\lambda = \frac{s-m}{M-m}. \tag{25}$$

Furthermore we have that

$$|\Delta_4(s)| \leq \frac{(M-m)^4}{720} \left\| f^{(4)} \right\|_{\infty}, \tag{26}$$

$\forall\ s \in [m, M]$.

Using property (P) and Theorem 13 we find

Theorem 14 *All terms and assumptions are according to Theorem 11-13. Then*

$$|\Delta_4(A)| \leq \frac{(M-m)^4}{24} I_4\left(\frac{A - m1_H}{M-m}\right) \left\| f^{(4)} \right\|_{\infty}, \tag{27}$$

where $I_4\left(\frac{A-m1_H}{M-m}\right)$ is given by (24).

Furthermore we have that

$$|\Delta_4(A)| \leq \left(\frac{(M-m)^4}{720} \left\| f^{(4)} \right\|_{\infty}\right) 1_H. \tag{28}$$

Next we follow [6].

Let $(P_n)_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $P_0 = 1$. Let $f : [m, M] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$. Setting

$$\overline{F_k} = \frac{(-1)^k (n-k)}{M-m} \left[P_k(m) f^{(k-1)}(m) - P_k(M) f^{(k-1)}(M)\right], \quad k = 1, ..., n-1, \tag{29}$$

and

$$k(t, s) = \begin{cases} t - m, & \text{if } t \in [m, s] \\ t - M, & \text{if } t \in (s, M], \end{cases} \tag{30}$$

we get that

$$\frac{1}{n}\left[f(s) + \sum_{k=1}^{n-1} (-1)^k P_k(s) f^{(k)}(s) + \sum_{k=1}^{n-1} \overline{F_k}\right] - \frac{1}{M-m}\int_m^M f(t)\, dt = \tag{31}$$

$$\frac{(-1)^{n-1}}{n(M-m)} \int_m^M P_{n-1}(t)\, k(t, s)\, f^{(n)}(t)\, dt,$$

$\forall\ s \in [m, M]$. The above sums are defined to be zero for $n = 1$.

For the harmonic sequence of polynomials

$$P_k(t) = \frac{(t-s)^k}{k!}, \quad k \geq 0 \tag{32}$$

identity (31) collapses to the Fink identity, see [9].

We may rewrite generalized Fink identity (31) as follows:

$$f(s)=\sum_{k=1}^{n-1}(-1)^{k+1}P_k(s)f^{(k)}(s)+ \tag{33}$$

$$\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)f^{(k-1)}(M)-P_k(m)f^{(k-1)}(m)\right]+$$

$$\frac{n}{M-m}\int_m^M f(t)\,dt+\frac{(-1)^{n+1}}{M-m}\int_m^M P_{n-1}(t)\,k(t,s)\,f^{(n)}(t)\,dt,$$

$\forall\ s\in[m,M]$, $n\in\mathbb{N}$, when $n=1$ the above sums are zero.

Next we integrate the representation formula (33) against projections E_s to derive the operator representation formula:

$$f(A)=\sum_{k=1}^{n-1}(-1)^{k+1}P_k(A)f^{(k)}(A)+ \tag{34}$$

$$\left[\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)f^{(k-1)}(M)-P_k(m)f^{(k-1)}(m)\right]+\right.$$

$$\left.\frac{n}{M-m}\int_m^M f(t)\,dt\right]1_H+\frac{(-1)^{n+1}}{M-m}\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(t,s)\,f^{(n)}(t)\,dt\right)dE_s.$$

The sequence of polynomials

$$P_k(t)=\frac{1}{k!}\left(t-\frac{m+M}{2}\right)^k,\ \ k\geq 0, \tag{35}$$

is also harmonic.

We mention

Theorem 15 *([6]) Let $f:[m,M]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\in\mathbb{N}$ and $f^{(n)}\in L_p([m,M])$, $1\leq p\leq\infty$. Then*

$$\left|\left[f(s)+\sum_{k=1}^{n-1}(-1)^k P_k(s)f^{(k)}(s)+\sum_{k=1}^{n-1}\overline{F_k}\right]-\frac{n}{M-m}\int_m^M f(t)\,dt\right|\leq \tag{36}$$

$$\frac{1}{M-m}\left\|P_{n-1}(\cdot)\,k(\cdot,s)\right\|_{p',[m,M]}\left\|f^{(n)}\right\|_p,$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

We observe that

$$\int_m^M\left|P_{n-1}(t)\,k(t,s)\right|^{p'}dt\leq\left\|P_{n-1}\right\|_{\infty,[m,M]}^{p'}\int_m^M\left|k(t,s)\right|^{p'}dt= \tag{37}$$

$$\|P_{n-1}\|_{\infty,[m,M]}^{p'}\left[\int_m^s (t-m)^{p'}\,dt+\int_s^M (M-t)^{p'}\,dt\right]=$$

$$\|P_{n-1}\|_{\infty,[m,M]}^{p'}\left[\frac{(s-m)^{p'+1}+(M-s)^{p'+1}}{p'+1}\right].$$

Therefore we obtain

$$\|P_{n-1}(\cdot)\,k(\cdot,s)\|_{p',[m,M]}\le\|P_{n-1}\|_{\infty,[m,M]}\left[\frac{(M-s)^{p'+1}+(s-m)^{p'+1}}{p'+1}\right]^{\frac{1}{p'}}. \tag{38}$$

Hence we have

Theorem 16 *Let $f:[m,M]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\in\mathbb{N}$ and $f^{(n)}\in L_p([m,M])$, $1\le p\le\infty$. Then*

$$\left|\left(f(s)+\sum_{k=1}^{n-1}(-1)^k P_k(s)\,f^{(k)}(s)\right)+\left(\sum_{k=1}^{n-1}\overline{F_k}\right)-\left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)\right|\le$$

$$\left(\frac{\left\|f^{(n)}\right\|_p}{M-m}\|P_{n-1}\|_{\infty,[m,M]}\right)\left[\frac{(M-s)^{p'+1}+(s-m)^{p'+1}}{p'+1}\right]^{\frac{1}{p'}}, \tag{39}$$

$\forall\ s\in[m,M]$, where $\frac{1}{p}+\frac{1}{p'}=1$.

We get the following operator inequality:

Theorem 17 *Let $f:[m,M]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\in\mathbb{N}$ and $f^{(n)}\in L_p([m,M])$, $1\le p\le\infty$. Then*

$$\left|\left(f(A)+\sum_{k=1}^{n-1}(-1)^k P_k(A)\,f^{(k)}(A)\right)+\left(\sum_{k=1}^{n-1}\overline{F_k}\right)1_H-\right.$$

$$\left.\left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)1_H\right|\le$$

$$\left(\frac{\left\|f^{(n)}\right\|_p}{M-m}\|P_{n-1}\|_{\infty,[m,M]}\right)\left[\frac{(M1_H-A)^{p'+1}+(A-m1_H)^{p'+1}}{p'+1}\right]^{\frac{1}{p'}}, \tag{40}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. By (P) property and (39). ■

We give

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Corollary 18 *(to Theorem 16) (see also [6]) We have*

$$\left|\left[f(s)+\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\left(s-\frac{m+M}{2}\right)^k f^{(k)}(s)+\right.\right.$$

$$\left.\sum_{k=1}^{n-1}\frac{(M-m)^{k-1}(n-k)}{k!2^k}\left[f^{(k-1)}(m)-(-1)^k f^{(k-1)}(M)\right]\right]$$

$$\left.-\frac{n}{M-m}\int_m^M f(t)\,dt\right|\leq$$

$$\left(\frac{\left\|f^{(n)}\right\|_p (M-m)^{n-2}}{2^{n-1}(n-1)!}\right)\left[\frac{(M-s)^{p'+1}+(s-m)^{p'+1}}{p'+1}\right]^{\frac{1}{p'}}, \tag{41}$$

$\forall\ s\in[m,M]$, *where* $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. Set $P_k(t)=\frac{1}{k!}\left(t-\frac{m+M}{2}\right)^k$, $k\geq 0$, in Theorem 16. ■

We finish with the operator inequality:

Corollary 19 *(to Theorem 17) We have*

$$\left|\left[f(A)+\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\left(A-\left(\frac{m+M}{2}\right)1_H\right)^k f^{(k)}(A)+\right.\right.$$

$$\left.\left(\sum_{k=1}^{n-1}\frac{(M-m)^{k-1}(n-k)}{k!2^k}\left[f^{(k-1)}(m)-(-1)^k f^{(k-1)}(M)\right]\right)1_H\right]$$

$$\left.-\left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)1_H\right|\leq$$

$$\left(\frac{\left\|f^{(n)}\right\|_p (M-m)^{n-2}}{2^{n-1}(n-1)!}\right)\left[\frac{(M1_H-A)^{p'+1}+(A-m1_H)^{p'+1}}{p'+1}\right]^{\frac{1}{p'}}, \tag{42}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. By Corollary 18 and (P) property. ■

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Integer and Fractional Self Adjoint Operator Opial type Inequalities

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Abstract

We present here several integer and fractional self adjoint operator Opial type inequalities to many directions. These are based in the operator order over a Hilbert space.

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Key Words and Phrases: Self adjoint operator, Hilbert space, Opial inequality, fractional derivative.

1 Motivation

In 1960, Z. Opial ([9]) proved the following famous inequality that motivates our work here.

Let $f \in C^1([0,h])$ be such that $f(0) = f(h) = 0$, and $f(t) > 0$ in $(0,h)$. Then

$$\int_0^h |f(t) f'(t)| \, dt \leq \frac{h}{4} \int_0^h (f'(t))^2 \, dt.$$

The constant $\frac{h}{4}$ is the best.

In this article we present integer and fractional self adjoint operator Opial type inequalities on a Hilbert space in the operator order.

2 Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted

$Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [6, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$. (the Banach algebra of all bounded linear operators from H into itself).

Equivalently, we use (see [5], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

We have also the formula

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x \rangle), \quad \forall\, x \in H.$$

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As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda.$$

Above, $m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \ \ \forall \lambda \in \mathbb{R},$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [8], pp. 256-266, and for more detalis see there pp. 157-266. See also [4].

Some more basics are given (we follow [5], pp. 1-5):

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x \rangle \in \mathbb{R}$, $\forall x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x \in H: \|x\|=1} |\langle Ax, x \rangle|.$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x \rangle \leq \langle Bx, x \rangle$, $\forall x \in H$.

In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k | n \geq 0, \, \alpha_k \in \mathbb{C}, \, 0 \leq k \leq n \right\}.$$

If $A \in \mathcal{B}(H)$ is selfadjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [5], p. 4, Theorem 7).

Hence it holds

$$\||\varphi(A)|\| = \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\}$$

$$= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|,$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|.$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)|\, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}.$$

3 Main Results

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, $m < M$; $m, M \in \mathbb{R}$.

In the next we obtain Opial type inequalities, both integer and fractional cases, in the operator order of $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators from H into itself).

Let the real valued function $f \in C([m, M])$, and we consider

$$g(t) = \int_m^t f(z)\, dz, \ \ \forall\, t \in [m, M], \tag{1}$$

then $g \in C([m, M])$.

We denote by

$$\int_{m1_H}^{A} f := \Phi(g) = g(A). \tag{2}$$

We understand and write that $(r > 0)$

$$g^r(A) = \Phi(g^r) =: \left(\int_{m1_H}^{A} f\right)^r.$$

Clearly $\left(\int_{m1_H}^{A} f\right)^r$ is a self adjoint operator on H, for any $r > 0$.

All of our functions in this article will be real valued. From [3] we mention the following basic version of Opial inequality:

Theorem 1 *Let $f \in C^1([m, M])$ with $f(m) = 0$. Then*

$$\int_m^{\lambda} |f(t)| |f'(t)| \, dt \leq \left(\frac{\lambda - m}{2}\right) \int_m^{\lambda} (f'(t))^2 \, dt, \quad \forall \lambda \in [m, M]. \tag{3}$$

When $f(t) = t - m$, $t \in [m, M]$, inequality (3) becomes equality.

By applying properties (P) and (ii) to (3) we obtain

Theorem 2 *Let $f \in C^1([m, M])$ with $f(m) = 0$. Then*

$$\int_{m1_H}^{A} |ff'| \leq \frac{1}{2}(A - m1_H)\left(\int_{m1_H}^{A} (f')^2\right). \tag{4}$$

We mention

Theorem 3 *([3]) Let $f \in C^1([m, M])$ with $f(m) = 0$, and $1 \leq p \leq 2$. Then*

$$\int_m^{\lambda} |f(t)|^p |f'(t)|^p \, dt \leq K(p)(\lambda - m)\left(\int_m^{\lambda} (f'(t))^2 \, dt\right)^p, \quad \forall \lambda \in [m, M], \tag{5}$$

where

$$K(p) = \begin{cases} \frac{1}{2}, & p = 1, \\ \frac{4}{\pi^2}, & p = 2, \\ \frac{2-p}{2p}\left(\frac{1}{p}\right)^{2p-2} I^{-p}, & 1 < p < 2, \end{cases} \tag{6}$$

with

$$I = \int_0^1 \left\{1 + \frac{2(p-1)}{2-p} z\right\}^{-2} \{1 + (p-1)z\}^{\frac{1}{p}-1} \, dz.$$

For $p = 1$, equality holds in (5) only for f linear.

By applying properties (P) and (ii) to (5) we derive

Theorem 4 *Here all are as in Theorem 3. It holds*

$$\int_{m1_H}^{A} |ff'|^p \leq K(p)(A - m1_H)\left(\int_{m1_H}^{A} (f')^2\right)^p. \tag{7}$$

We mention

Theorem 5 *([7]) Let $f \in C^1([m,M])$ with $f(m) = 0$, and $p, q \geq 1$. Then*

$$\int_m^\lambda |f(t)|^p |f'(t)|^q \, dt \leq \left(\frac{q}{p+q}\right)(\lambda - m)^p \int_m^\lambda |f'(t)|^{p+q} \, dt, \quad \forall\ \lambda \in [m,M]. \tag{8}$$

By applying properties (P) and (ii) to (8) we find

Theorem 6 *Let $f \in C^1([m,M])$ with $f(m) = 0$, and $p, q \geq 1$. Then*

$$\int_{m1_H}^{A} |f|^p |f'|^q \leq \left(\frac{q}{p+q}\right)(A - m1_H)^p \left(\int_{m1_H}^{A} |f'|^{p+q}\right). \tag{9}$$

We mention

Theorem 7 *([11]) Let $p > -1$. Let $f \in C^1([m,M])$, and $f(m) = 0$. Then*

$$\int_m^\lambda t^p |f(t) f'(t)| \, dt \leq \frac{1}{2\sqrt{p+1}} \int_m^\lambda \left(\lambda^{p+1} - mt^p\right)(f'(t))^2 \, dt \tag{10}$$

$$\leq \frac{1}{2\sqrt{p+1}} \int_m^\lambda \left(M^{p+1} - mt^p\right)(f'(t))^2 \, dt, \quad \forall\ \lambda \in [m,M]. \tag{11}$$

(inequality (11) is our derivation).

By applying properties (P) and (ii) to (10), (11) we obtain

Theorem 8 *Let $p > -1$. Let $f \in C^1([m,M])$ and $f(m) = 0$. Then*

$$\int_{m1_H}^{A} (id)^p |ff'| \leq \frac{1}{2\sqrt{p+1}} \left(\int_{m1_H}^{A} \left(M^{p+1} - m(id)^p\right)(f')^2\right). \tag{12}$$

We mention

Theorem 9 *([1], p. 20) Let $q(t)$ be positive continuous and non-increasing function on $[m,M]$. Further, let $f \in C^1([m,M])$, and $f(m) = 0$. Let $l \geq 0$, $w \geq 1$. Then*

$$\int_m^\lambda q(t)|f(t)|^l |f'(t)|^w \, dt \leq \left(\frac{w}{l+w}\right)(\lambda - m)^l \int_m^\lambda q(t)|f'(t)|^{l+w} \, dt, \tag{13}$$

$\forall\ \lambda \in [m,M]$.

By applying property (P) and (ii) to (13) we obtain

Theorem 10 *All as in Theorem 9. Then*

$$\int_{m1_H}^{A} q\,|f|^l\,|f'|^w \leq \left(\frac{w}{l+w}\right)(A-m1_H)^l \int_{m1_H}^{A} q\,|f'|^{l+w}. \tag{14}$$

We mention

Theorem 11 *(see [1], p. 68) Let $q(t)$ positive, continuous and non-increasing on $[m,M]$. Further let $f_1, f_2 \in C^1([m,M])$ with $f_1(m)=f_2(m)=0$. Let $l \geq 0$, $w \geq 1$. Then*

$$\int_{m}^{\lambda} q(t)\,|f_1(t)\,f_2(t)|^l \left[|f_1(t)\,f_2'(t)|^w + |f_1'(t)\,f_2(t)|^w\right] dt \leq$$

$$\frac{w}{2(l+w)}(\lambda-m)^{2l+w} \int_{m}^{\lambda} q(t)\left[(f_1'(t))^{2(l+w)} + (f_2'(t))^{2(l+w)}\right] dt, \tag{15}$$

$\forall\ \lambda \in [m,M]$.

By applying property (P) and (ii) to (15) we obtain

Theorem 12 *All as in Theorem 11. Then*

$$\int_{m1_H}^{A} q\,|f_1 f_2|^l \left[|f_1 f_2'|^w + |f_1' f_2|^w\right] \leq \tag{16}$$

$$\frac{w}{2(l+w)}(A-m1_H)^{2l+w} \int_{m1_H}^{A} q\left[(f_1')^{2(l+w)} + (f_2')^{2(l+w)}\right].$$

We mention

Theorem 13 *([10], p. 308) Let $f \in C^n([m,M])$, $n \in \mathbb{N}$, $f^{(i)}(m)=0$, for $i=0,1,2,...,n-1$. Then*

$$\int_{m}^{\lambda} \left|f(t)\,f^{(n)}(t)\right| dt \leq \frac{(\lambda-m)^n}{2} \int_{m}^{\lambda} \left(f^{(n)}(t)\right)^2 dt, \quad \forall\ \lambda \in [m,M]. \tag{17}$$

Using properties (P) and (ii) on (17) we derive

Theorem 14 *All as in Theorem 13. Then*

$$\int_{m1_H}^{A} \left|f \cdot f^{(n)}\right| \leq \frac{(A-m1_H)^n}{2}\left(\int_{m1_H}^{A} \left(f^{(n)}\right)^2\right). \tag{18}$$

We mention from [10], p. 309

Theorem 15 *Let $f_1, f_2 \in C^n([m,M])$ such that $f_1^{(k)}(m) = f_2^{(k)}(m) = 0$, for $k = 0, 1, ..., n-1$, $n \in \mathbb{N}$. Then*

$$\int_m^\lambda \left[\left| f_1(t) f_2^{(n)}(t) \right| + \left| f_2(t) f_1^{(n)}(t) \right| \right] dt \leq$$

$$B(\lambda - m)^n \int_m^\lambda \left[\left(f_1^{(n)}(t) \right)^2 + \left(f_2^{(n)}(t) \right)^2 \right] dt, \quad \forall \lambda \in [m, M], \tag{19}$$

where

$$B = \frac{1}{2n!} \left(\frac{n}{2n-1} \right)^{\frac{1}{2}}. \tag{20}$$

Using (19) and properties (P) and (ii) we obtain

Theorem 16 *All as in Theorem 15. Then*

$$\int_{m1_H}^A \left[\left| f_1 f_2^{(n)} \right| + \left| f_2 f_1^{(n)} \right| \right] \leq$$

$$B(A - m1_H)^n \left(\int_{m1_H}^A \left(\left(f_1^{(n)} \right)^2 + \left(f_2^{(n)} \right)^2 \right) \right). \tag{21}$$

Here we follow [2], p. 8.

Definition 17 *Let $\nu > 0$, $n := [\nu]$ (integral part), and $\alpha := \nu - n$ ($0 < \alpha < 1$). Let $f \in C([m,M])$ and define*

$$(J_\nu^m f)(z) = \frac{1}{\Gamma(\nu)} \int_m^z (z-t)^{\nu-1} f(t)\, dt, \tag{22}$$

all $m \leq z \leq M$, where Γ is the gamma function, the generalized Riemann-Liouville integral. We define the subspace $C_m^\nu([m,M])$ of $C^n([m,M])$:

$$C_m^\nu([m,M]) := \left\{ f \in C^n([m,M]) : J_{1-\alpha}^m f^{(n)} \in C^1([m,M]) \right\}. \tag{23}$$

So let $f \in C_m^\nu([m,M])$; we define the generalized ν-fractional derivative (of Canavati type) of f over $[m,M]$ as

$$D_m^\nu f := \left(J_{1-\alpha}^m f^{(n)} \right)'. \tag{24}$$

Notice that

$$\left(J_{1-\alpha}^m f^{(n)} \right)(z) = \frac{1}{\Gamma(1-\alpha)} \int_m^z (z-t)^{-\alpha} f^{(n)}(t)\, dt \tag{25}$$

exists for $f \in C_m^\nu([m,M])$, all $m \leq z \leq M$.

Also notice that $D_m^\nu f \in C([m,M])$.

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We need

Theorem 18 *([2], p. 15) Let* $f \in C_m^{\nu}([m,M])$, $\nu \geq 1$ *and* $f^{(i)}(m) = 0$, $i = 0,1,...,n-1$, $n := [\nu]$. *Here* $\lambda \in [m,M]$, *and* $l = 1,...,n-1$. *Let* $p,q > 1 :$ $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_m^{\lambda} \left| f^{(l)}(w) \right| \left| \left(D_m^{\nu} f \right)(w) \right| dw \leq$$

$$\frac{2^{-\frac{1}{q}} (\lambda - m)^{\frac{(\nu p - lp - p + 2)}{p}}}{\Gamma(\nu - l) \left((\nu p - lp - p + 1)(\nu p - lp - p + 2) \right)^{\frac{1}{p}}} \left(\int_m^{\lambda} \left| \left(D_m^{\nu} f \right)(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{26}$$

Using (26), properties (P) and (ii) we get

Theorem 19 *All as in Theorem 18. Then*

$$\int_{m1_H}^{A} \left| f^{(l)} \right| \left| \left(D_m^{\nu} f \right) \right| \leq$$

$$\frac{2^{-\frac{1}{q}} (A - m1_H)^{\frac{(\nu p - lp - p + 2)}{p}}}{\Gamma(\nu - l) \left((\nu p - lp - p + 1)(\nu p - lp - p + 2) \right)^{\frac{1}{p}}} \left(\int_{m1_H}^{A} \left| \left(D_m^{\nu} f \right) \right|^q \right)^{\frac{2}{q}}. \tag{27}$$

We need

Theorem 20 *([2], p. 26) Let* $\gamma_1, \gamma_2 \geq 0$, $\nu \geq 1$ *be such that* $\nu - \gamma_1, \nu - \gamma_2 \geq 1$ *and* $f \in C_m^{\nu}([m,M])$ *with* $f^{(i)}(m) = 0$, $i = 0,1,...,n-1$, $n := [\nu]$. *Here* $\lambda \in [m,M]$. *Let* q *be a nonnegative continuous functions on* $[m,M]$. *Denote*

$$Q(\lambda) := \left(\int_m^{\lambda} (q(w))^2 dw \right)^{\frac{1}{2}}, \quad \forall \ \lambda \in [m,M]. \tag{28}$$

Then

$$\int_m^{\lambda} q(w) \left| D_m^{\gamma_1}(f)(w) \right| \left| D_m^{\gamma_2}(f)(w) \right| dw \leq$$

$$K(q, \gamma_1, \gamma_2, \nu, \lambda, m) \left(\int_m^{\lambda} \left(D_m^{\nu} f(w) \right)^2 dw \right), \tag{29}$$

where

$$K(q, \gamma_1, \gamma_2, \nu, \lambda, m) := \frac{Q(\lambda)}{\sqrt[3]{6}} \frac{1}{\Gamma(\nu - \gamma_1)\Gamma(\nu - \gamma_2)} \cdot$$

$$\frac{(\lambda - m)^{2\nu - \gamma_1 - \gamma_2 - \frac{1}{2}}}{\left(\nu - \gamma_1 - \frac{5}{6} \right)^{\frac{1}{6}} \left(\nu - \gamma_2 - \frac{5}{6} \right)^{\frac{1}{6}} \left(4\nu - 2\gamma_1 - 2\gamma_2 - \frac{7}{3} \right)^{\frac{1}{2}}}. \tag{30}$$

Using (30) and Remark 3.4 of [2], p. 26, and properties (P) and (ii) to obtain

Theorem 21 *All terms and assumptions as in Theorem 20. Then*

$$\int_{m1_H}^{A} q\left|D_m^{\gamma_1}(f)\right|\left|D_m^{\gamma_2}(f)\right| \leq$$

$$K(q,\gamma_1,\gamma_2,\nu,A,m)\left(\int_{m1_H}^{A}\left(D_m^{\nu}f\right)^2\right), \tag{31}$$

where

$$K(q,\gamma_1,\gamma_2,\nu,A,m) := \frac{Q(A)}{\sqrt[3]{6}}\frac{1}{\Gamma(\nu-\gamma_1)\Gamma(\nu-\gamma_2)}\cdot$$

$$\frac{(A-m1_H)^{2\nu-\gamma_1-\gamma_2-\frac{1}{2}}}{\left(\nu-\gamma_1-\frac{5}{6}\right)^{\frac{1}{6}}\left(\nu-\gamma_2-\frac{5}{6}\right)^{\frac{1}{6}}\left(4\nu-2\gamma_1-2\gamma_2-\frac{7}{3}\right)^{\frac{1}{2}}}. \tag{32}$$

We need

Theorem 22 *([2], p. 30) Let $\gamma \geq 0$, $\nu \geq 1$, $\nu-\gamma \geq 1$, let q be a nonnegative continuous function on $[m,M]$. Let $f \in C_m^{\nu}([m,M])$ with $f^{(i)}(m)=0$, $i=0,1,...,n-1$, $n:=[\nu]$. Let $\lambda \in [m,M]$. Call*

$$Q(\lambda) := \left(\int_m^{\lambda}(q(w))^2(w-m)^{2\nu-2\gamma-1}dw\right)^{\frac{1}{2}}, \tag{33}$$

and

$$K(q,\gamma,\nu,\lambda,m) := \frac{Q(\lambda)}{\sqrt{2(2\nu-2\gamma-1)}\Gamma(\nu-\gamma)}. \tag{34}$$

Then

$$\int_m^{\lambda} q(w)\left|D_m^{\gamma}f(w)\right|\left|D_m^{\nu}f(w)\right|dw \leq K(q,\gamma,\nu,\lambda,m)\left(\int_m^{\lambda}\left(\left(D_m^{\nu}f\right)(w)\right)^2dw\right). \tag{35}$$

Using (33)-(35) and properties (P) and (ii) we derive

Theorem 23 *All as in Theorem 22. Denote by*

$$K(q,\gamma,\nu,A,m) := \frac{Q(A)}{\sqrt{2(2\nu-2\gamma-1)}\Gamma(\nu-\gamma)}. \tag{36}$$

Then

$$\int_{m1_H}^{A} q\left|D_m^{\gamma}f\right|\left|D_m^{\nu}f\right| \leq K(q,\gamma,\nu,A,m)\left(\int_{m1_H}^{A}\left(\left(D_m^{\nu}f\right)\right)^2\right). \tag{37}$$

We need

Theorem 24 *([2], p. 92) Let $\nu \geq 1$, $\gamma_1, \gamma_2 \geq 0$, such that $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$, and $f_1, f_2 \in C_m^\nu([m, M])$ with $f_1^{(i)}(m) = f_2^{(i)}(m) = 0$, $i = 0, 1, ..., n-1$, $n := [\nu]$. Here $\lambda \in [m, M]$. Let $\lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0$. Set*

$$\rho(\lambda) := \frac{(\lambda - m)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}. \tag{38}$$

Then

$$\int_m^\lambda \Big[|(D_m^{\gamma_1} f_1)(w)|^{\lambda_\alpha} |(D_m^{\gamma_2} f_2)(w)|^{\lambda_\beta} |(D_m^\nu f_1)(w)|^{\lambda_\nu} +$$

$$|(D_m^{\gamma_2} f_1)(w)|^{\lambda_\beta} |(D_m^{\gamma_1} f_2)(w)|^{\lambda_\alpha} |(D_m^\nu f_2)(w)|^{\lambda_\nu} \Big] dw \leq$$

$$\frac{\rho(\lambda)}{2} \Big[\|D_m^\nu f_1\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \|D_m^\nu f_1\|_\infty^{2\lambda_\beta} + \|D_m^\nu f_2\|_\infty^{2\lambda_\beta} + \|D_m^\nu f_2\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} \Big], \tag{39}$$

all $m \leq \lambda \leq M$.

Using (39) and properties (P) and (ii) we derive

Theorem 25 *All here as in Theorem 24. Set*

$$\rho(A) := \frac{(A - m1_H)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}. \tag{40}$$

Then

$$\int_{m1_H}^A \Big[|(D_m^{\gamma_1} f_1)|^{\lambda_\alpha} |(D_m^{\gamma_2} f_2)|^{\lambda_\beta} |(D_m^\nu f_1)|^{\lambda_\nu} +$$

$$|(D_m^{\gamma_2} f_1)|^{\lambda_\beta} |(D_m^{\gamma_1} f_2)|^{\lambda_\alpha} |(D_m^\nu f_2)|^{\lambda_\nu} \Big] \leq$$

$$\frac{\rho(A)}{2} \Big[\|D_m^\nu f_1\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \|D_m^\nu f_1\|_\infty^{2\lambda_\beta} + \|D_m^\nu f_2\|_\infty^{2\lambda_\beta} + \|D_m^\nu f_2\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} \Big]. \tag{41}$$

We give

Definition 26 *([2], p. 270) Let $\nu > 0$, $n := \lceil \nu \rceil$ (ceiling of ν), $f \in AC^n([m, M])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[m, M]$, that is in $AC([m, M])$). We define the Caputo fractional derivative*

$$(D_{*m}^\nu f)(z) := \frac{1}{\Gamma(n - \nu)} \int_m^z (z - t)^{n - \nu - 1} f^{(n)}(t)\, dt, \tag{42}$$

which exists almost everywhere for $z \in [m, M]$.

Notice that $D_{*m}^0 f = f$, and $D_{*m}^n f = f^{(n)}$.

We mention

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Theorem 27 *([2], p. 397) Let $\nu \geq \gamma+1$, $\gamma \geq 0$. Call $n := \lceil \nu \rceil$ and assume $f \in C^n([m,M])$ such that $f^{(k)}(m) = 0$, $k = 0,1,...,n-1$. Let $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $m \leq \lambda \leq M$. Then*

$$\int_m^\lambda |(D_{*m}^\gamma f)(w)| \, |(D_{*m}^\nu f)(w)| \, dw \leq$$

$$\frac{(\lambda - m)^{\frac{(p\nu - p\gamma - p + 2)}{p}}}{\left(\sqrt[q]{2}\right)\Gamma(\nu-\gamma)\left((p\nu - p\gamma - p+1)(p\nu - p\gamma - p + 2)\right)^{\frac{1}{p}}} \left(\int_m^\lambda |D_{*m}^\nu f(w)|^q \, dw\right)^{\frac{2}{q}}. \tag{43}$$

Note: By Proposition 15.114 ([2], p. 388) we have that $D_{*m}^\nu f, D_{*m}^\gamma f \in C([m,M])$.

Using (43) and Properties (P) and (ii) we give

Theorem 28 *All as in Theorem 27. Then*

$$\int_{m1_H}^A |(D_{*m}^\gamma f)| \, |(D_{*m}^\nu f)| \leq$$

$$\frac{(A - m1_H)^{\frac{(p\nu - p\gamma - p + 2)}{p}}}{\left(\sqrt[q]{2}\right)\Gamma(\nu-\gamma)\left((p\nu - p\gamma - p+1)(p\nu - p\gamma - p + 2)\right)^{\frac{1}{p}}} \left(\int_{m1_H}^A |D_{*m}^\nu f|^q\right)^{\frac{2}{q}}. \tag{44}$$

We need

Theorem 29 *([2], p. 398) Let $\nu \geq 2$, $k \geq 0$, $\nu \geq k+2$. Call $n := \lceil \nu \rceil$ and $f \in C^n([m,M]) : f^{(j)}(m) = 0$, $j = 0,1,...,n-1$. Let $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $m \leq \lambda \leq M$. Then*

$$\int_m^\lambda \left|\left(D_{*m}^k f\right)(w)\right| \left|\left(D_{*m}^{k+1} f\right)(w)\right| dw \leq$$

$$\frac{(\lambda - m)^{\frac{2(p\nu - pk - p + 1)}{p}}}{2\left(\Gamma(\nu-k)\right)^2 (p\nu - pk - p + 1)^{\frac{2}{p}}} \left(\int_m^\lambda |D_{*m}^\nu f(w)|^q \, dw\right)^{\frac{2}{q}}. \tag{45}$$

Using (45) and Properties (P) and (ii) we find

Theorem 30 *All as in Theorem 29. Then*

$$\int_{m1_H}^A \left|\left(D_{*m}^k f\right)\right| \left|\left(D_{*m}^{k+1} f\right)\right| \leq$$

$$\frac{(A - m1_H)^{\frac{2(p\nu - pk - p + 1)}{p}}}{2\left(\Gamma(\nu-k)\right)^2 (p\nu - pk - p + 1)^{\frac{2}{p}}} \left(\int_{m1_H}^A |D_{*m}^\nu f|^q\right)^{\frac{2}{q}}. \tag{46}$$

We need

Theorem 31 *([2], p. 399) Let $\gamma_i \geq 0$, $\nu \geq 1$, $\nu - \gamma_i \geq 1$; $i = 1, ..., l$, $n := \lceil \nu \rceil$, and $f \in C^n([m, M])$ such that $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$. Here $m \leq \lambda \leq M$; $q_1(\lambda), q_2(\lambda)$ continuous functions on $[m, M]$ such that $q_1(\lambda) \geq 0$, $q_2(\lambda) > 0$ on $[m, M]$, and $r_i > 0 : \sum_{i=1}^{l} r_i = r$. Let $s_1, s_1' > 1 : \frac{1}{s_1} + \frac{1}{s_1'} = 1$ and $s_2, s_2' > 1 : \frac{1}{s_2} + \frac{1}{s_2'} = 1$, and $p > s_2$.*

Denote by

$$Q_1(\lambda) := \left(\int_m^{\lambda} (q_1(w))^{s_1'} dw \right)^{\frac{1}{s_1'}} \tag{47}$$

and

$$Q_2(\lambda) := \left(\int_m^{\lambda} (q_2(w))^{\frac{-s_2'}{p}} dw \right)^{\frac{r}{s_2'}}, \tag{48}$$

$$\sigma := \frac{p - s_2}{p s_2}. \tag{49}$$

Then

$$\int_m^{\lambda} q_1(w) \prod_{i=1}^{l} \left| D_{*m}^{\gamma_i} f(w) \right|^{r_i} dw \leq$$

$$Q_1(\lambda) Q_2(\lambda) \prod_{i=1}^{l} \left\{ \frac{\sigma^{r_i \sigma}}{(\Gamma(\nu - \gamma_i))^{r_i} (\nu - \gamma_i - 1 + \sigma)^{r_i \sigma}} \right\} \cdot$$

$$\frac{(\lambda - m)^{\left(\sum_{i=1}^{l} (\nu - \gamma_i - 1) r_i + \sigma r\right) + \frac{1}{s_1}}}{\left(\left(\sum_{i=1}^{l} (\nu - \gamma_i - 1) r_i s_1\right) + r s_1 \sigma + 1\right)^{\frac{1}{s_1}}} \left(\int_m^{\lambda} q_2(w) \left| D_{*m}^{\nu} f(w) \right|^p dw \right)^{\frac{r}{p}}. \tag{50}$$

Using (50) and properties (P) and (ii) we obtain

Theorem 32 *All here as in Theorem 31. Set*

$$Q_1(A) := \left(\int_{m1_H}^{A} (q_1)^{s_1'} \right)^{\frac{1}{s_1'}} \tag{51}$$

and

$$Q_2(A) := \left(\int_{m1_H}^{A} (q_2)^{\frac{-s_2'}{p}} \right)^{\frac{r}{s_2'}}. \tag{52}$$

Then

$$\int_{m1_H}^{A} q_1 \prod_{i=1}^{l} \left| D_{*m}^{\gamma_i} f \right|^{r_i} \leq$$

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$$Q_1(A)\,Q_2(A)\prod_{i=1}^{l}\left\{\frac{\sigma^{r_i\sigma}}{\left(\Gamma\left(\nu-\gamma_i\right)\right)^{r_i}\left(\nu-\gamma_i-1+\sigma\right)^{r_i\sigma}}\right\}\cdot$$

$$\frac{\left(A-m1_H\right)^{\left(\sum_{i=1}^{l}(\nu-\gamma_i-1)r_i+\sigma r\right)+\frac{1}{s_1}}}{\left(\left(\sum_{i=1}^{l}\left(\nu-\gamma_i-1\right)r_is_1\right)+rs_1\sigma+1\right)^{\frac{1}{s_1}}}\left(\int_{m1_H}^{A}q_2\left|D_{*m}^{\nu}f\right|^p\right)^{\frac{r}{p}}. \qquad (53)$$

One can give many more operator Opial type (both integer and fractional) inequalities.

We choose to stop here.

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Numerical solution of the generalized Hirota-Satsuma coupled Korteweg-de Vries equation by Fourier Pseudospectral method

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Abstract

In this paper, an approximate solution of the generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation by the use of Fourier pseudospectral method is presented. A time discrete scheme is constructed by approximating the time derivative using forward difference formula, while the pseudospectral method is used in the space direction. The stability and convergence of the scheme are investigated using the energy method. The numerical results reveal that the Fourier pseudospectral method is a convenient, effective and accurate method to solve the generalized HS coupled KdV equation.

Key words: Generalized Hirota-Satsuma coupled Korteweg-de Vries equation, Fourier pseudospectral method, Stability, Convergence.

1 Introduction

The generalized HS coupled KdV equations are as follows [1, 2]:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u\frac{\partial u}{\partial x} + 3\frac{\partial}{\partial x}(vw), \qquad x \in \Omega, t \in [0,T], \tag{1.1}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial^3 v}{\partial x^3} + 3u\frac{\partial v}{\partial x}, \qquad x \in \Omega, t \in [0,T], \tag{1.2}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial^3 w}{\partial x^3} + 3u\frac{\partial w}{\partial x}, \qquad x \in \Omega, t \in [0,T] \tag{1.3}$$

with initial conditions

$$u(x,0) = f(x), \qquad v(x,0) = g(x), \qquad w(x,0) = h(x), \qquad x \in \Omega, \tag{1.4}$$

and boundary conditions

$$u(-L,t) = u(L,t) = 0,\ v(-L,t) = v(L,t) = 0,\ w(-L,t) = w(L,t) = 0,\ t \in [0,T], \tag{1.5}$$

where $\Omega = [-L, L]$. Hirota-Satsuma [1] introduced generalized the HS coupled KdV equations in 1976 and these equations are models of shallow water waves. The equations (1.1)–(1.5) have travelling wave solutions and multiple soliton solutions.

The equations (1.1)–(1.5) have attracted the attention of many researchers and a lot of work has already been carried out on solution methods. For example, the homotopy perturbation method (HPM) by Ganji and Rafei [3], homotopy analysis method (HAM) and Adomian's decomposition method (ADM) by Abbasbandy [4], modified extended tanh function method by Ali [5], direct algebraic method by Zhang Huiqun [6]. Rong Jihong et al. [7] used bifurcation theory technique. The auxiliary function method was used by Yang Feng and Hong-Qing [8], analytical technique by Ganji et al. [9], homogenous balance

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method by Adel Raly et al. [10]. Jacobi elliptic functions expansion method by Baojin Hong [11]. Travelling wave solutions of the above equations investigated by Zuo and Zhang [12], Xie and Ding [13], Feng and Li [14]. A differential transform method (DTM) and reduced differential transform method (RDTM) was used by Reze and Malek [15], Hirota's bilinear method and pfaffian techniques by Junchao Chen et al. [16], while the Lie group method was applied by Mina B. et al. [17].

1.1 A brief review of Fourier pseudospectral method

In the last two decades spectral methods have been extensively used in the field of numerical solution of nonlinear partial differential equations. The use of spectral methods for solving partial differential and integro-differential equations have the advantage that its accuracy is higher than other standard numerical methods. Spectral methods retain the exponential rate of convergence when the solutions of the problems is sufficiently smooth. Spectral methods have three different categories namely Galerkin method, collocation method and tau method. The pseudospectral method is a type of spectral method which is easy to apply for nonlinear partial differential equations with periodic boundary value problems. For a more detailed discussion of spectral methods, please see ([18, 19, 20, 21, 22]).

The Fourier pseudospectral method involves two steps. First, the discrete representation of the solution is constructed by using trigonometric polynomial to interpolate the solution at collocation points. Second, the equations for the discrete values of the solution are obtained from the original equations. This second step involves finding an approximation for the differential operator in terms of the discrete values of the solution at collocation points. For detailed, please see ([18, 19, 23, 26]).

1.2 The main aim of the paper

In this paper, a Fourier pseudospectral method is applied to solve the generalized HS coupled KdV equation. A finite difference method is used in the time direction and Fourier pseudospectral method in the space direction. The stability of the time discrete scheme and convergence of the approximate solution is investigated by the energy method [29]. Numerical results are shown to demonstrate the efficiency of the method. It should be noted that Darvishi et al. [27] solved the same equation by pseudospectral method and transformed the partial differential equation to ordinary differential equations. They found the numerical solution by using classical fourth-order Runge-Kutta method. There is no proof of stability and convergence. In our paper, we follow the approach of [23, 28].

The outline of the paper is as follows. In section 2 we present some preliminaries which will be used in next two sections. Section 3 is related to stability of the scheme for generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation. Convergence of the approximate solution is proved in section 4. Numerical results are presented for the applicability of the method section 5. Finally the conclusion is given in section 6.

2 Preliminaries

The inner product and norm are defined by $(u,v)=\int_\Omega u(x)v(x)dx$ and $\|u\|^2=(u,u)$ respectively. The maximum norm is denoted by $\|u\|_\infty$. The periodic Sobolev space is defined by [23]:

$$H^1=\left\{u\in L^2(R):\frac{du}{dx}\in L^2(R)\right\},\qquad H^1_p=\{u\in H^1(R):u(x-L)=u(x+L)\}.$$

The Sobolev norm and semi-norms are defined by [23]:

$$\|u\|=(u,u)^{1/2},\quad \|u\|_{H^1}=(\|u\|^2+\|\frac{\partial u}{\partial x}\|^2)^{1/2},\quad |u|_k=|u|_{H^k}=\sum_{|\beta|=k}\Big(\int_\Omega (D^\beta u)^2dx\Big)^{1/2}.$$

We define $t_n=n\tau,\quad n=0,1,...,N,$ where $\tau=T/N$ is the step size in time direction. The equation (1.1)–(1.3) is evaluated at the point (x,t_n), $n=0,1,\dots,N$. We denote $u^n=u(x,t_n)$, $v^n=v(x,t_n)$ and

$w^n = w(x, t_n)$, then equation (1.1), (1.2) and (1.3) can be written as:

$$u^{n+1} = u^n + \tau \left(\frac{1}{2} \frac{\partial^3}{\partial x^3} u^n - 3u^n \frac{\partial u^n}{\partial x} + 3 \frac{\partial}{\partial x}(v^n w^n) \right) + \tau R_1^n, \tag{2.1}$$

$$v^{n+1} = v^n + \tau \left(-\frac{\partial^3}{\partial x^3} v^n + 3u^n \frac{\partial v^n}{\partial x} \right) + \tau R_2^n, \tag{2.2}$$

$$w^{n+1} = w^n + \tau \left(-\frac{\partial^3}{\partial x^3} w^n + 3u^n \frac{\partial w^n}{\partial x} \right) + \tau R_3^n, \tag{2.3}$$

where R_1^n, R_2^n, and R_3^n are residual of the equation (2.1), (2.2) and (2.3) respectively. Furthermore $|R_1^n| < C_1\tau$, $|R_2^n| < C_2\tau$ and $|R_3^n| < C_3\tau$ for some positive constants C_1, C_2 and C_3. By ignoring the small terms R_1^n, R_2^n and R_3^n in the above equations, the time discrete scheme for the equation (2.1), (2.2) and (2.3) can be obtained as:

$$U^{n+1} = U^n + \tau \left(\frac{1}{2} \frac{\partial^3}{\partial x^3} U^n - 3U^n \frac{\partial U^n}{\partial x} + 3 \frac{\partial}{\partial x}(V^n W^n) \right), \tag{2.4}$$

$$V^{n+1} = V^n + \tau \left(-\frac{\partial^3}{\partial x^3} V^n + 3U^n \frac{\partial V^n}{\partial x} \right), \tag{2.5}$$

$$W^{n+1} = W^n + \tau \left(-\frac{\partial^3}{\partial x^3} W^n + 3U^n \frac{\partial W^n}{\partial x} \right), \tag{2.6}$$

where $U^n = U(x, t_n)$, $V^n = V(x, t_n)$ and $W^n = W(x, t_n)$. We present a lemma, which will be useful for the proof of stability and convergence.

Lemma 2.1 ([24]). *If $m \geq 1$, and $u, v \in H^m(\Omega)$, there exists a constant C independent of u, v and N, such that*

$$\|uv\|_m \leq C \|u\|_m \|v\|_m .$$

3 Stability

Assume $U^n(x,t)$ to be the approximate solution of $u^n(x,t)$, $V^n(x,t)$ to be the approximate solution of $v^n(x,t)$ and $W^n(x,t)$ be the approximate solution of $w^n(x,t)$. For simplicity we denote $u^n = u^n(x,t)$ and similarly for other variables. Let

$$\widetilde{u}^n = u^n - U^n, \qquad \widetilde{v}^n = v^n - V^n, \qquad \widetilde{w}^n = w^n - W^n.$$

Subtracting (2.4) from (2.1), (2.5) from (2.2) and (2.6) from (2.3) results in

$$\widetilde{u}^{n+1} = \widetilde{u}^n + \frac{\tau}{2} \frac{\partial^3}{\partial x^3} \widetilde{u}^n - 3\tau \left(u^n \frac{\partial u^n}{\partial x} - U^n \frac{\partial U^n}{\partial x} \right) + 3\tau \frac{\partial}{\partial x} \left(v^n w^n - V^n W^n \right), \tag{3.1}$$

$$\widetilde{v}^{n+1} = \widetilde{v}^n + \tau \left(-\frac{\partial^3}{\partial x^3} \widetilde{v}^n \right) + 3\tau \left(u^n \frac{\partial v^n}{\partial x} - U^n \frac{\partial V^n}{\partial x} \right), \tag{3.2}$$

$$\widetilde{w}^{n+1} = \widetilde{w}^n + \tau \left(-\frac{\partial^3}{\partial x^3} \widetilde{w}^n \right) + 3\tau \left(u^n \frac{\partial w^n}{\partial x} - U^n \frac{\partial W^n}{\partial x} \right). \tag{3.3}$$

Taking the inner product of (3.1), (3.2) and (3.3) with $\widetilde{u}^{n+1}$, $\widetilde{v}^{n+1}$ and $\widetilde{w}^{n+1}$ respectively. By applying Cauchy-Schwartz inequality, algebraic and Young's inequalities, we have

$$\begin{aligned} (1+3\tau)\|\widetilde{u}^{n+1}\|^2 + \tau \left\| \frac{\partial \widetilde{u}^{n+1}}{\partial x} \right\|^2 \leq \|\widetilde{u}^n\|^2 + \tau \left\| \frac{\partial^2 \widetilde{u}^n}{\partial x^2} \right\|^2 - 3\tau \left\| u^n \frac{\partial u^n}{\partial x} - U^n \frac{\partial U^n}{\partial x} \right\|^2 \\ + 3\tau \left\| v^n w^n - V^n W^n \right\|^2 , \end{aligned} \tag{3.4}$$

$$(1+3\tau)\|\widetilde{v}^{n+1}\|^2 + \tau \left\| \frac{\partial \widetilde{v}^{n+1}}{\partial x} \right\|^2 \leq \|\widetilde{v}^n\|^2 + \tau \left\| \frac{\partial^2 \widetilde{v}^n}{\partial x^2} \right\|^2 + 3\tau \left\| u^n \frac{\partial v^n}{\partial x} - U^n \frac{\partial V^n}{\partial x} \right\|^2 , \tag{3.5}$$

$$(1+3\tau)\|\widetilde{w}^{n+1}\|^2+\tau\left\|\frac{\partial\widetilde{w}^{n+1}}{\partial x}\right\|^2\leq\|\widetilde{w}^n\|^2+\tau\left\|\frac{\partial^2\widetilde{w}^n}{\partial x^2}\right\|^2+3\tau\left\|u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x}\right\|^2, \tag{3.6}$$

Now we are going to estimate nonlinear terms of (3.4), (3.5) and (3.6). Again we apply Cauchy-Schwartz inequality and lemma 2.1, we get

$$\begin{aligned}\left\|u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\| &= \left\|u^n\frac{\partial u^n}{\partial x}-u^n\frac{\partial U^n}{\partial x}+u^n\frac{\partial U^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\| \\ &= \left\|u^n\left(\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right)+\frac{\partial U^n}{\partial x}(u^n-U^n)\right\| \\ &\leq \|u^n\|_\infty\left\|\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right\|+\left\|\frac{\partial U^n}{\partial x}\right\|_\infty\|u^n-U^n\| \\ &\leq C_4\left(\left\|\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right\|+\|u^n-U^n\|\right)\end{aligned}$$

where $C_4=\left(\|\frac{\partial U^n}{\partial x}\|_\infty,\,\|u^n\|_\infty\right)$, we obtain

$$\left\|u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|^2\leq C_4\left(\left\|\frac{\partial\widetilde{u}^n}{\partial x}\right\|^2+\|\widetilde{u}^n\|^2\right)$$

Similarly we can apply Cauchy-Schwartz inequality and lemma 2.1, we get the estimation of nonlinear terms of (3.4), (3.5) and (3.6), we have

$$\begin{aligned}&\|v^nw^n-V^nW^n\|^2\leq C_5\left(\|\widetilde{v}^n\|^2+\|\widetilde{w}^n\|^2\right)\\ &\left\|u^n\frac{\partial v^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|^2\leq C_6\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{v}^n}{\partial x}\right\|^2\right),\\ &\left\|u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x}\right\|^2\leq C_7\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{w}^n}{\partial x}\right\|^2\right).\end{aligned}$$

where $C_5=\left(\|\frac{\partial V^n}{\partial x}\|_\infty,\,\|u^n\|_\infty\right)$, $C_6=\left(\|\frac{\partial W^n}{\partial x}\|_\infty,\,\|u^n\|_\infty\right)$, where $C_7=(\|v^n\|_\infty,\,\|W^n\|_\infty)$. Substituting the value of above values into (3.4), (3.5) and (3.6). Further more $\widetilde{C}=\max(C_4,C_5,C_6,C_8)$. We get

$$\begin{aligned}(1-3\tau)&\left(\|\widetilde{u}^{n+1}\|^2+\left\|\frac{\partial\widetilde{u}^{n+1}}{\partial x}\right\|^2+\|\widetilde{v}^{n+1}\|^2+\left\|\frac{\partial\widetilde{v}^{n+1}}{\partial x}\right\|^2+\|\widetilde{w}^{n+1}\|^2+\left\|\frac{\partial\widetilde{w}^{n+1}}{\partial x}\right\|^2\right)\\ &\leq(1+3\tau)\widetilde{C}\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{u}^n}{\partial x}\right\|^2+\|\widetilde{v}^n\|^2+\left\|\frac{\partial\widetilde{v}^n}{\partial x}\right\|^2+\|\widetilde{w}^n\|^2+\left\|\frac{\partial\widetilde{w}^n}{\partial x}\right\|^2\right)\end{aligned} \tag{3.7}$$

$$\begin{aligned}\|\widetilde{u}^{n+1}\|_{H^1}^2+\|\widetilde{v}^{n+1}\|_{H^1}^2+\|\widetilde{w}^{n+1}\|_{H^1}^2 &\leq\left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)\left(\|\widetilde{u}^n\|_{H^1}^2+\|\widetilde{v}^n\|_{H^1}^2+\|\widetilde{w}^n\|_{H^1}^2\right)\\ &\leq\left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)^2\left(\|\widetilde{u}^{n-1}\|_{H^1}^2+\|\widetilde{v}^{n-1}\|_{H^1}^2+\|\widetilde{w}^{n-1}\|_{H^1}^2\right)\\ &\;\;\vdots\\ &\leq\left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)^{n+1}\left(\|\widetilde{u}^0\|_{H^1}^2+\|\widetilde{v}^0\|_{H^1}^2+\|\widetilde{w}^0\|_{H^1}^2\right)\end{aligned}$$

Let

$$\lim_{n\longrightarrow\infty}\left(\frac{\widetilde{C}(1+3\tau)}{1-3\tau}\right)^{n+1}=\lim_{n\longrightarrow\infty}\left(\frac{\widetilde{C}(1+\frac{3\tau}{n+1})}{1-\frac{3\tau}{n+1}}\right)^{n+1}=\frac{\widetilde{C}e^{3\tau}}{e^{-3\tau}}=e^{6\widetilde{C}\tau} \tag{3.8}$$

Therefore
$$\|\widetilde{u}^{n+1}\|_{H^1}^2+\|\widetilde{v}^{n+1}\|_{H^1}^2+\|\widetilde{w}^{n+1}\|_{H^1}^2\leq\sqrt{e^{6\widetilde{C}\tau}}\left(\|\widetilde{u}^0\|_{H^1}^2+\|\widetilde{v}^0\|_{H^1}^2+\|\widetilde{w}^0\|_{H^1}^2\right)$$

Theorem 1. *Let u_0, v_0 and w_0 belong to $H^1(\Omega)$. Further, let u^n, v^n and w^n be the solution for initial boundary value problem (1.1)–(1.5) and U^n, V^n and W^n be the solution of the time discrete scheme (2.4)–(2.6). If $\tau<1/3$ then solution of the discrete scheme is stable in H^1 norm*

4 Convergence

In this section we consider the convergence of of approximate solution of generalized HS coupled KdV equation. Define
$$\widetilde{U}^n=u^n-U^n,\qquad\widetilde{V}^n=v^n-V^n,\qquad\widetilde{W}^n=w^n-W^n.$$
From equations (2.1)–(2.3) and (2.4)–(2.6), we obtain
$$\widetilde{U}^{n+1}=\widetilde{U}^n+\frac{\tau}{2}\frac{\partial^3\widetilde{U}^n}{\partial x^3}+3\tau\left(u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right)-3\tau\frac{\partial}{\partial x}\left(v^nw^n-V^nW^n\right)+\tau R_1^n,\tag{4.1}$$
$$\widetilde{V}^{n+1}=\widetilde{V}^n+\tau\left(-\frac{\partial^3\widetilde{V}^n}{\partial x^3}\right)+3\tau\left(u^n\frac{\partial v^n}{\partial x}-U^n\frac{\partial V^n}{\partial x}\right)+\tau R_2^n,\tag{4.2}$$
$$\widetilde{W}^{n+1}=\widetilde{W}^n+\tau\left(-\frac{\partial^3\widetilde{W}^n}{\partial x^3}\right)+3\tau\left(u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x}\right)+\tau R_3^n.\tag{4.3}$$

Taking the inner product of (4.1), (4.2) and (4.3) with $\widetilde{U}^{n+1}$, $\widetilde{V}^{n+1}$ and $\widetilde{W}^{n+1}$ respectively, yields
$$\|\widetilde{U}^{n+1}\|^2\leq\frac{1}{2}\|\widetilde{U}^n\|^2-\frac{\tau}{2}\left(\left\|\frac{\partial^2\widetilde{U}^n}{\partial x^2}\right\|^2+\left\|\frac{\partial\widetilde{U}^{n+1}}{\partial x}\right\|^2\right)+\tau|R_1^n|\|\widetilde{U}^{n+1}\|+G_1+G_2,\tag{4.4}$$
$$\|\widetilde{V}^{n+1}\|^2\leq\frac{1}{2}\|\widetilde{V}^n\|^2+\frac{\tau}{2}\left(\left\|\frac{\partial^2\widetilde{V}^n}{\partial x^2}\right\|^2+\left\|\frac{\partial\widetilde{V}^{n+1}}{\partial x}\right\|^2\right)+\tau|R_2^n|\|\widetilde{V}^{n+1}\|+G_3,\tag{4.5}$$
$$\|\widetilde{W}^{n+1}\|^2\leq\frac{1}{2}\|\widetilde{W}^n\|^2+\frac{\tau}{2}\left(\left\|\frac{\partial^2\widetilde{W}^n}{\partial x^2}\right\|^2+\left\|\frac{\partial\widetilde{W}^{n+1}}{\partial x}\right\|^2\right)+\tau|R_3^n|\|\widetilde{W}^{n+1}\|+G_4,\tag{4.6}$$
where
$$G_1=-3\tau\left(u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x},\widetilde{U}^{n+1}\right),\qquad G_2=3\tau\frac{\partial}{\partial x}\left(v^nw^n-V^nW^n,\widetilde{U}^{n+1}\right),$$
$$G_3=\tau\left(u^n\frac{\partial v^n}{\partial x}-U^n\frac{\partial V^n}{\partial x},\widetilde{V}^{n+1}\right),\qquad G_4=3\tau\left(u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x},\widetilde{W}^{n+1}\right).$$
By using the algebraic inequality and lemma 2.1, we get
$$|G_1|\leq3\tau\left\|u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|^2+\|\widetilde{U}^{n+1}\|^2\leq C_8\left(\left\|\frac{\partial\widetilde{u}^n}{\partial x}\right\|^2+\|\widetilde{u}^n\|^2\right)+\|\widetilde{U}^{n+1}\|^2,\tag{4.7}$$
$$|G_2|\leq3\tau\left\|v^nw^n-V^nW^n\right\|^2+\|\widetilde{U}^{n+1}\|^2\leq C_9\left(\|\widetilde{v}^n\|^2+\|\widetilde{w}^n\|^2\right)+\|\widetilde{U}^{n+1}\|^2,\tag{4.8}$$
$$|G_3|\leq3\tau\left\|u^n\frac{\partial v^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|^2+\|\widetilde{V}^{n+1}\|^2\leq C_{10}\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{v}^n}{\partial x}\right\|^2\right)+\|\widetilde{V}^{n+1}\|^2,\tag{4.9}$$
$$|G_4|\leq3\tau\left\|u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x}\right\|^2+\|\widetilde{W}^{n+1}\|^2\leq C_{11}\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{w}^n}{\partial x}\right\|^2\right)+\|\widetilde{W}^{n+1}\|^2,\tag{4.10}$$
where C_8, C_9, C_{10} and C_{11} are constants independent of τ and N. Let $\widetilde{M}=\max(C_8,C_9,C_{10},C_{11})$ Putting the values of (4.7) and (4.8) in to (4.4). Also substituting the values of (4.9) and (4.10) in to

(4.5) and (4.6) respectively. By using the same technique as in the previous section, we can obtain a equation similar to (3.7).

$$\begin{aligned}(1-3\tau)&\left(\|\widetilde{U}^{n+1}\|^2+\left\|\frac{\partial\widetilde{U}^{n+1}}{\partial x}\right\|^2+\|\widetilde{V}^{n+1}\|^2+\left\|\frac{\partial\widetilde{V}^{n+1}}{\partial x}\right\|^2+\|\widetilde{W}^{n+1}\|^2+\left\|\frac{\partial\widetilde{W}^{n+1}}{\partial x}\right\|^2\right)\\ &\le(1+3\tau)\widetilde{M}\left(\|\widetilde{U}^{n}\|^2+\left\|\frac{\partial\widetilde{U}^{n}}{\partial x}\right\|^2+\|\widetilde{V}^{n}\|^2+\left\|\frac{\partial\widetilde{V}^{n}}{\partial x}\right\|^2+\|\widetilde{W}^{n}\|^2+\left\|\frac{\partial\widetilde{W}^{n}}{\partial x}\right\|^2\right)\\ &+\tau\vartheta^2|R_1^n|^2+\tau\vartheta^2|R_2^n|^2+\tau\vartheta^2|R_3^n|^2.\end{aligned}\tag{4.11}$$

$$\begin{aligned}\|\widetilde{U}^{n+1}\|_{H^1}^2+\|\widetilde{V}^{n+1}\|_{H^1}^2+\|\widetilde{W}^{n+1}\|_{H^1}^2\le\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)&\left[\left(\|\widetilde{U}^{n}\|_{H^1}^2+\|\widetilde{V}^{n}\|_{H^1}^2+\|\widetilde{W}^{n}\|_{H^1}^2\right)\right.\\ &\left.+\left(\tau\vartheta^2|R_1^n|^2+\tau\vartheta^2|R_2^n|^2+\tau\vartheta^2|R_3^n|^2\right)\right]\end{aligned}$$

Let

$$\begin{aligned}\widetilde{E}^{n+1}&=\|\widetilde{U}^{n+1}\|_{H^1}^2+\|\widetilde{V}^{n+1}\|_{H^1}^2+\|\widetilde{W}^{n+1}\|_{H^1}^2\\ \widetilde{R}^{n}&=\tau\vartheta^2(|R_1^n|^2+|R_2^n|^2+|R_3^n|^2)\end{aligned}$$

Then equation (4.11) is written as

$$\begin{aligned}\widetilde{E}^{n+1}&\le\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)\left[\widetilde{E}^{n}+\tau\vartheta^2\widetilde{R}^{n}\right]\\ &\le\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^2\widetilde{E}^{n-1}+\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)\tau\vartheta^2\widetilde{R}^{n-1}+\tau\vartheta^2\widetilde{R}^{n}\\ &\vdots\\ &\le\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^n\widetilde{E}^{0}+\tau\vartheta^2\sum_{j=0}^{n}\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^j\widetilde{R}^{n-j}\end{aligned}$$

Since $\widetilde{E}^0=0$, we obtain

$$\widetilde{E}^{n+1}\le(n+1)\tau\vartheta^2\sum_{j=0}^{n}\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^j\widetilde{R}^{n-j}$$

Finally, using the result of (3.8) we get

$$\|u^n-U^n\|+\|v^n-V^n\|+\|w^n-W^n\|\le(n+1)\tau\vartheta^2e^{6\widetilde{M}t}|R^n|\le\widetilde{M}\sqrt{\vartheta^2e^{6\widetilde{M}t}}\tau$$

Theorem 2. *Let u^n, v^n and w^n be the solution for initial boundary value problem for (1.1)–(1.5) and let U^n, V^n and W^n be the solution of (2.4)–(2.6) time discrete scheme. If the conditions of Theorem 1 holds. Then the time discrete solution is convergent in H^1 and the convergence rate is $O(\tau)$.*

5 Numerical Results

In this section, we present numerical results to show the efficiency and accuracy of the method, mentioned in previous section. We define maximum error $\|E(u)\|_\infty$, $\|E(v)\|_\infty$ and $\|E(w)\|_\infty$ as follows

$$\begin{aligned}\|E(u)\|_\infty&=\max_{0\le j\le N}|u(x_j,t)-U(x_j,t)|,\\ \|E(v)\|_\infty&=\max_{0\le j\le N}|v(x_j,t)-V(x_j,t)|,\\ \|E(w)\|_\infty&=\max_{0\le j\le N}|w(x_j,t)-W(x_j,t)|,\end{aligned}$$

where u,v,w are the exact solutions of (1.1)–(1.5) and U,V,W are the approximate solutions.

5.1 Example 1

Consider the generalized HS coupled KdV equations (1.1)–(1.5) with the initial conditions [25]:

$$u(x,0) = \frac{\beta - 2\alpha^2}{3} + 2\alpha^2 \tanh^2(\alpha x),$$
$$v(x,0) = \frac{4\alpha^2(\beta + \alpha^2)}{3c_1}\left(\frac{c_0}{c_1} - \tanh(\alpha x)\right),$$
$$w(x,0) = c_0 + c_1 \tanh(\alpha x)$$

where c_0, c_1, α and β are arbitrary constants. For practical computation we choose the parameters as $c_0 = 1.5$, $c_1 = 0.1$, $\alpha = 0.1$, $\beta = 1.5$ and $N = 64$.

The absolute error of the U, V and W are given in Table-1, Table-2 and Table-3 respectively. The results of the present method are compared with the results of methods already available in the literature i.e., Reza and Malik [15], Xie and Ding [13] for the variable U, V and W at different values of t. We observe that the absolute error is less than 0.2×10^{-6}. The numerical results of the present method are better than the results obtained by Reza and Malik [15], Xie and Ding [13]. The space-time graphs of U, V and W are given in Figure-1, Figure-2 and Figure-3 respectively. The graph of exact and approximate solution are plotted in Figure-1 to Figure-3 at different values of t.

Table 1: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable U at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	3.290e-06	6.719e-10	6.739e-10	2.541e-06
0.4	5.252e-05	1.711e-07	1.719e-07	3.345e-07
0.7	1.597e-04	1.593e-06	1.603e-06	6.144e-07
1.0	3.227e-04	6.574e-06	6.625e-06	8.363e-07

Table 2: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable V at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	8.559e-11	3.320e-13	8.828e-11	1.430e-08
0.4	1.698e-10	8.490e-11	3.818e-08	2.234e-08
0.7	8.793e-10	7.951e-10	5.028e-07	5.933e-08
1.0	3.389e-09	3.306e-09	2.689e-06	7.474e-08

Table 3: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable W at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	5.349e-08	2.075e-10	4.385e-11	6.095e-08
0.4	1.061e-07	5.306e-08	1.896e-08	7.780e-08
0.7	5.496e-07	4.969e-07	2.497e-07	9.188e-08
1.0	2.118e-06	2.066e-06	1.335e-06	8.989e-08

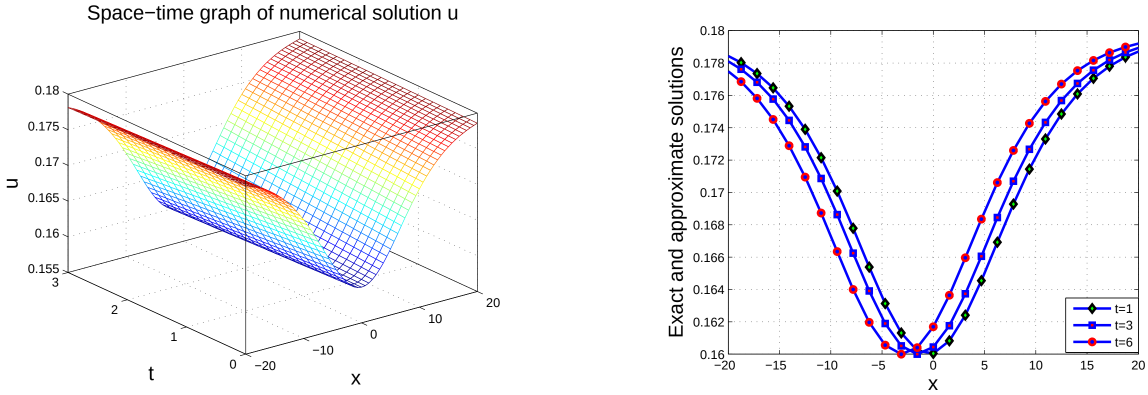

Figure 1: The left figure shows the space-time graphs of U, while the right figure shows the graph of U for different values of t.

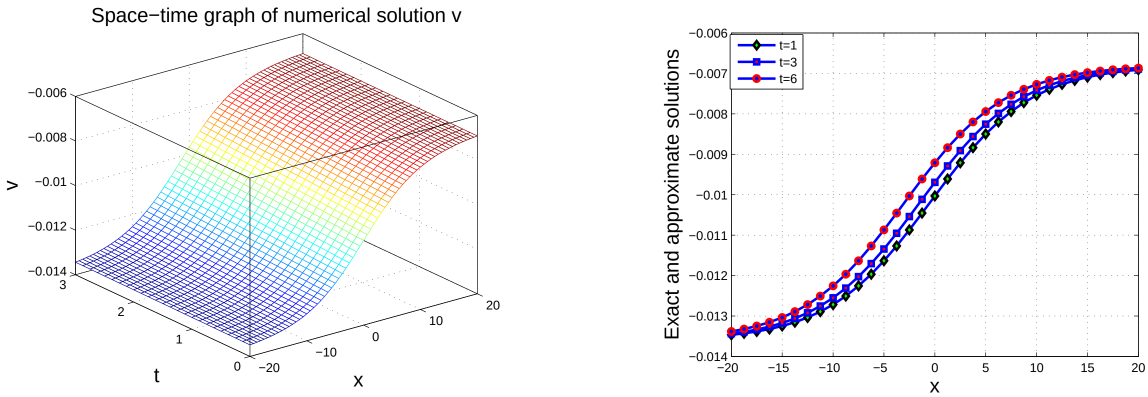

Figure 2: The left figure shows the space-time graphs of V, while the right figure shows the graph of V for different values of t.

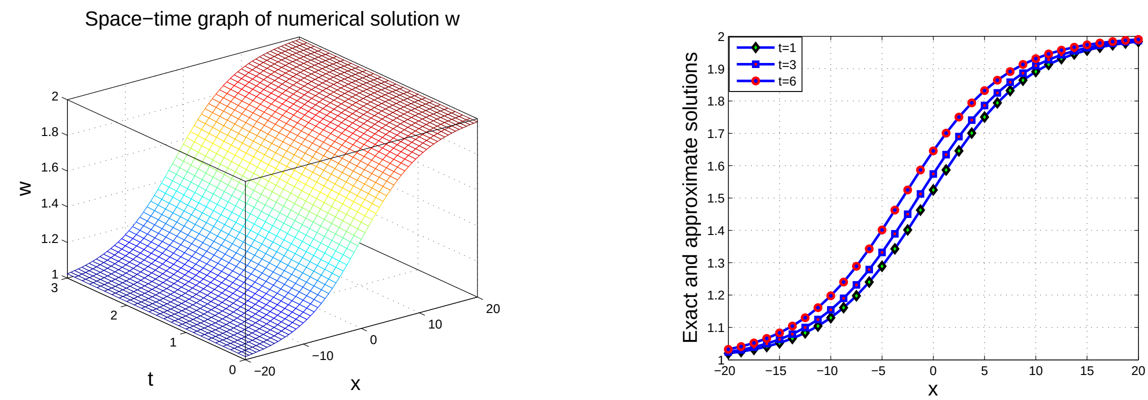

Figure 3: The left figure shows the space-time graphs of W, while the right figure shows the graph of W for different values of t.

5.2 Example 2

We consider the generalized HS coupled KdV equations (1.1)–(1.5) with the initial conditions [25]:

$$\begin{aligned}
u(x,0) &= \frac{\beta - 8\alpha^2}{3} + 4\alpha^2 \tanh^2(\alpha x), \\
v(x,0) &= -\frac{4}{3}\frac{\alpha^2(3\alpha^2 c_0 - 2\beta c_2 + 4\alpha^2 c_2)}{c_2^2} + \left(\frac{4\alpha^2}{c_2}\tanh^2(\alpha x)\right), \\
w(x,0) &= c_0 + c_2 \tanh^2(\alpha x)
\end{aligned}$$

where c_0, c_1, c_2, α and β are arbitrary constants. We choose the arbitrary constants for practical computation as, $c_0 = 1.5$, $c_1 = 0.1$, $c_2 = 0.5$, $\alpha = 0.1$, $\beta = 1.5$ and $N = 64$.

The absolute error of U, V and W are given in Table-4, Table-5 and Table-6 respectively. we compare the results of the present method with Reza and Malik [15], Xie and Ding [13] for the variable U, V and W at different value of t. The results are already available in the literature. We observe that the absolute error is less than 0.2×10^{-6}. The numerical results of the present method are comparatively better than the results obtained from Reza and Malik [15], Xie and Ding [13]. The space-time graphs of U, V and W are given in Figure-4, Figure-5 and Figure-6 respectively. The graph of exact and approximate solution are shown in Figure-4 to Figure-6 at different value of t.

Table 4: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable U at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	4.279e-09	1.660e-11	2.495e-05	3.762e-09
0.4	8.490e-09	4.245e-09	1.146e-04	4.677e-09
0.7	4.396e-08	3.975e-08	2.293e-04	5.366e-09
1.0	1.694e-07	1.653e-07	3.744e-04	7.595e-09

Table 5: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable V at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	8.559e-11	3.320e-13	8.828e-11	1.430e-08
0.4	1.698e-10	8.490e-11	3.818e-08	2.234e-08
0.7	8.793e-10	7.951e-10	5.028e-07	5.933e-08
1.0	3.389e-09	3.306e-09	2.689e-06	7.474e-08

Table 6: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable W at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	5.349e-08	2.075e-10	4.385e-11	6.095e-08
0.4	1.061e-07	5.306e-08	1.896e-08	7.780e-08
0.7	5.496e-07	4.969e-07	2.497e-07	9.188e-08
1.0	2.118e-06	2.066e-06	1.335e-06	8.989e-08

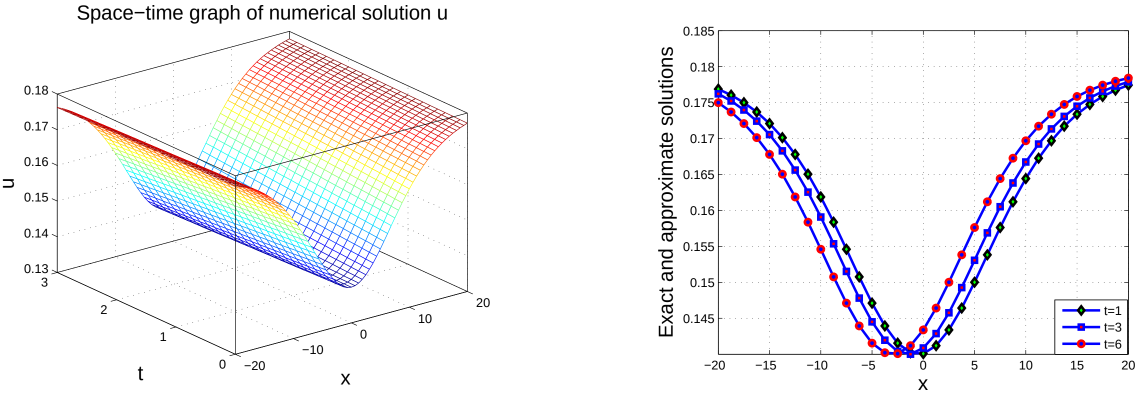

Figure 4: The left figure shows the space-time graphs of U, while the right figure shows the graph of U for different values of t.

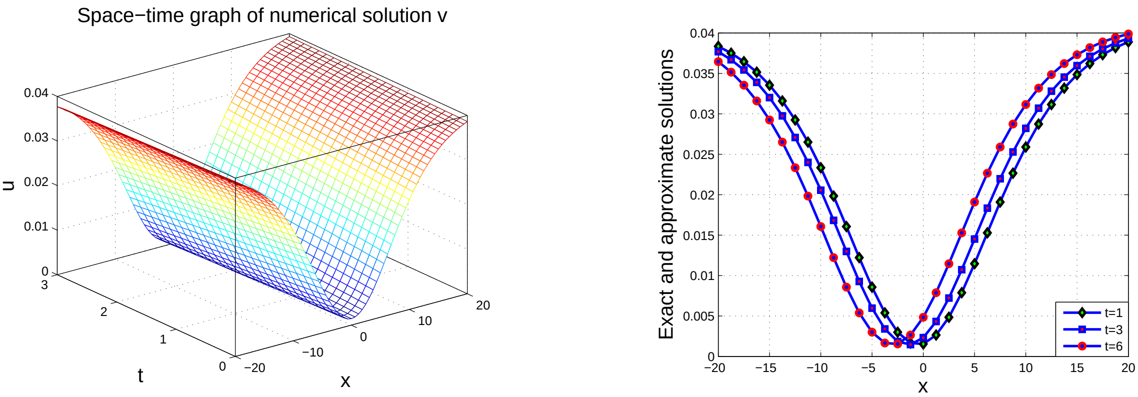

Figure 5: The left figure shows the space-time graphs of V, while the right figure shows the graph of V for different values of t.

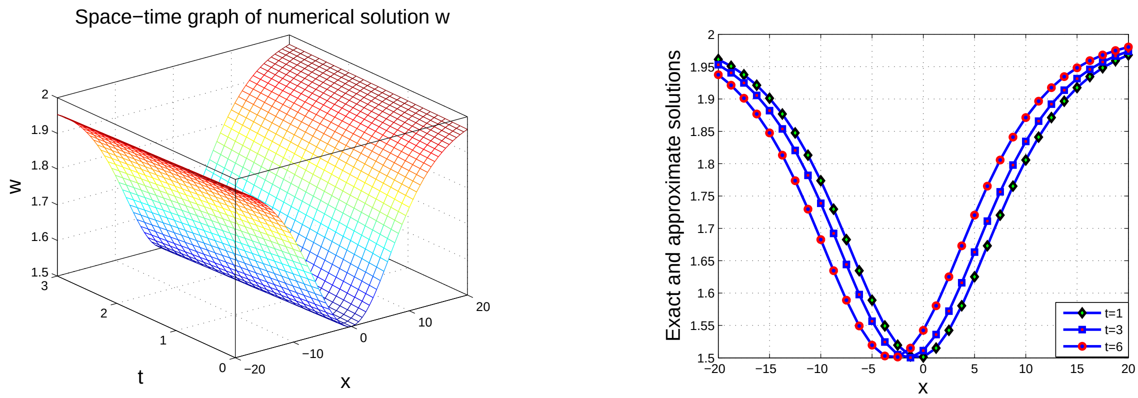

Figure 6: The left figure shows the space-time graphs of W, while the right figure shows the graph of W for different values of t.

6 Conclusion

In this paper, the generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation is solved numerically using the Fourier pseudospectral method. The time derivative of discrete scheme is approximated by the forward finite difference formula while the pseudospectral method is used in the space direction. The stability and convergence of the discrete scheme are proved by energy estimation method. The obtained solution is presented graphically at various time levels. The numerical results reveal that the Fourier pseudospectral method is convenient, effective and accurate to solve the generalized HS coupled KdV equations.

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